Quantum Dynamics as the Foundation of Deductive Theoretical Physics

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Abstract

There is a fundamental distinction between the two types of physical approaches, the dynamic, strictly causal kind, and the merely statistical kind. The Hilbert space-based formulation of quantum mechanics provides the most prominent example of a statistical order-from-disorder mechanism.

The most striking characteristics of a Hilbert space compared to classical or relativistic geometries are its internal geometry in the form of an inner product and the resulting property of being a closed manifold. Accordingly, a Hilbert space-based formulation of quantum dynamics that provides a statistical order-from-order mechanism must provide the Hilbert space structure with additional tools that continue to guarantee the property of a closed manifold. To ensure that the order-from-order mechanism applies to the laws of classical physics, additional requirements are necessary:

1. *the laws of classical mechanics can be causally described by the "interaction" of quantum-mechanical and quantum-dynamic systems. In particular, the concept of the "Coulomb potential" can ultimately be explained as an approximate model of underlying quantum dynamic models.*
2. *classical phenomena can be traced back to quantum-dynamic causes by means of suitable order-from-order mechanisms and quantum-dynamic fluctuations can be statistically distinguished from quantum-mechanical processes, (ScE).*

The two new concepts of the proposed theory of quantum dynamics are:

1. an extension of the statistical Hilbert space L_2 of quantum mechanics to $H_{-1/2}$
2. two Krein-space representations of the Hilbert space $H_{-1/2}$, which define the concept of a physical ether, consisting of two interconnected ground-state quantum fluctuation and plasma quantum fluctuation models with different superposition and non-superposition states.

The metaphysical prerequisite for both models is a mathematically justifiable "density" assumption regarding this concept of a „physical ether". It includes a concept of dynamic energy that is independent of mechanical energy and independent of the concepts of the theory of motion - that is, in particular, independent of the concepts of space, time, and continuity. The laws of classical mechanics may be interpreted as aproximation solutions in Hilbert scales of underlying variational represention of quantum-mechanical laws governed by the so-called energy method.

At first glance, the extended basis quantum-mechanical Hilbert space framework $H_{-1/2}$ appears to be a cheap magic trick proves to be a valid step toward resolving the problem of a missing physical interpretation of a mathematical "world-point state." The compact embedding of $H_{-1/2}$ into $H_{-\mathrm{n}/2-\varepsilon}$, $\varepsilon > 0$, which is independent of all spatial dimensions, ensures - by virtue of Sobolev's embedding theorem - that continuous fields in classical physics can be generated by quantum-mechanical causes, regardless of the dimension of the underlying Euclidean space. The systems of this extended Hilbert space can thus accommodate the mathematical concept of the Newtonian or Coulomb potential, both of which, for mathematical reasons, only make physical sense in 3D Euclidean space. The concept of phenomenon-specific "point charges" that generate continuous fields throughout 3D Euclidean space can thus be completely replaced. Furthermore, it has been shown that the variation equations of the 3D-NSE initial boundary value problem, based on a "test space" $H_{-1/2}$, avoid existing "blow-up" solutions by means of L_2 test functions. In classical physics, the systems of the extended "test space" $H_{-1/2}$ could therefore be physically interpreted as "dynamic fluid/gas elements."

Introduction

The mathematical framework of classical physics is provided by calculus, the theory of partial differential equations and integral equations. The expanded mathematical framework in cosmology includes additional concepts from topology, as well as the concepts of metric (Riemann) and linearly normed spaces (Minkowski). The mathematical framework of quantum theory is provided by probability theory and Hilbert space theory (von Neumann). The phenomenon of "gravitation" cannot be described within the framework of the theory of motion, neither in conjunction with Euclidean geometry nor in conjunction with Minkowski geometry. The mathematical solution concept here is "geometrodynamics" (i.e., based on methods of algebraic topology) without metaphysical foundations (PIP). The paradigm of quantum theory for deriving quantum-mechanical equations of motion from classical Lagrangian equations of motion is "quantization of the classical Lagrangian equations of motion." The two mathematical methods for transferring the laws of motion in quantum mechanics to the various gauge theories of quantum dynamics are "regularization" and "renormalization," (DeP).

The key physical concept that comes into play in all of the above theories is that of a "potential." The Newtonian potential and the Coulomb potential are the same mathematical concept. In physics, the Coulomb potential plays a central role not only in classical electrodynamics but also in all plasma theories and nuclear physics, (ChF), (BeH). It may seem both simple and easy to handle from a physicist's perspective, but from a mathematical standpoint, it is by no means so. It describes the so-called fundamental solution of the potential equation for a given "function" - the delta or Dirac "function" - whose "regularity" also depends on the dimension of space R^n.

The concept of transferring the laws of motion from classical physics to relativistic quantum mechanics in a physical vacuum (with the aim of enabling classical, so-called conserved quantities such as charge and current density) is associated with the terms "angular momentum" and "spin-orbit operator." This is an operator that unites two concepts of physical momentum: angular momentum (intrinsic rotation) and linear momentum. The stationary states of the Dirac equation are determined by separating the radial components, thereby yielding two determining equations for calculating the stationary states of the underlying Hamiltonian operator. When calculating the Sommerfeld energy formula for the electron in the hydrogen atom based on an assumed Coulomb potential, these two determining equations yield two power series with different asymptotic behavior for very large and very small radii. In order to predict finite energy values, both power series must converge, which can only be achieved by means of a suitably chosen constant - the Sommerfeld fine-structure constant.

The empirical "measurement method" of quantum mechanics is based on the mathematical methods of probability theory and statistics. The corresponding Hilbert space L_2 is the space of quadratically integrable Sobolev functions. (The integration refers to the Lebesgue integral. These integrals allow for so-called null sets; that is, the Lebesgue integral taken over a null set has the value zero. Such countable sets include, for example, the countably many eigenvalues of a Hamiltonian operator. From a physical perspective, this circumstance leads to the concept of a "wave packet" (which is a divergent integral in the L_2 Hilbert space, since the Hamilton operator is defined only on a subspace of the L_2 Hilbert space). The mathematical solution concept is that of the delta or Dirac distribution. The Sobolev embedding theorem guarantees continuous field solutions of $\Delta u = \delta(x) \in H_{-n/2-\varepsilon}$. Thus, a physical requirement determines the mathematical construct of the Dirac distribution. Multiplication by case-specific "potential constants" then defines the respective "source of causation" for the corresponding physical law. And this brings us full circle to the physical concept of a point charge in Euclidean space, whose mathematical concept is likewise the Dirac "function."

Infinitely small points with finite mass or charge do not exist in nature. Assuming point charges leads to infinite values (divergences) when calculating the interaction of a particle with its own field. At the Planck scale, the concept of continuous spacetime points loses its validity. A point distribution completely breaks down physically at this scale. The proposed model of quantum dynamics ensures continuous fields without having to resort to the concept of a Dirac "function." The related framework of empirical physics, in particular the concept of the "observer," can be described approximately using the methods of quantum mechanics. From a physical perspective, the shared concept of the "observer" corresponds to that of a common, statistically relevant Hamilton operator of both systems ("ground state" and "plasma"), which "exists" independently of the concepts of space, time, and continuity. "De-quantization" toward classical physics enables a subsequent assignment of measurement systems (geometries) appropriately selected for specific phenomena or theories. Experimentally measured physical constants are interpreted in this model not as fundamental fixed points, but

as approximation constants (approximate values) that arise from the reduction of the higher-order Hilbert space.

In quantum mechanics, the Lagrangian formulation depending on certain kinds of fields serves as a bridge to quantum field theory. In quantum field theories, the method of renormalization is used to describe how the dynamics of a quantum system change when we alter the scale (distance, energies) at which we probe it (DeP), p. 553. The fundamental mathematical problem underlying all quantum field theories is "infinite physical quantities (divergences) that arise when calculating different subatomic processes." The conceptual problem in theoretical physics is that each of these different subatomic processes requires its own theory. The basic concept of these renormalization theories is based on the two methods of "regularization" and "dimensional renormalization." Mathematically speaking, the "renormalization method" solves the problem of divergent integrals of all terms in a power series. The method of "dimensional regularization" ensures that the terms of such a series converge, but the series itself remains divergent in every neighborhood of zero - that is, it is undefined for the "physical vacuum region." A renormalization group is intended to explain why decoupling quantum mechanics and quantum dynamics - each with the same minimum number of scales (the "distance") - can work at all, (DeP) p. 560. The theoretical proof of the existence of "dynamic" elementary particles that cannot be directly observed statistically thus becomes a problem in the mathematics of a non-trivial quantized Yang-Mills theory: *"to prove that for any compact simple gauge group F, a non-trivial quantum Yang-Mills theory exists on* R^4 *and has a mass gap* $\Delta > 0$." https://www.claymath.org/millennium/navier-stokes-equation/.

The basic structure of the proposed theory of quantum dynamics is formed by an extended Hilbert space of the Hilbert space L_2 underlying J. von Neumann's quantum mechanical statistics, (NeJ). The basic requirements for such a theory are as follows

> *Requirement 1:*
>
> *The laws of classical mechanics can be causally described by the "interaction" of quantum-mechanical and quantum-dynamic systems. In particular, the concept of the "Coulomb potential" can ultimately be explained as an approximate model of underlying quantum dynamic models.*
>
> *Requirement 2:*
>
> *classical phenomena can be traced back to quantum-dynamic causes by means of suitable order-from-order mechanisms and quantum-dynamic fluctuations can be statistically distinguished from quantum-mechanical processes, (ScE).*

The proposed two connected basic models for a Hilbert-Krein space-based quantum dynamics in an extended basis quantum mechanical Hilbert space framework $H_{-1/2}$ consists of a physical ground-state model and a physical plasma model. What at first glance appears to be a cheap magic trick proves to be a valid step toward resolving the problem of a missing physical interpretation of a mathematical "world-point state." The metaphysical prerequisite for both mathematical models is a mathematically justifiable "density" assumption regarding this concept of a „physical ether". It includes a concept of dynamic energy that is independent of mechanical energy and independent of the concepts of the theory of motion - that is, in particular, independent of the concepts of space, time, and continuity. The laws of classical mechanics may be interpreted as aproximation solutions in Hilbert scales of underlying variational represention of quantum-mechanical laws governed by the so-called energy method, (VeW). The concept of related complementary variational methods is provided in (ArA).

Two ways to define potential operators in Hilbert-Krein spaces

The term "potential operator" in the context of Hilbert-Krein spaces comes up in the topic „Variational Methods for the Study of Nonlinear Operators, (VaM):

> **Definition 5.2** (VaM) p. 55: We say that the operator $\boldsymbol{F}$, from a Banach space E into the conjugate space E^*, is *potential* on some set $H \subset E$, if there exists a functional f such that $gradf(x) = \boldsymbol{F}(x)$ for every $x \in H$.

1. The concept of a potential operator in Hilbert spaces

The model problem of the potential theory is the Poisson equation

$$-\Delta u = f$$

for a given functions $f \in L_2$, where Δ defines the Laplacian (*potential*) operator. Expressed in terms of functional analysis, this means that there is a linear equation $Au = f$ in a Hilbert space H with inner product $(.,.)$ and the norm $\|..\|$ and a linear operator A with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact .

The operator A may be characterized by an orthonormal set of eigen-pairs (λ_n, φ_n). This orthogonal set of eigenpairs allows for the definition of finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \ , x_n := (x, \varphi_n)$$

and corresponding Hilbert scales H_α, $\alpha \in R$. Then, with $H := H_0$ the Poisson equation gets the form, (BrK)

$$u \in H_1: \qquad a(u,v) := (u,v)_1 := (f,v)_0 \ \forall v \in H_1.$$

2. The concept of a potential operator in Krein spaces

A Krein space is decomposed into an orthogonal sum of two spaces H_1 and H_2 ($H = H_1 \otimes H_2$) associated with two projection operators P_1 and P_2 onto H_1 and H_2, (AzT), (BoJ). From the functional, (VaM) p. 90

$$\varphi(x) := \big((x)\big)^2 = \|P_1 x\|^2 - \|P_2 x\|^2$$

we have

$$\|P_1 x\|^2 = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\big((x)\big)^2 \ ; \ \|P_2 x\|^2 = \tfrac{1}{2}\|x\|^2 - \tfrac{1}{2}\big((x)\big)^2.$$

The operator $J = P^+ - P^-$ is the *fundamental symmetry* belonging to the fundamenal decomposition $H = H_1 \otimes H_2$, (BoJ) p. 52. The gradient of $\varphi(x) := \big((x)\big)^2$ is given by, (VaM) p. 86

$$grad\big((x)\big)^2 = 2P_1 x - 2P_2 x.$$

Therefore, the *potential* operator

$$Wx := \tfrac{1}{2} grad\big((x)\big)^2 = P_1 x - P_2 x$$

on all of the space H corresponds to the symmetry operator J of a Krein space.

The appropriate choice $\alpha = \pm 1/2$ of a Hilbert scale H_α

The appropriate choice of a Hilbert scale H_α with respect to the requirements 1 & 2 follows from the relationship between all Hilbert scale norms $\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty$, $x_n := (x, \varphi_n)$ and a further extended special Hilbert scale norm

$$\|x\|_{(\tau)}^2 = \sum_1^\infty e^{-\sqrt{\lambda_n}\tau} x_n^2 < \infty, \tau > 0 \, .$$

The norm $\|x\|_{(\tau)}^2$ is weaken (exponential decay with respect to the eigenvalues λ_n) than all norms $\|x\|_\alpha^2$ (polynominal decay with respect to the eigenvalues λ_n). For corresponding approximation properties we refer to the appendix. The two types of norms are related to each other by the identity

$$(*) \quad \|x\|_{\alpha-1/2}^2 = \int_0^\infty \|x\|_{\alpha.(\tau)}^2 d\tau \, .$$

That means that the extension from $\alpha = 0$ to $\alpha = -1/2$ is consistent with the property (*). It proves to be sufficient in that, from the standpoint of empirical physics, it guarantees the existence of continuous fields as a result of the embedding $H_{-1/2} \subset H_{-n/2-\epsilon}$. (*) Furthermore, the identity (*) allows us to define a "physical instant" $t > 0$ in the following way

$$\text{(A)} \quad \|x\|_{-1/2}^2(t) := \sum_1^\infty \frac{1}{\sqrt{\lambda_n}} x_n^2(t) \text{ with } x_n^2(t) := x_n^2 \int_t^\infty e^{-\sqrt{\lambda_n}\tau} \, .$$

Note: In the extended Hilbert space framework the variational Poisson equation for a given *distribution* functions $f \in H_{-1/2}$ is given in the form (see also (BrK1)

$$u \in H_{1/2}: \ (\nabla u, \nabla v)_{-1/2} = (u, v)_{1/2} = (f, v)_{-1/2} \ \ \forall v \in H_{1/2} \, .$$

The distribution function $f \in H_{-1/2}$ can certainly be interpreted here as an alternative Newtonian gravitational potential or as an alternative Coulomb electric potential, subject to fewer mathematical restrictions and offering improved physical capabilities.

Note: If the symbol x' is interpreted as "momentum" and the parameter τ is interpreted as some form of "time" or "frequency" of quantum fluctuations the norm of $H_{(\tau)}$ is linked to the energy Hilbert space $H_{1/2} = H_1 \otimes H_1^\perp$ by the energy equation of a „vibrating string" governed by the norm of $H_{(\tau)}$ in the form

$$\|x\|_{1/2}^2 = \tfrac{1}{2} \int_0^\infty \left[\|x'\|_{(\tau)}^2 + \|\dot{x}\|_{(\tau)}^2 \right] d\tau \, .$$

Note: The concept of the Hilbert space extension $H_{-1/2} = H_0 \otimes H_0^\perp$ can also be applied to define an alternative Schrödinger operator (BrK1). Comparable physical "phenomena" appear in classical physics as turbulence in the form of nonlinear functions or differential operators. In the above context, it is therefore natural to refer to the nonlinear Schrödinger equation. This is characterized by a potential function V that depends on the magnitude of the wave amplitude ψ but not on its phase. "*This is to be expected, as far as electron plasma waves are concerned, because the nonlinearity arises from the ponderomotive force, which depends on the gradient of the wave* intensity," (ChF) p. 336.

Note: Mathematically, due to its compact embedding in $H_{-1/2}$, the closed subspace $H_0^\perp$, there exists a compact operator that can be interpreted as a compact perturbation of the H_1 -coercive Laplacian operator, where the "energetic" coerciveness condition can be applied:

> *A variational representation of an operator in the form $B = A + K$, where A is a H_α -coercive operator with a compact disturbance K, satisfies a coerciveness (Garding-type inequality) condition of the form*
>
> $$(Bu, v) \geq c \cdot \|u\|_\alpha \|v\|_\alpha - (Ku, v) \text{ or } (Bu, v) \geq c_1 \cdot \|u\|_\alpha^2 - c_2 \cdot \|u\|_\beta^2$$
>
> *where $H_\beta \subset H_\alpha$ is compactly embedded.* For related arguments concerning the Boltzmann-Landau equations we refer to (LiP), (LiP1), see also ((AzA).

Note: In the appendix a bounded-$H_{1/2}$ -energy norm estimate for the 3D-NSE problem is provided. If the systems of the underlying extended „test" space $H_{-1/2}$ are interpreted as dynamic fluid particles this means that such a concept avoids „blow up" solutions of the 3D Navier-Stokes initial boundary value problem. The $H_{-1/2}$-systems may be interpreted as dynamic fluid/gas particles.

Note: In (NaS) there is an interesting relationship of the $H_{1/2}$ space to „The Teichmüller theory and the universal period mapping via quantum calculus and the $H_{1/2}$ space on the unit circle".

(*) The embedding $H_{-1/2} \subset H_{-n/2-\epsilon}$, which is independent of all spatial dimensions, ensures - by virtue of Sobolev's embedding theorem - that continuous fields in classical physics can be generated by quantum-mechanical causes, regardless of the dimension of the underlying Euclidean space. The systems of this extended Hilbert space can thus accommodate the mathematical concept of the Newtonian or Coulomb potential, both of which, for mathematical reasons, only make physical sense in 3D Euclidean space. The concept of phenomenon-specific "point charges" that generate continuous fields throughout 3D Euclidean space can thus be completely replaced.

The extended quantum-mechnical Hilbert space $H_{-1/2}$ and
the "ground state" and "plasma" quantum fluctuation systems it contains

The concept of a „physical ether"

The quantum-mechanical basis Hilbert spaces $H_{-1/2}$ can be characterized by the norms of a "mechanical" Hamilton operator A with a countable number of eigenpair solutions. The two quantum fluctuation systems it contains are the connected "ground state" and "plasma" quantum fluctuation systems. Together, these two systems define the "physical ether," (EiA). They differ from $H_{-1/2}$ systems in that they have different indefinite norms (i.e., different invariants) and an additional structural element in the form of an extra (dynamic) potential operator. The methodological difference between the ground-state model and the plasma model is the additional *a priori* metaphysical assumption of a plasma density equal to twice the *a priori* metaphysical vacuum density (PlP).

The physical ether is endowed with a mathematically justifiable "density" $\frac{1}{4n-1}$ to define appropriate related quantum numbers:

> *The mathematical concept behind the construction of appropriate "implicate fluctuation" values is based on the different Schnirelmann densities of odd and even numbers. For odd natural numbers, the Schnirelmann density is ½; for even numbers, it is zero. The fundamental mathematical quantity that determines the respective quantum numbers is a density distribution of the odd and even numbers in the following form:* $\frac{2n-1}{4n-1}+\frac{2n}{4n-1}=1$ *. Accordingly, the distribution ratio* $\frac{1}{4n-1}$ *forms the basis for defining the "implicate fluctuation" values used to construct the dynamic quantum state or quantum energy norms*.

The physical properties of the"physical ether" are:

1. There is a concept of dynamic energy independent of the concept of mechanical energy, defined independently of the concepts of kinematics

2. The "physical ether" possesses the characteristics of a "ground state" of possible ponderable matter and a "plasma state" of possible non-ponderable matter [(*)]. This physical ether is independent of the concepts of kinematics - that is, in particular, independent of the concepts of space, time, and continuity. The characteristic property of a plasma state is the "potential" of two plasma systems that superimpose on every quantum-mechanical system.

3. The conceptual framework of empirical physics for physical measurements - in particular, the concept of the "observer" - can be described approximately using the methods of quantum mechanics and the methods of approximation theory in Hilbert scales. From a physical perspective, the concept of the "observer" corresponds to that of a common, statistically relevant Hamilton operator of the ground-state and plasma systems. "De-quantization" with respect to the downstream concepts of classical physics is achieved by means of a downstream assignment of suitably chosen metric systems (geometries) to the respectively compactly embedded approximation Hilbert spaces of the underlying basis Hilbert space.

[(*)] The requirements that a physical ether with the attributes "ground state" and "plasma" must meet in the basic model are based on Einstein's descriptive qualities of a physical ether, summarized in the requirement that a physical ether must exist without concepts from the mechanics of motion, (EiA) and the conceptual tension between the two groups of terms "matter" and "electromagnetism."
The fact that "plasma" should also be considered part of the concept of a physical ether follows from the fact that about 95% of the universe consists of the phenomenon known as the 'vacuum.' The same proportion applies to the empty space between a proton and an electron. The remaining 5% of the universe's vacuum consists roughly of 5% matter, 25% dark matter, and 70% dark energy. Nearly all (about 99%) of the 5% matter in the universe is in a 'plasma state.' A presumed sophisticated physical concept of "dark matter" "explains" the phenomenon of spiral shapes in the universe. A presumed sophisticated physical concept of "dark energy" explains the phenomenon of the cosmic microwave background radiation (LaM).

The mathematics of the ground state model and the plasma model

The basis quantum-mechanical Hilbert space framework is the Hilbert space $H_{-1/2} = H_0 \otimes H_0^{\perp}$ defined by the finite norms

$$\|x\|^2_{-1/2} = \int_0^{\infty} \|x\|^2_{(\tau)} d\tau = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} x_n \varphi_n \,, \quad x_n := (x, \varphi_n).$$

The quantum numbers of the two interconnected quantum fluctuation systems are based on the *a priori* density assumptions of a physical ether σ_n. The quantum numbers of the ground-state model correspond to this physical density σ_n of the ether.

	Ground-state model	Plasma model
	$\sigma_n := \frac{1}{4n-1}$	$2\sigma_n := \frac{2}{4n-1}$
Quantum fluctuation numbers	$\Gamma_n^{\pm} := \frac{1}{\sqrt{\lambda_n}} \pm \frac{\sigma_n}{2\sqrt{\lambda_n}} > 0$	$\Pi_n^{\pm} := \frac{1}{\sqrt{\lambda_n}} \pm \frac{2\sigma_n}{2\sqrt{\lambda_n}} = \frac{1}{\sqrt{\lambda_n}}(1 \pm \sigma_n) > 0$
	$\Gamma_n^+ + \Gamma_n^- = \frac{2}{\sqrt{\lambda_n}}$ $\Gamma_n^+ - \Gamma_n^- = \frac{\sigma_n}{\sqrt{\lambda_n}}$	$\Pi_n^+ + \Pi_n^- = \frac{2}{\sqrt{\lambda_n}}$ $\Pi_n^+ - \Pi_n^- = \frac{2\sigma_n}{\sqrt{\lambda_n}}$
	$x^+ := \frac{1}{2}\sum_{n=1}^{\infty} \Sigma_n^+ x_n \varphi_n$ $x^- := \frac{1}{2}\sum_{n=1}^{\infty} \Sigma_n^- x_n \varphi_n$ $x = x^+ + x^- = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} x_n \varphi_n$	$x^+ := \frac{1}{2}\sum_{n=1}^{\infty} \Pi_n^+ x_n \varphi_n$ $x^- := \frac{1}{2}\sum_{n=1}^{\infty} \Pi_n^- x_n \varphi_n$ $x = x^+ + x^- = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} x_n \varphi_n$
Indefinite norms functionals	$\|\|x^+\|\|^2_{\sigma} := \frac{1}{2}\sum_1^{\infty} \Sigma_n^+ x_n^2$ $\|\|x^-\|\|^2_{\sigma} := \frac{1}{2}\sum_1^{\infty} \Sigma_n^- x_n^2$ $\varphi_{\sigma}(x) := \|\|x^+\|\|^2_{\sigma} - \|\|x^-\|\|^2_{\sigma}$	$\|\|x^+\|\|^2_{2\sigma} := \|x\|^2_{-1/2} + \|\|x\|\|^2_{2\sigma}$ $\|\|x^-\|\|^2_{2\sigma} = \|x\|^2_{-1/2} - \|\|x\|\|^2_{2\sigma}$
Norms	$\|\|x\|\|^2_{\sigma} = \sum_{n=1}^{\infty} \frac{\sigma_n}{\sqrt{\lambda_n}} x_n^2$ $\|\|x\|\|^2_{\sigma,\perp} := \sum_{n=1}^{\infty} \frac{1-\sigma_n}{\sqrt{\lambda_n}} x_n^2$	
	$\|\|x\|\|^2_{\sigma} = (W_{\sigma}x, x) = \sum_{n=1}^{\infty} \frac{\sigma_n}{\sqrt{\lambda_n}} x_n^2$ with	$\|\|x\|\|^2_{2\sigma} := 2(W_{\sigma}x, x) = 2\|\|x\|\|^2_{\sigma}$ with
	$W_{\sigma}x := \frac{1}{2} grad\varphi_{\sigma}(x)$ $= x^+ - x^- = \sum_{n=1}^{\infty} \frac{\sigma_n}{\sqrt{\lambda_n}} x_n \varphi_n$	$W_{2\sigma}x := \frac{1}{2} grad\varphi_{2\sigma}(x)$ $= x - x^- = \sum_{n=1}^{\infty} \frac{1-\sigma_n}{\sqrt{\lambda_n}} x_n \varphi_n$
Norm estimates	$\|\|x\|\|^2_{\sigma} \le \frac{1}{3}\|x\|^2_{-1/2} < \frac{1}{2}\|x\|^2_{-1/2} <$ $< \frac{2}{3}\|x\|^2_{-1/2} \le \|\|x\|\|^2_{\sigma,\perp}$	$\frac{1}{3}\|x\|^2_{-1/2} \le \|\|x^-\|\|^2_{2\sigma}$ $\|\|x^-\|\|^2_{2\sigma} < \|x\|^2_{-1/2} - \|\|x\|\|^2_{\sigma}$

The characteristic feature of the ground-state system is that it allows for three non-superimposable states, which can be characterized by the quantum spectrum range $\left[\frac{1}{3}, \frac{2}{3}\right]$.

The characteristic feature of the plasma system is that it allows only a dynamic quantum-antiquantum plasma pair. It is characterized by the fact that it does not allow for any non-superimposed quantum-mechanical states. From a mathematical perspective, it is interesting that the additional *a priori* physical fundamental condition of a plasma state at the quantum level simplifies probability-theory-based quantum mechanics (ponderable matter) and, at the same time, this plasma state can serve as the basis for continuous fields in the sense of a Maxwell-Mie field theory, (WeH) p. 171 ff., (WEH1) p. 206 ff..

Appendix

A bounded-$\mathbf{H_{1/2}}$ -energy norm estimate for the 3D-NSE problem enabled by the concept of dynamic fluids

For space dimensions $n = 3$, there is no global boundedness estimate for the variational representation of the homogeneous 3D-NSE system in the form, (BrK2), (see also (NiJ3))

$$u \in H_1\colon\ (\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} = 0 \text{ for all } v \in H_{-1/2}$$
$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} \text{ for all } v \in H_{-1/2}\ .$$

For $-1 \le \alpha \le 1$, the Stokes operator A can be generalized to the operators $A^\alpha := \int_0^\infty \lambda^\alpha dE_\lambda$ with domain

$$D(A^\alpha) := \{u \in L^2_\sigma(\Omega\ ;\ \|A^\alpha u\|_2^2 = \int_0^\infty \lambda^{2\alpha} d\|E_\lambda u\|_2^2\}$$

which are positive and self-adjoint, (SoH) p. 133. It turned out that the nonlinear energy term of the 3D nonlinear, nonstationary NSE system is bounded with respect to the norm $\|u\|_{1/2}$ as a simple consequence of the Sobolevskii inequality (GiY):

Substituting $B(u) := P(u, grad)u)$ into the NSE and assuming $Pu_0 = u_0$, the NSE initial-boundary equation is given by

$$\frac{du}{dt} + Au + Bu = Pf\ , u(0) = u_0\ .$$

By formally multiplying the homogeneous equation by $A^\alpha v$ with appropriately selected α, this can be expressed in the variational form

$$u \in H_{1/2}\colon\ (\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} = 0 \qquad \text{for all } v \in H_{-1/2}$$
$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} \qquad \text{for all } v \in H_{-1/2}\ .$$

Setting $v := u$ yields

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{-1/2} + \|u\|^2_{1/2} \le \left|(Bu, u)_{-1/2}\right| \le \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0.$$

The last term can be estimated using the Sobolevskii estimates in the form (*)

$$\left\|A^{-1/4}Bu\right\|_0 \le c\|u\|_1^2.$$

Setting $y(t) := \|u\|^2_{-1/2}$, we obtain $y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$, and therefore, by Gronwall's lemma

$$\|u(t)\|_{-1/2} \le \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\{\|u_0\|_{-1/2} + \|u_0\|_0^2\},$$

which guarantees global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

(*) Lemma 3.2 (GiY): For $0 \le \delta < 1/2 + n \cdot (1 - 1/p)/2$, it holds that $\left|A^{-\delta}P(u, grad)v\right|_p \le M \cdot \left|A^\theta u\right|_p \cdot |A^\rho u|_p$ with a constant $M := M(\delta, \theta, \rho, p)$ if $\delta + \theta + \rho \ge n/2p + 1/2\ , \theta, \rho > 0\ , \theta + \rho > 1/2$. Setting $p := 2\ , \delta := 1/4\ , \theta := \rho := 1/2$, and satisfying $\theta + \rho \ge \frac{1}{4}(n + 1) = 1$, it follows

$$\left\|A^{-\delta}P(u, grad)u\right\| \le c\left\|A^\theta u\right\| \cdot \|A^\rho u\| = c\|u\|_{2\theta} \cdot \|u\|_{2\rho} = c\|u\|_1^2$$

or

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{-1/2} + \|u\|^2_{1/2} \le \left|(Bu, u)_{-1/2}\right| \le c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

Approximation theory in Hilbert scales

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$ and the norm $\|..\|$. Let A be a linear operator with the properties, (NiJ), (NiJ1)

iii) A is self-adjoint, positive definite

iv) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, one may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A).$$

The Fredholm-Riesz-Schauer theory states that there are at most a countable number of eigenvalues λ_n of A^{-1} with $\lambda_n \neq \lambda_m, n \neq m, |\lambda_n| \geq \lambda > 0$, and corresponding eigenelements x_n. Any eigen-element of the compact operator $K = A^{-1}$ is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_n \to \lambda_n^{-1}$ we have that there is a countable sequence $\{\lambda_n, \varphi_n\}$ with

$$A\varphi_n = \lambda_n \varphi_n \ , \ \ (\varphi_n, \varphi_m) = \delta_{n,m} \ \text{ and } \ \lim_{n\to\infty} \lambda_n \to \infty$$

and any $x \in H$ is represented by

$$x = \sum_{i=1}^{\infty} (x, \varphi_n)\, \varphi_n \ \text{ and } \ \ \|x\|^2 := \sum_1^{\infty} x_n^2, \ x_n := (x, \varphi_n).$$

The x_n are called generalized Fourier coefficients. Based on this basic concept of a linear operator with properties i) and ii), further Hilbert spaces $\{H_\alpha | \alpha \in R\}$ can be defined by the corresponding convergence condition of their underlying norms

$$\|x\|_\alpha^2 = \sum_1^{\infty} \lambda_n^\alpha x_n^2 < \infty \ , \ x_n := (x, \varphi_n)$$

accompanied by the inner product $(x, y)_\alpha = \sum_1^{\infty} \lambda_n^\alpha \, x_n \, y_n$. A characteristic feature of these norms is their polynomial convergence behavior with respect to the exponent α. There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

i) Let $\alpha < \beta$. Then $\|x\|_\alpha \leq \|x\|_\beta$ and the embedding $H_\beta \subset H_\alpha$ is compact

ii) Let $\alpha < \beta < \gamma$. Then $\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu$ for $x \in H_\gamma$ with $\mu = \frac{\gamma-\beta}{\gamma-\alpha}$ and $\nu = \frac{\beta-\alpha}{\gamma-\alpha}$

iii) Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

$$\|x - y\|_\alpha \leq t^{\beta-\alpha} \|x\|_\beta$$

$$\|x - y\|_\beta \leq \|x\|_\beta \ , \ \|y\|_\beta \leq \|x\|_\beta$$

$$\|y\|_\gamma \leq t^{-(\gamma-\beta)} \|x\|_\beta$$

iv) Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

$$\|x - y\|_\rho \leq t^{\beta-\rho} \|x\|_\beta \quad \text{for } \ \alpha \leq \rho \leq \beta \ , \ \|y\|_\sigma \leq t^{-(\sigma-\beta)} \|x\|_\beta \qquad \text{for } \ \beta \leq \sigma \leq \gamma \ .$$

For $\tau > 0$ one may introduce the Hilbert space $H_{(\tau)}$ by an additional inner product resp. norm in the form

$$(x,y)_{(\tau)} = \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n}\tau}(x,\varphi_n)(y,\varphi_n)$$

$$\|x\|_{(\tau)}^2 = (x,x)_{(\tau)}^2 .$$

Now the factor has exponential decay $e^{-\sqrt{\lambda_n}\tau}$ instead of a polynomial decay in case of λ_n^{α}.

Obviously it holds

$$\|x\|_{(t)} \le c(\alpha,\tau)\|x\|_{\alpha} \text{ for } x \in H_{\alpha}$$

with $c(\alpha,\tau)$ depending only from α and $\tau > 0$. Thus the (τ)-norm is weaker than any α-norm. On the other hand any negative norm index, i.e. $\|x\|_{\alpha}$ with $\alpha < 0$, is bounded by the 0-norm and the newly introduced (τ)-norm. For $\alpha > 0$ be fixed and $\delta > 0$ being arbitrary the α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{\tau/\delta}\|x\|_{(t)}^2, \quad \|x\|_{-1/2}^2 \le \delta\|x\|_0^2 + e^{\frac{\tau}{\delta}}\|x\|_{(\tau)}^2 .$$

For $\tau, \delta > 0$ be fixed the counterpart of the properties i)-iv) above is that for any $x \in H_0$ there is a $y = y_\tau(x)$ according to

i) $\|x - y\| \le \|x\|$

ii) $\|y\|_1 \le \delta^{-1}\|x\|$

iii) $\|x - y\|_{(\tau)} \le e^{-\tau/\delta}\|x\|.$

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