Consolidated Overview of the Digamma Concept

Additive Number Theory - Circle Method - Type II - Large Sieve - Gram/Hilbert-Space Structure
with updated plus/minus classification

Purpose. This document consolidates the Digamma variants developed so far into one architecture. It distinguishes exact identities, classical tools, model-based scaling arguments, and transfers to prime or Type-II sums that remain open.

1. Starting point: Digamma zeros and the natural scale

Let w_n be the n-th negative zero of the digamma function ψ, with $-n < w_n < -n+1$, and define $y_n := n+w_n \in (0,1)$. The central quantity is the variation of y_n.

$$\psi(w_n)=0, \quad y_n=n+w_n, \quad y_n \sim 1/\log n$$

$$y_n-y_{(n+1)} \sim 1/(n \log^2 n)$$

$$H_L \sim L/\log^2 L.$$

$$\text{For } M \sim L \sim \sqrt{X}: \quad H_L/L \sim 1/\log^2 X.$$

Status: The Digamma asymptotics provide the logarithmic scale. Their conversion into a genuine minor-arc gain is not automatic.

2. Digamma block lemma and discrete geometry

Using the implicit curve $\pi \cot(\pi y)=\psi(x+1-y)$, one obtains on a suitable tail a monotone derivative of size $1/(x \log^2 x)$. After scaling and rounding, the increments are 0 or 1 beyond a transition threshold. Thus the rounded values form complete integer blocks, possibly with multiplicities.

Benefit: the analytic zero geometry can be combined with congruence conditions, parity, and local residue classes. Limitation: the block structure does not detect primes by itself.

$$0 < -y'(x) \le (1+\varepsilon)/(x \log^2 x)$$

$$r_n(N)=2N\, y_n, \quad m_n(N)=\lfloor r_n(N)+1/2 \rfloor.$$

3. Circle-method framework

For a smoothed prime exponential sum we use

$$S(\alpha)=\Sigma_n \Lambda(n)\, \eta(n/X) e(n\alpha)$$

$$R_3(N)=\int_0^1 S(\alpha)^3 e(-N\alpha) d\alpha = R_M(N)+R_m(N).$$

Role: The ternary problem is used as a controlled benchmark. Success here would not prove binary Goldbach, but would test whether the new mechanism produces an additional minor-arc gain.

4. Type-II structure as the main test field

After a Vaughan-type decomposition, a bilinear term has the form

$$B(\alpha)=\Sigma_{m\sim M} a_m \Sigma_{n\sim L} b_n e(\alpha mn), \quad ML \sim X.$$

$$\text{Critical balanced range:} \quad M \sim L \sim \sqrt{X}.$$

Status: Type II is the natural place to test whether the Digamma scale survives genuine arithmetic oscillation.

5. Concept A: Digamma on the correlation shift h

Cauchy-Schwarz generates shifts $h=n_1-n_2$ and residue-wise correlation vectors. The normalized double projection gives an exact variance identity.

$$C_b(h)=\Sigma_n b_{n+h}\ \text{conjugate}(b_n).$$

$$E_q=\Sigma_h p_h||u_{q,h}||^2-||\Sigma_h p_h u_{q,h}||^2$$

$$=(1/2)\Sigma_{h,j}p_h p_j||u_{q,h}-u_{q,j}||^2 \geq 0.$$

Classification: ± Structurally useful for local residue classes and variance, but not an independent minor-arc gain.

6. Local prime structure and admissible projection

For pair correlations, the natural admissible residue set is

$$A_q(h)=\{r \bmod q : (r,q)=1 \text{ and } (r+h,q)=1\}.$$

$$\nu_q(h)=q \prod_{p|q,p|h}(1-1/p) \prod_{p|q,p\nmid h}(1-2/p).$$

$$(P_{adm}\, v)_r=1_{A_q(h)}(r)[v_r-(1/\nu_q(h))\Sigma_{s\in A_q(h)}v_s].$$

Classification: + Arithmetically appropriate projection: it removes known local major-arc/singular-series structure, but does not replace a global minor-arc estimate.

7. Partial local conditioning

Full conditioning $q|\Delta$ is impossible at the critical Digamma scale because $H \ll q$. Introduce $d|q$ and require $d|\Delta$, freezing selected small prime divisors while retaining nontrivial shifts.

$$F_q(d)=(d/H)^2+\Sigma_{p|q,p\nmid d}1/p.$$

$$F_q^{\min}=\min_{d|q,d\leq H}F_q(d).$$

$$\text{Within the model:} \quad F_q^{\min} \ll 1/\log\log X \to 0.$$

Classification: +/± Useful for controlling changes of local admissibility projectors; insufficient for the full rational exponential phase.

8. Concept B: Direct Digamma difference on a bilinear variable

Define $B_j(\alpha)=\Sigma_{n\sim L}b_n e(\alpha m_{jn})$. Then the difference is exact and telescopes across a block.

$$B_{j+1}(\alpha)-B_j(\alpha)=\Sigma_n b_n e(\alpha m_{jn})[e(\alpha\Delta_{jn})-1],$$

$$\Delta_j = m_{(j+1)} - m_j.$$

$$\Sigma_j [B_{(j+1)} - B_j] = B_{end} - B_{start}.$$

Classification: ++ Strongest exact algebraic connection between the Digamma displacement and the Type-II structure.

9. Decisive falsification test: $\alpha = a/q + \beta$

The difference factor splits exactly into a continuous part and a rational part.

$$e(\alpha\Delta n) - 1 = e(a\Delta n/q)[e(\beta\Delta n) - 1] + [e(a\Delta n/q) - 1].$$

9.1 The β-part

For $|\beta\Delta n| \ll 1$, the continuous factor is small. On the critical scale it gives the natural logarithmic gain.

$$|e(\beta\Delta n) - 1| \leq 2\pi|\beta\Delta n|.$$

$$|\beta| \sim X^{(-1)}, \quad \Delta \sim \sqrt{X}/\log^2 X, \quad n \sim \sqrt{X}$$

$$|\beta|\Delta n \sim 1/\log^2 X, \quad \text{energy scale} \sim 1/\log^4 X.$$

Classification: ++ Natural and algebraically stable candidate for an additional logarithmic gain.

9.2 The rational part

Let $g = (\Delta, q)$ and $q' = q/g$. For $q' > 1$ the rational multiplier is not small in quadratic mean.

$$(1/q')\Sigma_{(n \bmod q')} |e(a\Delta' n/q') - 1|^2 = 2.$$

Classification: −− Central negative result: the strong Digamma version cannot control the rational minor-arc phase.

10. Concept C: Hybrid division of labor - Large Sieve + Digamma

The refined hypothesis is: treat a/q by classical circle-method/Large-Sieve machinery and apply Digamma only to the continuous β-variation. On coefficient space, the rational multiplier and β-difference multiplier commute.

$$a/q \rightarrow \text{classical arithmetic control}$$

$$\beta \rightarrow \text{Digamma difference operator}$$

$$R_{(q,a,m)} D_{(\beta,\Delta)} = D_{(\beta,\Delta)} R_{(q,a,m)}.$$

Classification: ++ Currently the most plausible framework. It remains open whether the β-gain survives as a genuinely additional gain after the full Type-II/Large-Sieve argument.

11. Coherent mode and explicit projection

For positive weights p_j with $\Sigma p_j = 1$ define the weighted mean and the coherent rank-one projection.

$$\bar{F} = \Sigma_j p_j F_j, \quad P_{coh} F = \bar{F} \cdot 1.$$

$$\|(I - P_{coh})F\|_p^2 = \Sigma_j p_j |F_j|^2 - |\Sigma_j p_j F_j|^2.$$

Classification: + Clean Hilbert-space decomposition. The classical rational machinery should control the coherent component, while Digamma is assigned to its orthogonal complement.

12. Concept D: Local Digamma path

For nearest-neighbor differences the weighted operator is a path Laplacian. For J approximately equally weighted points its first positive eigenvalue has the scale $\lambda_1 \sim \pi^2/J^2$.

$$(\nabla_D F)_j = F_{(j+1)} - F_j.$$

$$\lambda_1 \sim \pi^2/J^2, \quad \lambda_1^{(-1)} \sim J^2.$$

$$||(I - P_{coh})F||^2 \lesssim (J^2/\log^4 X) E_{classical}.$$

Classification: − A long nearest-neighbor chain is unfavorable: the J^2 Poincaré loss can consume the Digamma gain.

13. Concept E: Full Digamma Gram structure

The normalized Gram energy measures the entire orthogonal complement directly, without a J^2 Poincaré reconstruction loss.

$$E_{Gram}(F) = (1/2)\Sigma_{(j,k)} p_j p_k ||F_j - F_k||^2$$

$$= \Sigma_j p_j ||F_j - \bar{F}||^2.$$

Classification: +/++ Spectrally favorable. The counter-cost is that distant pairs have larger Δ_{jk}, which can destroy the local $\log^{-2} X$ translation advantage.

14. Concept F: Band-Gram model

The current compromise is to connect neither only nearest neighbors nor all pairs, but only $1 \le |j-k| \le R(X)$. This produces a direct optimization between translation cost and spectral-gap gain.

$$\Phi_X(R) = (R/\log^2 X)^2 \lambda_R^{(-1)}.$$

Decisive question: $\inf_R \Phi_X(R) \overset{?}{\to} 0.$

Classification: ++/open. This is currently the sharpest mathematical yes/no test of the hybrid Digamma approach.

15. Consolidated plus/minus classification

Concept	Rating	Reason
Digamma as a prime detector	−−	No direct Λ/primality information.
Digamma as a replacement for minor-arc theory	−−	The rational phase a/q is not small.
Digamma correlation projection in h	±	Useful structural diagnostic, but no automatic Type-II gain.
Admissible-residue projection	+	Matches the local singular-series structure.

Partial conditioning d|Δ	+/±	Local projector variation tends to zero, but the full rational phase remains.
Direct bilinear Digamma difference	++	Exact algebra and telescoping.
Digamma applied only to β	++	Natural log^−2X scale at the correct analytic location.
Local path/Poincaré structure	−	Spectral gap ~J^−2 makes reconstruction expensive.
Full Gram structure	+/++	No J^2 penalty, but long pair distances are costly.
Large Sieve + Digamma Gram	++/open	Most plausible division of labor.
Large Sieve + Digamma Band-Gram	++/open	Current leading candidate; R(X)-optimization remains to be proved.

16. What has been gained so far

- The natural Digamma scale $1/\log^2 X$ has been located in a genuine bilinear β-difference.
- The strong version that attempted to control the rational phase has been falsified by an exact mean-square obstruction.
- Local singular-series structure has been separated from the global exponential phase.
- The coherent component has been isolated as a rank-one mode.
- The long path-Poincaré variant has been identified as too expensive.
- The full Gram structure explains how to remove the coherent mode without a J^2 loss.
- The remaining problem has been reduced to a concrete bandwidth/spectral-gap optimization.

17. Central open test

The next decisive step is the analysis of an R-range band Laplacian/Gram operator in the critical Type-II block.

$$\Phi_X(R)=(R/\log^2 X)^2\,\lambda_R^{(-1)}.$$

One must determine whether there is a choice R=R(X) for which Φ_X(R)→0 while the coherent mode remains controlled by the classical rational/Large-Sieve structure.

If this succeeds, it would identify a genuinely additional asymptotic mechanism. If even the optimized Band-Gram variant fails to produce an o(1) factor, Digamma should probably be retained in the Goldbach part only as an auxiliary or comparison structure.

18. Recommended overall architecture

Circle Method

↓

Vaughan / bilinear Type-II decomposition

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$\alpha = a/q + \beta$

↙ ↘

a/q: classical arithmetic β: Digamma differences

Large Sieve / local structure natural log^−2X scale

↘ ↙

P_coh ⊕ (I−P_coh)

↓

Band-Gram optimization R(X)

↓

Test: additional o(1) or log^−γX gain?

19. Final assessment

Currently negative: Digamma as a prime detector, Digamma as a replacement for minor-arc theory, and the long local path-Poincaré variant.

Currently positive: the direct bilinear Digamma difference, β-translation, coherent projection, and the full Gram variance.

Most promising but still open: the hybrid Large Sieve + Digamma Band-Gram construction with optimized range R(X).

Decision criterion: whether the optimized construction yields a genuinely additional o(1) or log^−γX factor relative to the classical Type-II scale.

Working research note - no claim of a proof of Goldbach or the Riemann Hypothesis.