

The Positivity of a Sequence of Numbers and the Riemann Hypothesis

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In this note, we prove that the Riemann hypothesis for the Dedekind zeta function is equivalent to the nonnegativity of a sequence of real numbers.

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1. THE RIEMANN ZETA FUNCTION

Let $\{\lambda_n\}$ be a sequence of numbers given by

$$(n-1)! \lambda_n = \frac{d^n}{ds^n} [s^{n-1} \log \zeta(s)]_{s=1} \quad (1.1)$$

for all positive integers n , where

$$\zeta(s) = s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

with $\zeta(s)$ being the Riemann zeta function.

THEOREM 1. *A necessary and sufficient condition for the nontrivial zeros of the Riemann zeta function to lie on the critical line is that λ_n is non-negative for every positive integer n .*

Proof. Define

$$\varphi(z) = \zeta\left(\frac{1}{1-z}\right) = 4 \int_1^\infty [x^{3/2} \psi'(x)]' (x^{-1/2} x^{1/2(1-z)} + x^{-1/2(1-z)}) dx \quad (1.2)$$

for z in the unit disk, where

$$\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

Write

$$\zeta(s) = \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad (1.3)$$

where the product is taken over all nontrivial zeros of the Riemann zeta function with ρ and $1 - \rho$ being paired together. It follows that

$$\varphi(z) = \prod_{\rho} \frac{1 - (1 - (1/\rho))z}{1 - z}.$$

A necessary and sufficient condition for the nontrivial zeros of the Riemann zeta function to lie on the critical line is that $\varphi'(z)/\varphi(z)$ is analytic in the unit disk. Put

$$\frac{\varphi'(z)}{\varphi(z)} = \sum_{n=0}^{\infty} \lambda_{n+1} z^n$$

for $|z| < \frac{1}{4}$, where

$$\lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n \right] \quad (1.4)$$

for every positive integer n . On the other hand, by (1.3) we have

$$\begin{aligned} \frac{1}{(n-1)!} \frac{d^n}{ds^n} [s^{n-1} \log \zeta(s)]_{s=1} &= - \sum_{\rho} \sum_{k=0}^{n-1} \binom{n}{k} (\rho-1)^{k-n} \\ &= \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n \right], \end{aligned}$$

and hence λ_n is also given by the expression (1.1). Let

$$\varphi(z) = 1 + \sum_{j=1}^{\infty} a_j z^j. \quad (1.5)$$

We find that

$$\lambda_n = n \sum_{l=1}^n \frac{(-1)^{l-1}}{l} \sum_{\substack{1 \leq k_1, \dots, k_l \leq n \\ k_1 + \dots + k_l = n}} a_{k_1} \cdots a_{k_l}$$

for every positive integer n . Expanding the right side of (1.2) in power series (1.5), we find that

$$a_j = 2 \sum_{n=0}^{\infty} \frac{(j+n) \cdots (j+1)}{n! (n+1)! 2^n} \int_1^{\infty} [x^{3/2} \psi'(x)]' (x^{-1/2} + (-1)^{n+1}) (\log x)^{n+1} dx \quad (1.6)$$

for every positive integer j . By (1.6) we can write

$$\begin{aligned} a_j &= 4 \sum_{n=0}^{\infty} \frac{(n+j) \cdots (n+1)}{j! (n+1)! 2^{n+1}} \int_1^{\infty} [x^{3/2} \psi'(x)]' (x^{-1/2} + (-1)^{n+1}) (\log x)^{n+1} dx \\ &= \frac{4}{j!} \frac{d^j}{dt^j} \left\{ t^{j-1} \sum_{n=0}^{\infty} \frac{(t/2)^{n+1}}{(n+1)!} \int_1^{\infty} [x^{3/2} \psi'(x)]' \right. \\ &\quad \left. \times (x^{-1/2} + (-1)^{n+1}) (\log x)^{n+1} dx \right\}_{t=1} \\ &= \frac{4}{j!} \frac{d^j}{dt^j} \left\{ t^{j-1} \int_1^{\infty} [x^{3/2} \psi'(x)]' (x^{-1/2} [e^{(t/2) \ln x} - 1] + [e^{-(t/2) \ln x} - 1]) \right\}_{t=1} \\ &= \frac{4}{j!} \frac{d^j}{dt^j} \left\{ t^{j-1} \int_1^{\infty} [x^{3/2} \psi'(x)]' (x^{-1/2} e^{(t/2) \ln x} + e^{-(t/2) \ln x}) \right\}_{t=1} \\ &= 4 \sum_{l=1}^j \binom{j-1}{j-l} \frac{1}{l!} \int_1^{\infty} [x^{3/2} \psi'(x)]' \left(\frac{1}{2} \log x \right)^l [1 + (-1)^l x^{-1/2}] dx. \end{aligned}$$

This expression implies that a_j is a positive real number for every positive integer j . Since the identity

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \left(\sum_{i=0}^{\infty} a_i z^i \right) \left(\sum_{j=0}^{\infty} \lambda_{j+1} z^j \right)$$

holds, we have the recurrence relation

$$\lambda_n = n a_n - \sum_{j=1}^{n-1} \lambda_j a_{n-j}$$

for every positive integer n .

By (1.1), λ_n is a real number for every positive integer n . If the nontrivial zeros of $\zeta(s)$ lie on the critical line, then $|1 - (1/\rho)| = 1$ for every nontrivial

zero ρ of $\zeta(s)$. Put $1 - (1/\rho) = \exp(i\theta_\rho)$ for some real number θ_ρ . Then by (1.4) we have

$$\lambda_n = \sum_{\rho} (1 - e^{in\theta_\rho}) = \sum_{\rho} (1 - \cos n\theta_\rho).$$

This implies that the number λ_n is nonnegative for every positive integer n .

Conversely, if the number λ_n is nonnegative for every positive integer n , then

$$\lambda_n \leq na_n$$

for every positive integer n . It follows that

$$\sum_{n=1}^{\infty} |\lambda_n z^{n-1}| \leq \sum_{n=1}^{\infty} na_n |z|^{n-1} = \varphi'(|z|) < \infty$$

for z in the unit disk. This implies that $\varphi'(z)/\varphi(z)$ is analytic in the unit disk.

This completes the proof of the theorem.

2. THE DEDEKIND ZETA FUNCTION

Let k be an algebraic number field with r_1 real places and r_2 imaginary places. The Dedekind zeta function $\zeta_k(s)$ of k is defined by

$$\zeta_k(s) = \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1}$$

for $\text{Re } s > 1$, where the product is taken over all the finite prime divisors of k . Put $G_1(s) = \pi^{-s/2} \Gamma(s/2)$ and $G_2(s) = (2\pi)^{1-s} \Gamma(s)$. Define

$$Z_k(s) = G_1(s)^{r_1} G_2(s)^{r_2} \zeta_k(s).$$

By Theorem 3 of Chapter VII, Section 6, of [4], the function $Z_k(s)$ is analytic in the complex plane except for simple poles at $s=0$ and $s=1$, and satisfies the functional identity

$$Z_k(s) = |\mathfrak{d}|^{(1/2)-s} Z_k(1-s)$$

where \mathfrak{d} is the discriminant of k . Its residues at $s=0$ and $s=1$ are respectively $-c_k$ and $|\mathfrak{d}|^{-1/2} c_k$ with $c_k = 2^{r_1} (2\pi)^{r_2} hR/e$, where h , R , and e are respectively the number of ideal classes of k , the regulator of k , and the number of roots of unity in k . Let $\xi_k(s) = c_k^{-1} s(s-1) |\mathfrak{d}|^{s/2} Z_k(s)$. Then $\xi_k(s)$ is an entire function and $\xi_k(0) = 1$.

Let $\{\lambda_n\}$ be a sequence of numbers given by

$$(n-1)! \lambda_n = \frac{d^n}{ds^n} [s^{n-1} \log \zeta_k(s)]_{s=1}$$

for all positive integers n . The aim now is to prove the following theorem.

THEOREM 2. *A necessary and sufficient condition for the nontrivial zeros of the Dedekind zeta function $\zeta_k(s)$ to lie on the critical line is that λ_n is non-negative for every positive integer n .*

3. PROOF OF THE THEOREM 2

LEMMA 3.1. *The identity*

$$\lambda_n = \sum_{\rho} \left(1 - \left(1 - \frac{1}{\rho} \right)^n \right)$$

holds for every positive integer n , where summation is taken over all non-trivial zeros of the Dedekind zeta function $\zeta_k(s)$ with ρ and $1-\rho$ being paired together.

Proof. By Theorem 2 of Barner [1], we have the formula (cf. Chapter 2 of [2])

$$\zeta_k(s) = \prod_{\rho} \left(1 - \frac{s}{\rho} \right), \quad (3.1)$$

where the product is taken over all zeros of $\zeta_k(s)$ with ρ and $1-\rho$ being always paired together. An argument similar to that made for the Riemann zeta function in Chapter 2 of [2] shows that the convergence of the product (3.1) is uniform on compact subsets of the complex plane.

Since $\zeta_k(s) = \zeta_k(1-s)$, we have

$$\frac{d^n}{ds^n} [s^{n-1} \log \zeta_k(s)]_{s=1} = (-1)^n \frac{d^n}{ds^n} [(1-s)^{n-1} \log \zeta_k(s)]_{s=0}. \quad (3.2)$$

Since $\zeta_k(s)$ does not vanish at $s=0$, we can write

$$\log \zeta_k(s) = - \sum_{\rho} \sum_{m=1}^{\infty} \frac{\rho^{-m}}{m} s^m \quad (3.3)$$

where $|s| < \varepsilon$ for a sufficiently small positive number ε , where ρ and $1-\rho$ are paired together in the summation over ρ . Since the product (3.1)

converges uniformly, the series (3.3) converges uniformly for $|s| < \varepsilon$. It follows that

$$\frac{1}{(n-1)!} \frac{d^n}{ds^n} [(1-s)^{n-1} \log \zeta_k(s)]_{s=0} = -\sum_{\rho} \sum_{m=1}^n (-1)^{n-m} \binom{n}{m} \rho^{-m}.$$

This formula together with (3.2) implies the stated identity. ■

Define

$$\varphi(z) = \zeta_k \left(\frac{1}{1-z} \right)$$

for z in the unit disk. Since the function $\zeta_k(s)$ is analytic in the complex plane of s , the function $\varphi(z)$ is analytic in the unit disk.

LEMMA 3.2. *Let*

$$\varphi(z) = 1 + \sum_{j=1}^{\infty} a_j z^j.$$

Then the coefficient a_j is a positive real number for every positive integer j .

Proof. Define ε_v to be one when v is a real place of k and to be two when v is an imaginary place of k . Let $x = \prod x_v$ be the variable in the half space $\mathbb{R}_+^{r_1+r_2}$. Denote by $|x|$ the product $\prod x_v^{\varepsilon_v}$, which is taken over all infinite places of k . If $N = r_1 + 2r_2$, then the Hecke theta function $\Theta_k(x)$ is defined by

$$\Theta_k(x) = \sum_{\mathfrak{b}} \exp \left(-\pi |\mathfrak{d}|^{-1/N} (N\mathfrak{b})^{2/N} \sum_v \varepsilon_v x_v \right)$$

where the summation over \mathfrak{b} is taken over all nonzero integral ideals of k and where the summation over v is taken over all infinite places of k . Put $dx = \prod dx_v$. It follows from Theorem 3 of Chapter XIII, Section 3, in [3] that

$$\zeta_k(s) = 1 + c_k^{-1} s(s-1) \int_{|x| \geq 1} \Theta_k(x) (|x|^{s/2} + |x|^{(1-s)/2}) \frac{dx}{x}. \quad (3.4)$$

Let

$$\int_{|x| \geq 1} \Theta_k(x) (|x|^{1/2(1-z)} + |x|^{1/2} |x|^{-1/2(1-z)}) \frac{dx}{x} = \sum_{m=0}^{\infty} b_m z^m. \quad (3.5)$$

It is clear that b_0 is a positive number. We have

$$b_m = \sum_{n=0}^{\infty} \frac{(m+n) \cdots (m+1)}{n! (n+1)! 2^{n+1}} \int_{|x| \geq 1} \Theta_k(x) (1 + |x|^{1/2} (-1)^{n+1}) (\log |x|)^{n+1} \frac{dx}{x}$$

for every positive integer m . By computation, we find that

$$\begin{aligned} b_m &= \frac{1}{m!} \sum_{n=0}^{\infty} \frac{(n+m) \cdots (n+1)}{(n+1)! 2^{n+1}} \\ &\quad \times \int_{|x| \geq 1} \Theta_k(x) (1 + |x|^{1/2} (-1)^{n+1}) (\log |x|)^{n+1} \frac{dx}{x} \\ &= \frac{1}{m!} \frac{d^m}{dt^m} \left(t^{m-1} \int_{|x| \geq 1} \Theta_k(x) (e^{(t/2) \log |x|} + |x|^{1/2} e^{-(t/2) \log |x|}) \frac{dx}{x} \right)_{t=1}. \end{aligned}$$

It follows that

$$b_m = \sum_{l=1}^m \binom{m-1}{m-l} \frac{1}{l!} \int_{|x| \geq 1} \Theta_k(x) \left(\frac{1}{2} \log |x| \right)^l (|x|^{1/2} + (-1)^l) \frac{dx}{x} \quad (3.6)$$

for every positive integer m . Since $\Theta_k(x)$ is positive for every x in $\mathbb{R}_+^{r_1+r_2}$, it follows from (3.6) that the coefficients b_m are positive real numbers for all nonnegative integers m .

The identity

$$\frac{z}{(1-z)^2} = \sum_{q=1}^{\infty} qz^q$$

holds for z in the unit disk. It follows from (3.4) and (3.5) that

$$c_k a_j = \sum_{m=0}^{j-1} (j-m) b_m \quad (3.7)$$

for every positive integer j . Since b_m are positive numbers for all nonnegative integers m , we see that a_j is a positive real number for every positive integer j . ■

Proof of the Theorem. Since $\xi_k(1) = 1$ and $\xi_k(s) = \xi_k(1-s)$, it follows from the product formula (3.1) that

$$\varphi(z) = \prod_{\rho} \frac{1 - (1 - (1/\rho))z}{1 - z}. \quad (3.8)$$

Since $\zeta_k(s)$ does not vanish at $s=1$, we can write

$$\varphi'(z)/\varphi(z) = \sum_{n=0}^{\infty} \lambda_{n+1} z^n \quad (3.9)$$

by using the formula (3.8) when $|z| < \varepsilon$ for a sufficiently small positive number ε . Since

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \left(\sum_{i=0}^{\infty} a_i z^i \right) \left(\sum_{j=0}^{\infty} \lambda_{j+1} z^j \right),$$

we have

$$\lambda_n = n a_n - \sum_{j=1}^{n-1} \lambda_j a_{n-j} \quad (3.10)$$

for $n=2, 3, \dots$, where $\lambda_1 = a_1$ and $a_0 = 1$.

If the nontrivial zeros of $\zeta_k(s)$ lie on the critical line, it follows from Lemma 3.1 that the numbers λ_n are nonnegative for all positive integers n .

Conversely, assume that the number λ_n is nonnegative for every positive integer n . It follows from (3.10) and Lemma 3.2 that

$$\lambda_n \leq n a_n$$

for every positive integer n . This inequality together with Lemma 3.2 implies that

$$\sum_{n=1}^{\infty} |\lambda_n z^{n-1}| \leq \sum_{n=1}^{\infty} n a_n |z|^{n-1} = \varphi'(|z|) \quad (3.11)$$

for z in the unit disk. Since $\varphi'(z)$ is analytic in the unit disk, $\varphi'(|z|)$ is finite for z in the unit disk. It follows from (3.9) and (3.11) that $\varphi'(z)/\varphi(z)$ is analytic in the unit disk. It is clear that a necessary and sufficient condition for the nontrivial zeros of the Dedekind zeta function $\zeta_k(s)$ to lie on the critical line is that $\varphi'(z)/\varphi(z)$ is analytic in the unit disk. Therefore, the nontrivial zeros of the Dedekind zeta function $\zeta_k(s)$ lie on the critical line.

This completes the proof of the theorem. \blacksquare

Remark. We know from the proof of Theorem 2 that $\lambda_1 = a_1$, which is a positive number by Lemma 3.2. An explicit expression for λ_n is implicit in the recurrence relation (3.10) together with formulas (3.6) and (3.7).

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