A Hyperfunction Duality Argument to prove the Riemann Hypothesis

Dr. Klaus Braun
79106 Freiburg i. Br.
www.riemann-hypothesis.de

February 29, 2010

Abstract

The Riemann duality equation is equivalent to the Theta relation, which can be interpreted as an identity of a strong analytical function and its strong, non analytical "dual" function. This can be seen as root cause, while the Berry conjecture hasn't proven so far: the idea of this paper is to transform the Theta relation into L_2 by applying the Hilbert transform, whereby a weak Theta function property is still valid, but where the corresponding dual weak Theta function belongs also to L_2:

specific properties of the Fourier transform of the Gauss-Weierstrass density function $f(x):=e^{-\pi x^2}$ with its non-analytical counterpart e^{-1/x^2} enables Jacobi's $\vartheta -$ relation

$$1+\psi(x^2):=G(x):=\sum_{-\infty}^{\infty} f(nx)=\frac{1}{x}G(\frac{1}{x})$$

resp. Riemann's duality equation ([11] H.M. Edwards, 1.6ff), valid for all complex $s\in C$,

$$\xi(s):=\zeta(s)\Omega(s)=\xi(1-s) \quad \text{with} \quad \Omega(s):=(s-1)\int_0^\infty x^s(xf'(x))d\log x .$$

There is an only formally valid representation of Riemann's duality equation as transform of an integral operator in the form ([11] H.M. Edwards, 10.3)

$$\zeta(s)\int_0^\infty x^s(xf'(x))d\log x=\frac{\xi(s)}{s-1}=\frac{s}{2}\int_0^\infty x^{1-s}G(x)\frac{dx}{x} .$$

This operator has no transform at all, as the integral does not converge for any s. The integral would converge at infinity if the constant term $f(0)=\hat{f}(0)=1$ is absent. Hyperfunctions (23] R. Penrose, 9.2, 9.7, [24] B. E. Petersen, 1.16) are distributions allowing to treat "functions" by Fourier transform, which can transmit unexpected (non-analytical!) signals, represented by a Laurent-series description with vanishing constant Fourier term. Applying the Hilbert transform to f gives a Cauchy principle-valued function with vanishing constant Fourier terms, Applying Müntz formula then defines a distribution valued holomorphic Zeta fake function $\xi^*(s)$ fulfilling the relation

$$(\xi_s^*,\varphi)=(\xi_{1-s}^*,\varphi) \qquad \text{for each } \varphi\in C_c^\infty$$

with

$$\xi^*(s):=\zeta(s)\Omega^*(s)=\int_0^\infty x^s G^*(x)d\log x=\int_0^\infty x^{1-s}G^*(x)d\log x \quad , 0<\operatorname{Re}(s)<1 ,$$

and

$$\Omega^*(s):=\int_0^\infty x^s(Hf)(x)d\log x \quad \text{and} \quad G^*(x)=\frac{1}{x}G^*(\frac{1}{x}) \quad \text{in the distribution sense.}$$

As a consequence ξ_s^* is the weak form of the Zeta function $\xi(s)$, now represented as transform of a Pseudo Differential (integral) operator in the critical stripe, where the spectral theory can be applied to. The several equivalent criteria to the Riemann Hypothesis can be investigated, using the "dual" structure of $\psi^*(x^2)$ as an enabler to overcome current singularity handicaps, when trying to verify those criteria in a non distribution sense.

One criteria requires a representation of $\Xi^*(t):=\xi^*(1/2+it)$ as a convolution of distributions, which is provided by the Hilbert transform structure itself ([7] D.A.Cardon). The underlying Polya criteria ([27] G. Polya) is also applied in [6] D. Bump, et. al., showing that the zeros of the transforms of the Hermite polynomials lie all on the critical line. Specific properties of the Hilbert transform guarantee, that the same is true also for the Hilbert transforms of the Hermite polynomials and therefore also for $\psi^*(x^2)\in L_2$ in a distribution sense. This is the Riemann Hypothesis.

0. Introduction

Our terminology follows those of [11] H.M. Edwards, [24] B. E. Petersen and [36] R.S. Strichartz.

Specific properties of the Gauss-Weierstrass density function

(0.1) $$f(x) := e^{-\pi x^2}$$

with its non-analytical "dual" counterpart e^{-1/x^2} and the Mellin transform in the form

(0.2) $$\tilde{\Pi}(s) := \Gamma(1+s/2)\pi^{-s/2} = \int_0^\infty x^s f'(x)dx$$

enables Jacobi's $\vartheta -$relation resp. Riemann's duality equation in the form ([11] H.M. Edwards, 1.6ff)

$$\vartheta(x^2) := G(x) := \sum_{-\infty}^{\infty} f(nx) = G(1/x)/x$$

$$\xi(s) := \zeta(s)(s-1)\tilde{\Pi}(s) = \xi(1-s)\,,\, s \in C \quad .$$

[11] H.M. Edwards, chapter 10): *Let V be the vector space of all complex-valued functions on R^+ with the inner product*

$$(u,v) := \int_0^\infty u(x)v(x)dx \cdot$$

By $I : v(x) \mapsto \int_0^\infty v(ux)F(u)du$ *an integral operator* $I : V \to V$ *is defined. An operator is said to be invariant if it commutes with all translation operators* $T_u : v(x) \mapsto v(ux)$. *The transform of an invariant operator is the function whose domain is the set of complex numbers* s *such that the function* $v(x) := x^{-s}$ *lies in the domain of the operator and whose value for such an s is the factor by which the operator multiplies* $v(x) := x^{-s}$ *. Thus e.g. the Zeta function* $\zeta(s)$ *for* $\operatorname{Re}(s) > 1$ *is the transform of the summation operator*

$$v(x) \mapsto \sum_1^\infty v(nx) \cdot$$

When defining the adjoint of an invariant operator on V the inner product is defined on a rather small subset of V, whenever both side of $(Lu,v) = (u, L^* v)$ *are defined.*

There is an only formally valid representation of Riemann's duality equation as transform of an integral operator

(0.3a) $$I : v(x) \mapsto \int_0^\infty v(ux)G(u)du$$

in the form ([11] H.M. Edwards, 10.3, (2.8) below)

(0.3b) $$\int_0^\infty x^{1-s} G(x) \frac{dx}{x} = \frac{2\xi(s)}{s(s-1)} \ .$$

But the operator (0.3a) has no transform at all, as the integral does not converge for any s. The integral would converge at ∞ if the constant term $f(0) = \hat{f}(0) = 1$ is absent.

If one would find an integral operator in the from (0.3a) satisfying the same functional equation than G does and if

(0.3c) $$\int_0^\infty x^{1-s} G(x) \frac{dx}{x} \text{ converges and } \int_0^\infty x^{1-s} G(x) \frac{dx}{x} = \int_0^\infty x^{s} G(x) \frac{dx}{x}$$

Then this operator would be self-adjoint in the sense of above ([11] H.M. Edwards, 10.2, 10.3).

Hyperfunctions are distributions, allowing to treat "functions" by Fourier transform, which can transmit unexpected (non-analytic!) signals, represented by a Laurent-series description with vanishing constant "Fourier term" ([23] R. Penrose, 9.2). In the one-dimensional case hyper-functions are the distributions of the dual space $C^{-\omega}$ of the real-analytical functions of a real variable C^{ω}, defined on some connected segment $\subset R$ ([23] R. Penrose, 9.7, [24] B. E. Petersen, 1.16) and appendix). This gives the link of our approach to Penrose's thoughts and ideas moving forward "the road to reality". In the one-dimensional case the concept of hyperfunctions enables a link between distributions and a holomorphic, i.e. a complex-analytical function, as any distribution f on R can be realized as the "jump" of the corresponding in $C - R$ holomorphic Cauchy integral function

$$F(x) := \frac{1}{2\pi i} \oint \frac{f(t)dt}{t-x}$$

across the real axis, given by

$$(f, \varphi) = \lim \int_{-\infty}^{\infty} F(x+iy) - F(x-iy))\varphi(x)dx \quad \text{for } y \to 0^+ \ .$$

The Hilbert transform H (see (1.1) below) gives a Cauchy principle-valued function with Fourier terms

(0.4) $$(Hu)_\nu = -i \operatorname{sgn}(\nu) u_\nu \ .$$

The study of the Hilbert transform and the study of operational calculus for non-commuting operators in quantum mechanics (e.g. the Weyl operator) contain some of the basic ingredients of the theory of pseudo differential operators ([24] B. E. Petersen, 3.1). Freeding the Hilbert transform from its too intimate link connection with complex variables techniques Calderon and Zygmund introduced the algebra of singular integral operators (modulo compact operators) based on salient features of the Hilbert transform ([24] B. E. Petersen, 2.9). This also stimulated the study of the algebra generated by singular differential operators and partial differential operators ([24] B. E. Petersen, 4.1ff), which all leads into the concept of pseudo differential operator.

The Hilbert transform (1.1), which is a classical Pseudo differential operator, transforms the Gauss-Weierstrass density function into a "P.v. distribution" ([24] B. E. Petersen, 1.7,) in the form

(0.5) $$Hf(x) = f(x) * \frac{1}{\pi x} = \hat{f}(x)\left[\frac{1}{\pi x}\right]^{\wedge} \quad \text{with} \quad \hat{f}(x) = e^{-\pi x^2}$$

resp. $$FHF^{-1} = 2\pi F(\frac{1}{\pi x}) = -i\,\mathrm{sgn}(x)\cdot$$

With reference to (0.4) we mention Euler's famous formula ([22] N. Nielsen, chapter IX, §51)

$$sign(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(tx)dt}{t} \quad .$$

As link to a well known Zeta function constant we mention ([24] B. E. Peterson, 1.15, [5] R. P. Brent)

$$\left[P.f.\frac{1}{|x|}\right]^{\wedge} = -2\gamma - 2\log|\xi| \quad ,$$

whereby P.f. denotes Hadamard's "partie finie" or "finite part". The P.v. distribution (0.5) can be calculated, which we state in

Lemma 0.1: The Hilbert transform of the Gauss-Weierstrass density function (0.1) and its related Fourier transform are given by

i) $$[H(f)](x) = 4\pi\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi \quad ,$$

ii) $$[H(f)]^{\wedge}(\omega) = 2\pi i\int_0^\infty f(\xi)[\delta(\omega-2\pi\xi)-\delta(\omega+2\pi\xi)]d\xi$$

iii) $$H(\frac{1}{x}f(\frac{1}{x})) = \frac{1}{x}(Hf)(\frac{1}{x}) \quad .$$

Proof of lemma 0.1: is given in the appendix.

With respect to lemma 0.1 ii) we refer to the definition of hyperfunctions (appendix B). With respect to lemma 0.1iii) we refer to [18] J.M. Hill.

[24] B. E. Petersen, chapter 1, §15: *Let* $z \to g_z$ *be a function defined on a open subset* $U \subset C$ *with values in the distribution space. Then* g_z *is called a holomorphic in* $U \subset C$ *(or* $g(z) := g_z$ *is called holomorphic in* $U \subset C$ *in the distribution sense), if for each* $\varphi \in C_c^\infty$ *the function* $z \to (g_s, \varphi)$ *is holomorphic in* $U \subset C$ *in the usual sense*.

The constant Fourier term of $(Hf)(x)$ vanishes. This enables the definition of the distribution complex-valued function

(0.6) $$\Omega^*(s) := (s-1)\tilde{\Pi}_s^* := (s-1)\tilde{\Pi}^*(s) := \int_0^\infty x^s (Hf)(x)\frac{dx}{x} \ .$$

The Müntz formula (lemma 2.1) builds representations of the Zeta in the form

$$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s\left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0 < \mathrm{Re}(s) < 1 \ ,$$

when $\omega(x)$ satisfies certain conditions. We apply this formula to the function

(0.7) $$G^*(x) := 2\psi^*(x^2) := 2\sum_1^\infty \omega(nx) := 2\sum_1^\infty H\left[f(nx)\right] \in L_2 \ ,$$

to build a distribution valued holomorphic function

$$\xi_s^* = \zeta(s)\Omega^*(s) \quad \text{for } 0 < \mathrm{Re}(s) < 1.$$

Jacobi's $\vartheta-$relation and the vanishing constant Fourier term of $H\left[f(x)\right]$ then imply

(0.8) $$\psi^*(x^2) = \frac{1}{x}\psi^*(\frac{1}{x^2}) \qquad \text{in the distribution sense.}$$

The distribution framework ensures the convergence of the Mellin transform integrals, when applying the variable transform $x \to \frac{1}{y}$ resp. $\frac{dx}{x} = \frac{dy}{y}$. This then leads to the relation

(0.9) $$(\xi_s^*, \varphi) = (\xi_{1-s}^*, \varphi) \qquad \text{for each } \varphi \in C_c^\infty \ .$$

Thus, in the sense of (0.3), the corresponding integral operator

$$I : v(x) \mapsto \int_0^\infty v(ux)\psi^*(u^2)du$$

is self-adjoint and spectral theory can be applied. (0.8), (0.9) is the main result of this manuscript, which we summarize in

Proposition 0.3: The distribution valued holomorphic Zeta fake function ξ_s^* fulfills the Riemann duality equation in the critical stripe, i.e. ξ_s^* is the weak representation of $\xi(s)$ represented as Mellin transform of a self-adjoint integral operator.

For the rest of this section we give the link to an appropriate Hilbert space environment.

The weighted Hermite polynomials (e.g. [36] R.S. Strichartz, 7.6)

(0.10) $$\varphi_n(x) := \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \quad \text{with} \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \ , \quad H_0(x) = 1 \, , \quad H_1(x) = x \, ,$$

form a set of orthonormal functions in $L_2(-\infty, \infty)$, i.e. the Hermite polynomials have only real zeros. The relation to (0.1) is given by

$$f(x) = \pi^{1/4} \varphi_0(\sqrt{2\pi} x) \ .$$

The Hilbert transforms of Hermite polynomials are given in the appendix. Lemma 1.3 indicates the orthogonality relations to the original weighted Hermite polynomials $\varphi_n \in L_2$ leading to $H\varphi_n \in L_2$ and

(0.11) $$L_2 := H := span[\varphi_n(x)] = span[H(\varphi_n(x))] \ .$$

In [6] D. Bump, et. al. it's shown that all zeros of the Mellin transforms of the weighted Hermite polynomials lie on the critical line using the recursion formula of the Hermite polynomials in combination with an argument from Polya ([27]), which he developed to analyze the zeros of $q(s) := K_s(x)$ (see [37] E. C. Titchmarsh, 10.23 and appendix part B (5.1)):

Lemma 0.4: If $-\infty < c < \infty$ and $K(z)$ is an entire function of genus 0 or 1 that assumes real values for real z, has only real zeros and has at least one real zero, then the function

$$K(z - ic) + K(z + ic)$$

also has only real zeros.

We note the relations

$$\sin(z - ic) + \sin(z + ic) = 2 \sin x \cosh c$$

$$\cos(z - ic) + \cos(z + ic) = 2 \cos x \cosh c \, .$$

Applying the arguments from [6] D. Bump et.al., the Mellin transforms of the functions $H(\varphi_n(x))$ have their zeros on the critical line.

By (0.9) a hermitian operator is defined. From spectral theory it then follows that its spectrum is real, i.e. it holds

Corollary 0.5: The zeros of the complex-valued distribution Mellin transform of $\psi^*(x^2)$, which are the zeros of ξ_s^* , lie all on the critical line. This proves the RH in the distribution sense. Applying standard density arguments this then proves the RH itself.

In [7] D.A. Cardon an analysis is given to study convolution operators and the zeros of corresponding entire function. For a special class of probability (!) distribution (in the sense of probability theory!!) functions F there is a generalization of Polya's lemma such that the convolution ([7,8] D. A. Cardon),

(0.12) $$(K*dF)(z):=\int_{-\infty}^{\infty}K(z-iu)dF(u)$$

has only real zeros. Being especially $F(x)$ the normal distribution ([7] D. A. Cardon)

(0.13) $$F(x):=\int_{-\infty}^{x}e^{-\pi u^2}du=\int_{-\infty}^{x}f(u)du \quad \text{and} \quad K(z):=z^n$$

the corresponding convolution (0.9) enables a representation of the Hermite polynomials in the form

(0.14) $$h_n(z):=(K_n*dF)(z)=\int_{-\infty}^{\infty}(z-iu)^n dF(u)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(z-iu)^n e^{-u^2}du=2^{-n}H_n(z) \quad .$$

We refer to

Lemma 0.6 ([7] D.A.Cardon): If the function $\Xi(t):=\xi(1/2+it)$ with

$$\xi(s):=\zeta(s)(s-1)\tilde{\Pi}(s)=\xi(1-s)$$

can be realized as a convolution $\Xi(t)=(K*dF)(t)$ where $K(t)\in LP^*$, i.e. is a entire function from the Laguerre-Polya class of order <2, i.e. $\Phi(z)=c^z me^{\alpha z-\beta z^2}\prod_k(1-z/\alpha_k)e^{z/\alpha_k}$, where c,α,α_k are real, $\beta\geq 0$ and m is a nonnegative integer, this would prove the RH.

Hypothesis: On the critical line the representation (0.7) fulfills the assumptions of lemma 0.6 below in the distribution sense due to the convolution structure of the Hilbert transform (see also appendix (B.1)).

1. Hyperfunctions and Pseudo Differential Operators,

Mellin-, Fourier-, Wavelet-, Hilbert-Transforms

The Fourier transform plays a key role in the theory of signal processing. For real functions $u(t)$ the positive real frequency axis already contains all information about the waveform in the time domain. In the one-dimension case the Riesz operator is identical with Hilbert transform ([24] B. E. Petersen, 2.9), that is a Cauchy principle-valued function, expressed in the form

(1.1) $$(R_1 u)(x) := (Hu)(x) := \lim \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{u(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy \qquad \text{for } \varepsilon \to 0$$

fulfilling $$(Hu)_\nu = -i \operatorname{sgn}(\nu) u_\nu \ .$$

The Hilbert transform is a classical pseudo-differential operator ([24] B. E. Petersen, 3.6) with symbol $i\operatorname{sgn}(s)$. The principle value $P.v.(1/x)$ of the not locally integrable function $1/x$ is the distribution g defined by ([24] B. E. Petersen, 1.7)

$$(g,\varphi) := \lim \int_{x \geq \varepsilon} \varphi(x) \frac{dx}{x} = \int_{-\infty}^{\infty} \log|x| \varphi'(x) dx \qquad \text{for each } \varphi \in C_c^\infty \ .$$

The relation of this specific principle value to the Fourier and Hilbert transform (1.1) is given by ([24] B. E. Petersen, 2.9)

(1.2) $$\left[P.v.(\frac{1}{x}) \right]^\wedge = -i\pi \operatorname{sgn}(s) \quad \text{and} \quad \left[P.v.(\frac{1}{x}) \right]^{\wedge\wedge} = -2\pi P.v.(\frac{1}{x}) \cdot$$

For example the function $\sin(\omega t)$ is the Hilbert transform of $\cos(\omega t)$. This gives a $\pm \pi/2 -$ phase-shift operator, which is a basic property of the Hilbert transform. It can be used to remove the not needed negative frequency axis.

To make a rigorous presentation of the Hilbert transform theory one have to apply distribution theory. We state some main properties of the Hilbert transform in

Lemma 1.1: For the Hilbert transform it holds

i) $\|H\| = 1$, $H^* = -H$, $H^2 = -I$, $H^{-1} = H^3$,

ii) $H(f * g) = f * Hg = Hf * g$, $f * g = -Hf * Hg$

iii) If $(\varphi_n)_{n \in N}$ is an orthogonal system, so it is for the system $(H(\varphi_n))_{n \in N}$, i.e.

$$(H\varphi_n, H\varphi_n) = -(\varphi_n, H^2 \varphi_n) = (\varphi_n, \varphi_n).$$

iv) $\|Hu\|^2 = \|u\|^2$, i.e. if $u \in L_2$ then $Hu \in L_2$.

For other properties related e.g. to rotations we refer to [35] E.M. Stein.

A complex function is called Hermitian if its real part is even and its imaginary part is odd. If $g(t)$ is a real function, then e.g. $\hat{g}(\xi)$ is Hermitian and therefore $|\hat{g}(\xi)|^2$ is even.

A complex signal u is called a strong analytical signal if it fulfills $Hu = iu$. For strong analytical signals u it holds that $H(\mathrm{Re}(u)) = \mathrm{Im}(u(x))$, i.e.

$$z(t) = u(t) + iH(u(t))$$

is a strong analytical signal.

The specific properties of the operator (1.1), which are essential for our arguments, we summaries in

Lemma 1.3: The Hilbert Operator (1.1) fulfills

i) The Fourier term for $\nu = 0$ is $(Hu)_0 = 0$

ii) $H(xu(x)) = xH(u(x)) - \frac{1}{\pi}\int_{-\infty}^{\infty} u(y)dy$

iii) for odd functions it hold $H(xu(x)) = x(Hu)(x)$

iv) $Hu(x) = u(x) * \frac{1}{\pi x}$, $\frac{1}{\pi x} = \lim \frac{x}{\pi(x^2+\rho^2)}$ $\rho \to 0$

v) If $u, Hu \in L_2$ then u and Hu are orthogonal, i.e. $\int_{-\infty}^{\infty} u(y)(Hu)(y)dy = 0$.

Proof of lemma 1.3 is given in the appendix.

2. A Distribution valued Zeta Function as Transform of a Pseudo Differential Operator

The Gauss-Weierstrass density function

(2.1) $$f(x) := e^{-\pi x^2}$$

with its Mellin transform in the form

(2.2) $$\tilde{\Pi}(s) := \Gamma(1+\frac{s}{2})\pi^{-s/2} = \int_0^\infty x^s (xf'(x)) \frac{dx}{x}$$

enables Jacobi's $\vartheta -$relation (see e.g. [17] H. Hamburger)

(2.3) $$1+2\psi(x^2) := G(x) := \sum_{-\infty}^{\infty} f(nx) = \frac{1}{x} G(\frac{1}{x})$$

and Riemann's duality equation

(2.4) $$\xi(s) := \zeta(s)(s-1)\tilde{\Pi}(s) = \xi(1-s)$$ valid for all complex $s \in C$.

The $\vartheta -$relation (2.3) is a direct consequence of Poisson summation formula ([24] B. E. Petersen, 2.11)

(2.5) $$\sum_{n=-\infty}^{\infty} \hat{\varphi}(2\pi n) = \sum_{n=-\infty}^{\infty} \varphi(n)$$

and the Fourier transform of the Gauss-Weierstrass kernel ([24] B. E. Petersen, 2.3)

(2.6) $$\frac{1}{2\pi}\left[e^{-\varepsilon|x|^2}\right]^{\wedge} = \frac{1}{\sqrt{4\pi\varepsilon}} e^{-|\xi|^2/(4\varepsilon)} \ .$$

The constant Fourier term $(\xi = 0)$ doesn't vanish. Therefore it (unfortunately) doesn't hold

$$\psi(x^2) \neq \frac{1}{x}\psi(\frac{1}{x^2}) \ .$$

There is an only formally valid representation ([11] H.M. Edwards, 10.3) of (1.7) as transform of an integral operator in the form

(2.7) $$\int_0^\infty x^{1-s} G(x) \frac{dx}{x} = \frac{2\xi(s)}{s(s-1)} \ .$$

The operator (2.7) has no transform at all, as the integral does not converge for any s. The integral would converge at ∞ if the constant term $f(0)=\hat{f}(0)=1$, ($n=0$) is absent. Roughly speaking solves the measure $xf'(x)dx$ the convergence issue of (2.7).

The function ([11] H.M. Edwards, 10.3)

(2.8) $$H(x):=\frac{d}{dx}\left[x^2\frac{d}{dx}G(x)\right]=\frac{d}{dx}\left[x^2\frac{d}{dx}\left[G(x)-1\right]\right]=2\sum_1^\infty(2\pi^2n^4x^4-3\pi n^2x^2)e^{-\pi n^2x^2}>0$$

overcomes the convergence issue of (2.7) by "differentiation" to get ride off the "jeopardizing" non-vanishing constant Fourier term, fulfilling both (self-adjoint) conditions, i.e. it holds

$$H(x)=\frac{1}{x}H(\frac{1}{x})$$ and $$\int_0^\infty x^{1-s}H(x)\frac{dx}{x}=\int_0^\infty y^{-(-s)}\frac{1}{y}H(\frac{1}{y})\frac{dy}{y}=\int_0^\infty x^sH(x)\frac{dx}{x}$$ converge for any s,

defining an entire function, which is invariant concerning $s\leftrightarrow 1-s$, i.e.

(2.9) $$\int_0^\infty x^{1-s}H(x)\frac{dx}{x}=s(s-1)\int_0^\infty 2x^{1-s}\psi(x^2)\frac{dx}{x}=2\xi(1-s)\cdot$$

The operator (2.8) is a "differentiation" operator, balancing with respect to (2.9) the move to a higher regularly Hilbert scale by combining it with a multiplication operator, but the prize to be paid for that "move" is a corresponding change of underlying zeros resp. eigenvalues behavior.

On the other side looking at the Hilbert transform of (2.1) and its related Fourier transform there is an only formally valid representation of the corresponding Poisson summation formula in the strong sense. But as the Fourier transform exists in the distribution sense the weak $\vartheta-$relation

(2.10) $$G^*(x):=\sum_{-\infty}^\infty H\left[f(nx)\right]=\frac{1}{x}G(\frac{1}{x})\quad ,$$

fulfills as well

(2.11) $$\psi^*(x^2):=\sum_1^\infty H\left[f(nx)\right]=\frac{1}{x}\psi^*(\frac{1}{x^2})$$ in the distribution sense.

Combining lemma 0.1 with (2.11) gives

(2.12) $$\psi^*(x^2)=4\pi\sum_1^\infty\int_0^\infty f(\xi)\sin(2\pi n\xi x)d\xi$$ in the distribution sense.

The fact, that in opposite to (2.6) the Fourier transform of the Hilbert transform of (2.1) vanishes at $x=0$ (see (0.8)) suggests to replace

(2.13) $$xf'(x) \to Hf(x) = 4\pi \int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi \ .$$

The adjoint of the differential operator $v(x) \mapsto xv'(x)$, found by integration by parts, is $u(x) \mapsto \frac{d}{dx}[xu(x)]$ ([11] H.M. Edwards, 10.32). In terms of transforms this operation is related to the substitution $s \mapsto (1-s)$.

A vanishing constant Fourier term plays also a key role in the wavelet theory. A wavelet transform is similar as a Fourier transform. A Fourier transform delivers the frequency spectrum of a timely signal $\varphi(t)$ without any loss of information, although it gives the frequencies without any information about the points in time, when the frequencies occur. The wavelet transform delivers this sort of information in a better distinguishing form: one gets both the frequency analysis and the points in time, when those frequencies happen. In this sense the wavelet transform describes the music of an orchestra on a 2-dimensional instead of a 1-dimensional paper. This reminds to [32] M. du Sautoy's statement (p. 120ff), that "the primes have music in them", where the behavior of the zeros of the Zeta function gives the melody and the loudness of its music.

We recall Müntz' formula, which gives a representation of the Zeta function in the critical stripe:

Lemma 2.1 (**Müntz' formula**) For $\omega(x), \omega'(x)$ continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ and $\omega(x) = o(x^{-\beta})$ for $x \to \infty$ and $\alpha, \beta > 1$ it holds

(2.14) $$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s \left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt \right] \frac{dx}{x} \qquad \text{for } 0 < \mathrm{Re}(s) < 1.$$

Proof: is given in the appendix. •

Applying Müntz formula with $\omega(x) := \psi^*(x^2)$ (2.11) it follows the

Corollary 2.2: in the critical stripe there exists a holomorphic function $\xi_s^* := \xi^*(s)$ in the distribution sense represented by

(2.15) $$\xi^*(s) := \zeta(s)\int_0^\infty x^s Hf(x)\frac{dx}{x} = \int_0^\infty x^s \psi^*(x^2)\frac{dx}{x} \qquad \text{for } 0 < \mathrm{Re}(s) < 1,$$

which is dual in the distribution sense, i.e. it fulfills

(2.16) $$(\xi_s^*, \varphi) = (\xi_{1-s}^*, \varphi) \qquad \text{for each } \varphi \in C_c^\infty \ .$$

Applying standard density arguments to corollary 2.2 leads to

Corollary 2.3: The holomorphic function $\xi_s^* := \xi^*(s)$ is the weak distribution representation of the entire function $\xi(s)$ and the standard Zeta function theory can be applied to $\xi_s^* := \xi^*(s)$ accordingly e.g. it holds ([11] H.M. Edwards, 1.8)

(2.17) $$\xi^*(s) = \int_0^\infty x^s \psi^*(x^2)\frac{dx}{x} = \frac{1}{2}\int_1^\infty \sqrt{x}\cosh\left[(s-\frac{1}{2})\log x\right]\psi^*(x^2)\frac{dx}{x}$$ in the distribution sense,

i.e. there is a series expansion in the form $\xi^*(s) = \sum_1^\infty a_{2n}(s-\frac{1}{2})^{2n}$ with

$$a_{2n} := \frac{1}{2}\int_1^\infty \sqrt{x}\cosh\left[(s-\frac{1}{2})\log x\right]\psi^*(x^2)\frac{(\log x)^{2n}}{(2n)!}\frac{dx}{x}$$

$$a_{2n} := 2\pi\sum_{k=1}^\infty\int_0^\infty\int_1^\infty x^{-1/2}\cosh\left[(s-\frac{1}{2})\log x\right]\frac{(\log x)^{2n}}{(2n)!}f(\xi)\sin(2\pi\xi kx)dxd\xi \ .$$

For the abbreviation

$$g(x) := \frac{1}{2}x^{1/2}\psi^*(x^2) \quad \text{and} \quad G(u) := g(e^u)$$

it holds with (2.11)

$$g(\frac{1}{x}) := \frac{1}{2}x^{1/2}\frac{1}{x}\psi^*(\frac{1}{x^2}) = \frac{1}{2}x^{1/2}\psi^*(x^2) = g(x) \ .$$

Substituting the variable $u = \log x$ (2.17) resp. $x = 1/y$ (2.17) can be re-written in the form

Corollary 2.4: The complex-valued distribution holomorphic function $\Xi^*(t) := \xi^*_{1/2+it} := \xi^*(1/2+it)$ can represented in the form

(2.18) $$\Xi^*(t) := \int_1^\infty g(x)\cos(t\log x)\frac{dx}{x} = \int_1^\infty G(u)\cos(tu)du$$

and fulfills

$$\Xi^*(t) = \int_1^\infty g(x)\cos(t\log x)\frac{dx}{x} = \int_0^1 g(y)\cos(t\log y)\frac{dy}{y} = \frac{1}{2}\int_0^\infty g(x)\cos(t\log x)\frac{dx}{x} = \frac{1}{2}\int_0^\infty G(u)\cos(tu)du \ .$$

Riemann's Hypothesis is saying, that all zeros of (2.18) are real, which is the fulfilled, if the underlying integral operator of (2.18) is self-adjoint ([11] H.M. Edwards, 10.3), as a hermitian operator has a real spectrum only.

Appendix Part A: 1. Proofs

Proof of lemma 0.1

i) The Fourier transform of $\varphi_0(t) := \pi^{-1/4} e^{-t^2/2}$ is given by $\hat{\varphi}_0(t) := \sqrt{2}\pi^{1/4} e^{-\omega^2/2}$. With (0.4) we get

$$[H(\varphi_0)]^\wedge(\omega) = -i\,\mathrm{sgn}(\omega)\hat{\varphi}_0(\omega) \;.$$

Applying the inverse Fourier transform then gives

$$[H(\varphi_0)](t) = \sqrt{2}\pi^{1/4} \int_{-\infty}^{\infty} (-i\,\mathrm{sgn}(\omega)) e^{-\omega^2/2} e^{-i\omega t} d\omega \;.$$

Since $\mathrm{sgn}(\omega) e^{-\omega^2/2}$ is odd we have

$$[H(\varphi_0)](t) = 2\sqrt{2}\pi^{1/4} \int_0^{\infty} e^{-\omega^2/2} \sin(\omega t) d\omega \;.$$

With $f(x) = \pi^{1/4}\varphi_0(\sqrt{2\pi}x)$ it follows

$$\pi^{1/4}[H(\varphi_0)](\sqrt{2\pi}x) = 2\sqrt{2\pi} \int_0^{\infty} e^{-\omega^2/2} \sin(\sqrt{2\pi}\omega x) d\omega \;.$$

Substituting the variables $\omega = \sqrt{2\pi}\xi$ then leads to

$$[H(f)](x) = 4\pi \int_0^{\infty} e^{-\pi\xi^2} \sin(2\pi\xi x) d\xi \;.$$

ii) We recall the Fourier transforms

$$g_1(x) := \sin(2\pi a x) \qquad \hat{g}_1(\omega) = \frac{i}{2}[\delta(\omega - 2\pi a) - \delta(\omega + 2\pi a)]$$

$$g_2(x) := \begin{cases} \frac{i}{2}\pi sign(x) & |x| \le 2a \\ 0 & |x| > 2a \end{cases} \qquad \hat{g}_2(\omega) = \frac{\sin 2(a\omega)}{\omega} \;,$$

which leads to ii)

iii) As $f(x)$ is an even function is follows that $[H(f)](x)$ is even. From lemma1.3 below it then follows

$$H(\frac{1}{x} f(\frac{1}{x})) = \frac{1}{x}(Hf)(\frac{1}{x}) \qquad \bullet$$

Lemma (**Müntz' formula**) For $\omega(x), \omega'(x)$ continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ and $\omega(x) = o(x^{-\beta})$ for $x \to \infty$ and $\alpha, \beta > 1$ it holds

$$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s \left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0 < \operatorname{Re}(s) < 1.$$

Proof: recalled from [37] E. C. Titchmarsh:

i) because $\omega(x)$ is continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ it holds

$$\sum_1^\infty \frac{1}{n^s}\left|\int_0^\infty x^{s-1}\omega(x)\right|dx \qquad \text{exists for} \quad 1 < \sigma < \alpha \ ,$$

i.e. the inversion leading to the left hand side of (4.3) is justified.

ii) $$\sum_1^\infty \omega(nx) - \int_0^\infty \omega(xt)dt = x\int_0^\infty \omega'(t)(t-[t])dt = x\int_0^{1/x} O(1)dt + x\int_{1/x}^\infty O((xt)^{-\beta})dt = O(1)$$

The first summand is justified, because $\omega(x)$ is continuous and bounded in any finite interval the second summand is justified, because $\omega(x) = o(x^{-\alpha})$, i.e. it holds

$$\sum_1^\infty \omega(nx) = O(1) + \frac{c}{x} \ \text{ with } \ c = \int_0^\infty \omega(t)dt \ .$$

Hence

$$\int_0^\infty x^s \sum_1^\infty \omega(nx) + \frac{dx}{x} = \int_0^1 x^s\left[\sum_1^\infty \omega(nx) - \frac{c}{x}\right]\frac{dx}{x} + \int_1^\infty x^s \sum_1^\infty \omega(nx)\frac{dx}{x} + \frac{c}{s-1}$$

for $\sigma > 0$ except $s = 1$. Also

$$-c\int_1^\infty x^{s-2}dx = \frac{c}{s-1} \qquad \text{for } \sigma < 1$$

and therefore the result for $0 < \sigma = \operatorname{Re}(s) < 1$ •

Proof of lemma 1.3

i) is given in [24] B. E. Petersen, 2.9

ii) Consider the Hilbert transform of $xu(x)$

$$H(xu(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yu(y)}{x-y}\,dy \cdot$$

The insertion of a new variable $z = x - y$ yields

$$H(xu(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-z)u(x-z)}{z}\,dz = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(xu(x-z)}{z}\,dz - \frac{1}{\pi}\int_{-\infty}^{\infty} u(x-z)dz = xH(u(x)) - \frac{1}{\pi}\int_{-\infty}^{\infty} u(y)dy$$

iii) follows directly from i) and ii)

iv) is given in [24] B. E. Petersen, 2.9

v) $\int_{-\infty}^{\infty} u(y)(Hu)(y)dy = \frac{i}{2\pi}\int_{-\infty}^{\infty} sign((\omega)\left|\hat{u}(\omega)\right|^2 d\omega$ with $\left|\hat{u}(\omega)\right|^2$ is even gives the result •

2. Hermite Polynomials

The Hermite polynomials $H_n(x)$ fulfill the recursion formula

(2.9) $$H_n(\sqrt{2\pi}x) = 2xH_{n-1}(\sqrt{2\pi}x) - (n-1)b_n\varphi_{n-2}(x) - 2(n-1)H_{n-2}(\sqrt{2\pi}x) \cdot$$

Using the abbreviation

$$a_n := \sqrt{\frac{2(n-1)!}{n!}} \qquad b_n := \sqrt{\frac{(n-2)!}{n!}}$$

this gives the recursion formula

(2.10) $$\varphi_n(x) := a_n x\varphi_{n-1}(x) - (n-1)b_n\varphi_{n-2}(x)\,, \quad \varphi_0(x) := \pi^{-1/4}e^{-\frac{x^2}{2}}, \quad \varphi_1(x) := 2^{-1/2}\pi^{-1/4}xe^{-\frac{x^2}{2}} \quad ,$$

from which the recursion formula for the corresponding Hilbert transforms can be calculated

(2.11) $$\hat{\varphi}_n(x) := a_n\left[x\hat{\varphi}_{n-1}(x) - \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_{n-1}(y)dy\right] - (n-1)b_n\hat{\varphi}_{n-2}(x)$$

$$\hat{\varphi}_0(x) = \pi^{1/4}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}}\sin(\omega x)d\omega \cdot$$

Appendix Part B: General Background Information

1. Hyperfunctions

In the one-dimensional case hyperfunctions are the distributions of the dual space $C^{-\omega}$ of the real-analytical functions of a real variable C^{ω}, defined on some connected segment $\subset R$. Any real-analytical function is $\in C^{\infty}$, but not every function $\in C^{\infty}$ is analytical, e.g. it holds

$$e(x) := \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x = 0 \end{cases} \in C^{\infty} \text{ but } e(x) \notin C^{\omega} .$$

From $e^{(n)}(0) = 0$ for all n for the Taylor series it follows $\sum_0^{\infty} \frac{0}{n!} x^n = 0$, what's different to $e(x)$ except at $x = 0$, i.e. $e(x) \notin C^{\omega}$ is not an analytical function. The situation is different in case of complex-analytical functions, which are holomophic and analytical at the same time.

In the one-dimensional case the concept of hyperfunctions (see e.g. [24] B. E. Petersen, 1.16) enables a link between distributions and a holomorphic, i.e. a complex-analytical function, as any distribution f on R can be realized as the "jump" of the corresponding in $C - R$ holomorphic Cauchy integral function

$$F(x) := \frac{1}{2\pi i} \oint \frac{f(t)dt}{t - x}$$

across the real axis, given by

$$(f, \varphi) = \lim \int_{-\infty}^{\infty} F(x+iy) - F(x-iy))\varphi(x)dx \quad \text{for } y \to 0^+ .$$

This means that the dual (distribution) space $C^{-\omega}$ of the space of the real-analytical functions C^{ω} characterizes the so-called hyperfunctions [[23] R. Penrose, [24] B. E. Petersen).

A hyperfunction of one variable $f(x)$ on an open set $\Omega \subset R$ is a formal expression of the form $F_+(x+i0) - F_-(x-i0)$, where $F_{\pm}(z)$ is a function holomorphic on the upper, respectively lower, half-neighborhood $U_{\pm} = U \cap \{z | \operatorname{Im}(z) > 0\}$, for a complex neighborhood $U \supset \Omega$ satisfying $U \cap R = \Omega$. The expression $f(x)$ is identified with 0 if and only if $F_{\pm}(z)$ agrees on Ω as a holomorphic function.

If the limits exist in distribution sense, the formula gives the natural imbedding of the space of distributions into that of hyperfunctions. Hyperfunctions can be defined on real-analytic manifolds. Fourier series are typical examples of hyperfunctions on a manifold:

(B.1) $\sum_{\nu \in Z} a_\nu e^{i\nu x}$ converges as a hyperfunction if and only if $a_\nu = O(e^{\varepsilon|\nu|})$ for all $\varepsilon > 0$.

Some examples of generalized functions interpreted as hyper functions are

a. Dirac's delta function $\delta(x) = -\frac{1}{2\pi i}\left[\frac{1}{x+i0} - \frac{1}{x-i0}\right] = \pi \lim \int_0^\infty e^{-ak} \cos kx dk$, $a \to 0$

b. Heaviside's function $Y(x) = -\frac{1}{2\pi i}\left[\log(-x-i0) - \log(-x+i0)\right] = -\frac{1}{2\pi i}\log(-z)$.

The Heaviside function can be characterized ([23] B. E. Petersen, 1.16) by

$\lim \log(x+iy) = \log x + i\pi\hat{Y}$ for $y \to 0^+$ and $\hat{Y}(x) = Y(-x)$

c. $x_\pm^\lambda = \frac{\pm(\mp z)^\lambda}{2i \sin \pi\lambda}$ for $\lambda \notin Z$

$x_\pm^m = \pm\frac{1}{2\pi i} \mp (z)^m \ln(\mp z)$ for $\lambda = m \in Z$

d. Feynmann propagator (Green's function), i.e. the solution $\frac{1}{2\pi i}(S^\vee - S^\wedge)$

with
$$2\pi(2\pi)^m S^\wedge(t,x) = \iint \frac{e^{-i\omega t + ikx} dk d\omega}{(\omega - |k| - i\varepsilon)(\omega - |k| - i\varepsilon)}$$

$$2\pi(2\pi)^m S^\vee(t,x) = \iint \frac{e^{-i\omega t + ikx} dk d\omega}{(\omega - |k| + i\varepsilon)(\omega - |k| + i\varepsilon)}$$

of the distribution wave equation

$$(\frac{\partial^2}{\partial t^2} - \Delta)S(t,x) = \delta(t)\delta^m(x) \quad .$$

We mention the famous Gamma identity for $0 < \mathrm{Re}(z) < 1$

$$g(z)g(1-z) = \frac{\pi}{\sin \pi z} \quad \text{with} \quad g(z) := \Gamma(z) \ .$$

Its proof is using the "Haar measure" property on the multiplicative group of positive real numbers $(R_{>0}^*, *)$) (see [8] H.M. Edwards 10.2 for the corresponding Fourier analysis technique and handicaps in the context of self adjoint operator and its transforms, see ([24] B. E. Petersen 2.10 for a link of Haar measure and pseudo differential operators) of

$$d\mu(x) := \frac{dx}{x} \quad \text{with} \quad d\mu(cx) = \frac{d(cx)}{x} = d\mu(x) \cdot$$

to be applied to

$$\Gamma(z) := \int_0^\infty x^z e^{-x} d\mu(x) \quad .$$

With reference to this in relation to an alternative condition for the Riemann duality equation (([11] H.M. Edwards 10.3, [16] H. Hamburger) we mention ([14] I.S. Gradshteyn, I.M. Ryzhik, 1.217)

$$i\pi\cot i\pi x=\pi\coth\pi x=\pi(1+\sum_{1}^{+\infty}e^{-2\pi nx})=\frac{1}{x}+\sum_{1}^{\infty}\frac{2x}{n^2+x^2}=\frac{1}{x}+\sum_{1}^{\infty}\left[\frac{i}{n+ix}-\frac{i}{n-ix}\right].$$

2. Conformal mapping on the unit disk

The Riemann mapping theorem asserts that any open region in the complex plane, bounded by a simple closed loop, can be mapped holomorphically to the interior of the unit circle

$$D:=\{z\|z|<1\};$$

the boundary being also mapped accordingly.

Let $u(s)$ being a $2\pi-periodic$ function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense, i.e. $u\in H:=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$ and u_ν the Fourier coefficients

$$u_\nu:=\frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx$$

defining for real β the norms

$$\|u\|_\beta^2:=\sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 .$$

There is a natural representation of the Fourier decomposition

$$u(x)=\frac{a_0}{2}+\sum_{1}^{\infty}a_\nu\cos(\nu x)+\sum_{1}^{\infty}b_\nu\sin(\nu x):=\sum_{-\infty}^{\infty}u_\nu e^{i\nu x}\in L_2$$

as Laurent series description in terms of a complex variable, defined on a circle $z.=e^{ix}$:

$$u(z):=\tilde{u}(z):=u(x)=\sum_{-\infty}^{\infty}u_\nu z^\nu\in H:=L_2^*(\Gamma) \cdot$$

with $u_0:=\frac{a_0}{2}$, $u_\nu:=\frac{1}{2}(a_\nu-ib_\nu)$, $c_{-\nu}:=\frac{1}{2}(a_\nu+ib_\nu)$, $\nu>0$.

There is also an interpretation of Jacobi's Theta function as probability density of the position of a Brownian motion on the circle started at 0 at time 0 and run for time t ([3] Ph. Biane, et.al., see also [23] R. Penrose, 21.9, for general remarks about probability distribution in a wave function in the context of "quantum jumps").

The Poisson kernel is given by

$$P_r(\theta):=\sum_{-\infty}^{\infty}r^{|\nu|}e^{i\nu\theta}=\frac{1-r^2}{1-2r\cos\theta+r^2}=\mathrm{Re}(\frac{1+re^{i\theta}}{1+re^{i\theta}})\quad \text{for}\quad 0\le r<1 .$$

The corresponding Poisson integral

$$v(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)u(e^{i\varphi})d\varphi$$

gives the extension of a function $u(e^{i\varphi})$ defined on the unit circle to a harmonic function on the unit disk $v(re^{i\varphi}) = v(z)$.

There is a conformal map of the unit disk to the upper half-plane by means of certain bilinear, i.e. Möbius transform, i.e. for $f \in L_p(R)$ there is a harmonic extension into the upper half-plane given by

$$u(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_y(x-t)f(t)dt$$

with

$$P_y(x) := \frac{y}{x^2 + y^2}$$

and $\|u\|_{H_p} = \|f\|_{L_p}$ (see Hardy space below).

The singular integral operator $[u] := Au$, defined by

$$[u](x) := \frac{1}{2\pi}\oint \cot\frac{x-y}{2}u(y)dt = -\lim\frac{1}{2\pi}\int_\varepsilon^\pi [u(x+y) - u(x-y)]\cot\frac{y}{2}dy \quad \text{for } \varepsilon \to 0 \ ,$$

is being used in conformal mapping theory providing also a framework to construct Green functions. Its properties show the same as the Hilbert transform ([12] D. Gaier), i.e.

Lemma: For the integral operators $[u] := Au$ it holds

i) A is skew-symmetric in the space $L_2(0,2\pi)$ and $[u]' = [u']$.

ii) A maps the space $H := L_2(0,2\pi) - R$ isometric onto itself,

i.e. $\|Au\| = \|u\|$ and $A^2 = -I$ i.e. $(Au, v) = -(u, Av)$.

iii) For the Fourier coefficients it holds

$$[u]_\nu = -isign(\nu)u_\nu \qquad \text{i.e.} \qquad [u](x) = i\sum_1^\infty \left[u_{-\nu}e^{-i\nu x} - u_\nu e^{i\nu x}\right] \in L_2 \ .$$

From ([20] N.I. Muskhelishvili §18,19, we recall another property of A, that is, that A is a bijective mapping of the space $C^{0.\lambda}$ $(0 < \lambda < 1)$ onto itself, whereby $C^{0.\lambda}$ denotes the space of periodic, continuous functions, fulfilling a Hölder condition according to the exponent λ modulo R.

3. Hardy Spaces

Let $u(z)$ a regular, analytical function on the open disk $D := \{z \| z| < 1\}$. Due to a result from Hardy the mean function

$$\mu_\delta(r) := \frac{1}{2\pi}\int_0^{2\pi} \left|u(re^{i\varphi})\right|^\delta d\varphi \quad , \quad \delta > 0$$

is increasing, i.e. it's either divergent or is bounded, as $r \to 1$. In case it's bounded we write

$$\tilde{u}(e^{i\varphi}) := \lim u(re^{i\varphi}) \; .$$

The Hardy space $H_2(D)$ consists of those functions, whose mean square value on the circle of radius remains bounded as $r \to 1$. For $L_2^*(\Gamma)$ and its closed vector subspace $H_2^*(\Gamma) \subset L_2^*(\Gamma)$, the following characterization holds true

$$u \in H_2^*(\Gamma) \quad \text{if and only if} \quad u_\nu = 0 \text{ for } \nu < 0 \quad .$$

Supposing that $\tilde{u} \in H_2^*(\Gamma)$, i.e. that $\tilde{u}$ has Fourier coefficients with $\tilde{u}_\nu = 0$ for $\nu < 0$, then the element u of the Hardy space associated to $\tilde{u}$ is the holomorphic function

$$u(z) = \sum_0^\infty u_\nu z^\nu \quad , \; |z| < 1 \; .$$

When $1 \le p < \infty$ the real Hardy spaces $H_p(D)$ are easy to describe: A real function f on the unit circle belongs to the real Hardy space $H_p(\Gamma)$ if it is the real part of a function in $H_p(\Gamma)$, and a complex function f belongs to the real Hardy space if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ belong to the space.

For $p < 1$, such tools as Fourier coefficients, Poisson integral, conjugate function, are no longer valid, as the following example shows

$$F(z) := \frac{1+z}{1-z} \text{ for } |z| < 1$$

for which

$$f(e^{i\theta}) = \tilde{F}(e^{i\theta}) = i\cot(\frac{\theta}{2}) \; .$$

The function $F(z)$ is in H_p for every $p < 1$, the radial limit f is in $H_p(\Gamma)$, but its real part $\mathrm{Re}(f)$ is 0 almost everywhere. It is no longer possible to recover $F(z)$ from $\mathrm{Re}(f)$, and one cannot define real-$H_p(\Gamma)$ in the simple way above.

For the same function $F(z)$, let $f_r\,(e^{i\theta}) := \tilde{F}(re^{i\theta})$ The limit when $r \to 1$ of $\mathrm{Re}(f_r)$, in the sense of distributions on the circle, is a non-zero multiple of the Dirac distribution at $Z = 1$. The Dirac distribution at any point of the unit circle belongs to real-$H_p(\Gamma)$ for every $p < 1$.

4. Wavelet Transform

A wavelet transform is similar as a Fourier transform, which delivers the frequency spectrum of a timely signal $\varphi(t)$ without any loss of information, although the Fourier transform itself gives the frequencies without any information about the points in time, when the frequencies occur. The wavelet transform delivers this sort of information in a better distinguishing form: one gets both the frequency analysis and the points in time, when those frequencies happen, which reminds to [32] M. du Sautoy's words about "the primes have music in them", that the wavelet transform describes the music of an orchestra on a 2-dimensional paper.

A wavelet ([36] R.S. Strichartz, 7.8) is a function $\psi(x) \in L_2(R)$ with a Fourier transform which fulfills

$$0 < c_\psi := 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \ .$$

Classical Hilbert spaces in complex analysis are examples of wavelets, like Hardy space of L_2 functions on the unit circle with analytical continuation inside the unit disk.

The wavelet transform of a function $f(x) \in L_2(R)$ with the wavelet $\psi(x) \in L_2(R)$ is the function

$$W_\psi[f](a,b) := \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty} f(t)\overline{\psi}_{b,a}(t)dt = \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{a}}\overline{\psi}(\frac{t-b}{a})dt\,, \quad a \in R - \{0\}, b \in R$$

For a wavelet $\psi(x) \in L_1(R)$ its Fourier transform is continuous and fulfills

$$0 = \hat{\psi}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t) d\omega$$

The wavelet transform to the wavelet $\psi(x) \in L_2(R)$

$$W_\psi : L_2(R) \to L_2(R^2, \frac{dadb}{a^2}),$$

is isometric and for the adjoint operator

$$W_\psi^* : L_2(R^2, \frac{dadb}{a^2}) \to L_2(R)$$

$$W_\psi^*[g](a,b) := \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t)\frac{1}{\sqrt{a}}\psi(\frac{t-b}{a})g(a,b)\frac{dadb}{a^2}$$

it holds $W_\psi^* W_\psi = Id$ and $W_\psi W_\psi^* = P_{range(W_\psi)}$.

The continuous wavelet transform is known in pure mathematics as Calderón's reproducing formula, i.e. for $\psi(x) \in L_1(R^n)$ real and radial with vanishing mean, i.e.

$$\int_0^\infty \frac{|\hat{\psi}(a\omega)|^2}{a} da \equiv 1 \ .$$

For

$$\psi_a(x) := \frac{1}{a^n} \psi(\frac{x}{a})$$

it holds Calderón's formula

$$f = \int_0^\infty \psi_a * \psi_a * f \frac{da}{a} \ .$$

We mention $e^{2\pi i} = 1$, $e^{\pi i} + 1 = 0$, $i = e^{\pi i/2}$ and $i^i = e^{i \log i} = e^{i(\pi i/2)} = e^{-\pi/2} = 0.207879576..$

5. Zeta function as a Transform of a Self-adjoint Operator

The Riemann function $\Xi(t)$ is an even entire function ([11] H.M. Edwards, 1.8, [36] E. C. Titchmarsh, 2.15, 10.2) with integral representations

$$\Xi(t) := \xi(1/2 + it) := \int_{-\infty}^\infty \Phi(u) e^{itu} du = 2\int_0^\infty \Phi(u) \cos(ut) du = 4\int_1^\infty \frac{d}{dx}\left[x^{3/2}\psi'(x)\right] x^{-1/4} \cos(\frac{t}{2} \log x) dx \ ,$$

with

$$\Phi(\log u) := 2\sqrt{u} \sum_1^\infty (2\pi^2 n^4 u^4 - 3\pi n^2 u^2) e^{-\pi n^2 u^2} = \sqrt{x} H(x) > 0 \ .$$

The Riemann hypothesis is equivalent to the statement that all zeros of $\Xi(t)$ are real.

In [26] G. Polya investigated the fake function

$$\Xi^*(t) := 2\int_0^\infty \Phi^*(u) \cos(ut) du$$

with
$$\Phi^*(\log u) := 8\pi^2 \cosh(\frac{9}{2}u) e^{-2\pi \cosh(2u)} \ ,$$

which shows similar asymptotic behavior as $\Phi(\log u)$ and proved that all zeros of the fake function $\Xi^*(t)$ lie on the critical line.

From Polya we recall his comments to the strong requirements for the function $\varphi(t) := \varphi(\log u)$:

"Zur Erläuterung sei beigefügt, dass von der Funktion $\varphi(t)$ ziemlich viel gefordert wird, nämlich nicht bloss, dass sie die Realität der Nullstellen von einem oder einigen Integralen von der Form

$$\int_{-\infty}^{\infty} \varphi(t) F(t) e^{ist} dt$$

erhält, sondern von überhaupt allen, die nur reelle Nullstellen haben. Eben weil diese Forderung so stark ist, kann die Natur von $\varphi(t)$ so scharf präzisiert werden. Der Begriff „vom Geschlecht 1" ist so aufzufassen, dass die ganzen Funktionen vom Geschlecht 0, die ganzen rationalen Funktionen und die Konstanten als spezielle Funktionen vom Geschlecht 2 gelten. Die notwendige und hinreichende Bedingung, die für $\varphi(t)$ angegeben wird, besteht eigentlich darin, dass $\varphi(it)$ ein Polynom mit nur reellen Nullstellen oder ein Grenzwert solcher Polynome sei.

Remark: Polya's lemma, being applied to

(3.4) $$F(u) := u^{-1/2} e^{-x(u^2+u^{-2})} \quad \text{i.e.} \quad \int_0^{\infty} u^{1-s} F(u) \frac{du}{u} = \int_0^{\infty} u^{-1/2-s} e^{-x(u^2+u^{-2})} dt \ ,$$

shows that all zeros of the fake function

(3.5) $$\Xi^*(z) = 2\int_0^{\infty} \Phi^*(u) \cos(uz) du = 16\pi^2 \int_0^{\infty} \cosh(\frac{9}{2}u) e^{-2\pi \cosh(2u)} \cos(uz) du$$

lie on the critical line, as this is the case for all zeros of the Bessel function

(5.1) $$K_x(z) := \int_0^{\infty} \cosh(uz) e^{-x \cosh u} du \quad \text{for } x > 0 \ . \bullet$$

In [37] E. C. Titchmarsh, 2.15 are cases evaluated for integrals in the form

$$\Phi(u) = \int_0^{\infty} \varphi(t) \Xi(t) \cos(ut) dt \ , \quad \text{i.e.} \quad \Xi(t) = \int_0^{\infty} \varphi(u) \Phi(u) \cos(ut) du$$

which includes the examples from above as special case of:

Lemma 3.1: If $f(t)$ has a representation in the form $f(t) = |\varphi(it)| = \varphi(it)\varphi(-it)$ with φ analytic, it holds for the even function $\Xi(z) := \xi(\frac{1}{2} + iz) = \Xi(-z)$

$$f(t)\Xi(t) = \int_0^{\infty} \Phi(x) \cos(xt) dx \quad \text{resp.} \quad \Phi(x) = \int_0^{\infty} f(t) \Xi(t) \cos(xt) dt$$

with $$\Phi(x) = \frac{1}{2i\sqrt{y}} \int_{1/2-i\infty}^{1/2+i\infty} \varphi(s-\frac{1}{2}) \varphi(\frac{1}{2}-s) \xi(s) y^s ds \ , \ y = e^x \ .$$

Lemma 3.2 ([6] D. Bump, et. al.): An operator which takes an even function $q(\nu)$ and replaces it by $\frac{q(\nu+1)-q(\nu-1)}{\nu}$ has the property of moving the zeros of a function closer on the imaginary axis, and so an eigenfunction of this operator should have its zeros on the imaginary axis.

Lemma 3.3: ([6] D. Bump, et. al.): Let $q(s)$ be a polynomial, and assume that the zeros of $q(s)$ lie in the closed stripe $\{\mathrm{Re}(s) \in [-c,c]\}$ with $c>0$. Then if $a>0$, the zeros of

$$r(s)=(s+a)q(s+2)-(s-a)q(s-2)$$

lie in the open strip $\{\mathrm{Re}(s) \in (-c,c)\}$.

Lemma 3.4: (Jordan's lemma) If $m>0$ and P/Q is the quotient of two polynomials such that $\deg ree Q \geq 1+\deg ree(P)$, then

$$\lim \int_{C_\rho^+} \frac{P(z)}{Q(z)})e^{imz} dz = 0 \quad \text{for } \rho \to \infty \ ,$$

where C_ρ^+ is the upper half-circle with radius ρ .

Jordan's lemma might be useful to link Polya's "polynomial" ideas with our ideas from the paper "Braun, A Bessel polynomial argument to prove the Riemann Hypothesis" with the "vanishing constant Fourier term " requirements.

6. Analytical "Dual" Functions and Ramanujan's Main theorem

One relation of Ramanujan's Master Theorem to the Zeta function is the non self-adjoint integral representation

$$\frac{\pi}{\sin \pi s}\zeta(s)=\int_0^\infty \left[\log x+\gamma+\sum_0^\infty \zeta(n+1)(-x)^n \right] x^{1-s}\frac{dx}{x} \quad , \quad 0<\mathrm{Re}(s)<1 \ .$$

With $\varphi(k):=\frac{1}{\zeta(2k+1)}$ the Hardy/Littlewood resp. the Riesz equivalence criteria of the Riemann Hypothesis are

RH holds if and only if $F(x)=\sum_0^\infty \frac{\varphi(k)}{k!}(-x)^k = O(x^{-1/4})$.

RH holds if and only if $\sum_1^\infty \frac{(-1)^{k+1}}{(k-1)!\zeta(2k)} x^k = O(x^{1/4+\varepsilon})$.

We recall Ramanujan's Master Theorem ([1] B. C. Berndt, The first quarterly report, 1.2 Theorem I (Ramanujan's Master Theorem) in the form

$$\int_0^\infty F(x)x^{s-1}dx=\Gamma(s)\varphi(-s) \quad \text{for } F(x)=\sum_0^\infty \frac{\varphi(k)}{k!}(-x)^k \text{ in the neighborhood of } x=0 .$$

The motivation of Ramanujan for this formula is given in [1] B. C. Berndt, section 9, chapter 4, Entry 8), which we recall below. To the author's understanding in the context of this manuscript this is about a building principle for analytical duals for analytical function.

"<u>Statement</u>: If two functions of x be equal, then a general theorem can be formed by simply writing $\varphi(n)$ instead of x^n in the original theorem

<u>Solution</u>: "Put $x=1$ and multiply it by $f(0)$ then change x to x, x^2, x^3, x^4.... and multiply $\frac{f'(0)}{1!}, \frac{f''(0)}{2!}, \frac{f'''(0)}{3!}, \ldots$ respectively and add up all the results. Then instead of x^n we have $f(x^n)$ for positive as well as for negative values of n . Changing $f(x^n)$ to $\varphi(n)$ we can get the result."

He underline this statement with the following example. Of course, this formal building procedure is fraught with numerous difficulties.

$$\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2}$$

$$f(0)[\arctan 1+\arctan 1]=\frac{\pi}{2}f(0) \text{ , } \frac{f'(0)}{1!}\left[\arctan x+\arctan\frac{1}{x}\right]=\frac{f'(0)}{1!}\frac{\pi}{2} \text{ , } \frac{f''(0)}{2!}\left[\arctan x+\arctan\frac{1}{x}\right]=\frac{f''(0)}{2!}\frac{\pi}{2}$$

Replace $\arctan z$ by its Maclaurin series in z , where z is any integral power of x . Now add all the equalities above. On the left side one obtains two double series. Invert the order of summation in each double series to find that

$$\sum_0^\infty (-1)^n \frac{f(x^{2n+1})+f(x^{-2n-1})}{2n+1}=\frac{\pi}{2}f(1) \text{ .}$$

Replace $f(x^n)$ by $\varphi(n)$ to conclude that

$$\sum_0^\infty (-1)^n \frac{\varphi(2n+1)+\varphi(-2n-1)}{2n+1}=\frac{\pi}{2}\varphi(0) \text{ .}$$

Following the same process Ramanujan concluded

$$\frac{x}{1+x}+\frac{1/x}{1+1/x}=1 \quad \rightarrow \quad \sum_1^\infty (-1)^n\varphi(n)+\varphi(-n)=\varphi(0)$$

$$\log(1+x)-\log(1+\frac{1}{x})=\log x \quad \rightarrow \quad \sum_1^\infty (-1)^n \frac{\varphi(n)-\varphi(-n)}{n}=\varphi'(0)$$

7. Some Links to physical Models

Remark A

The simplest version of the harmonic oscillator is the Hamiltonian system with Hamiltonian

$$H(p,q)=\frac{1}{2}(p^2+\omega^2 q^2) \quad \text{and} \quad \dot{q}=p\,, \quad \dot{p}=-\omega q\,, \ddot{q}=-\omega^2 q$$

Identifying $R^2 \cong C$ by putting $z=p+i\omega q$ a solution to $H(p,q)=\frac{1}{2}|z|^2$ is given in the form

$$z(t)=Ce^{i\omega t} \quad .$$

The Hermite polynomials are used to model the energy states of the harmonic quantum oscillator. The distribution functions from lemma 0.1 might be provide an appropriate model for the zero point energy in the same Hilbert space framework

$$H := span\left[H(\varphi_n(x))\right]$$

but using alternatively the eigenpairs $(\lambda_n^*, H\varphi_n(x))$ to model the energy states, which might overcome current inconsistencies between the Casimir effect (i.e. existing radiation at absolute zero point of the temperature) and the calculated infinite energy density from the harmonic quantum operator model.

The Maxwell equations follow the U(1)-symmetry. It basically says that a photon is symmetric to itself. U(1) is diffeomorph to the unit circle, consisting of all complex numbers with absolute value 1 under the multiplication operation. U(1) is the rotation group in the (q,p)-plane, which plays a key role for the quantum harmonic oscillator. The invariance of U(1) gives the root cause of the existence of the Leiter (creation or annihilation) operators:

$$H_{klassig}=\frac{1}{2}(p-ix)(p+ix)=a^* a$$

$$H_{quantum}=\frac{1}{2}\left(P^2+Q^2\right)=A^*A+\frac{1}{2}\vec{1} \quad , \quad H_{quantum}=\frac{1}{2}\left(P^2+Q^2\right)=AA^*-\frac{1}{2}\vec{1} \ .$$

A reference to lemma 3.3 is given in [36] R.S. Strichartz, 7.6, by the following

Lemma A.1: Suppose φ is an eigenfunction of H with eigenvalue λ. Then $A^*\varphi$ is an eigenfunction with eigenvalue $\lambda+2$, and $A\varphi$ is an eigenfunction (as long as $\lambda \neq 0$) with eigenvalue $\lambda-2$.

We mention only one link of “Hamiltonian function” to (complementary) variational principles refering to [39] W. Velte, 6.2.4, which is called the method of Noble.

Let $(E,\langle,\rangle)$ and $(E',(,))$ be Hilbert spaces and $T : E \to E'$, $T^* : E' \to E$ linear operators fulfilling $(u',Tu) = \langle T^* u',u \rangle$ and let $W : E'xE \to R$ a functional fulfilling

$$T = \frac{\partial W(u',)}{\partial u'} \quad \text{and} \quad T^* = \frac{\partial W(.,u)}{\partial u}$$

i.e. the operators T and T^* are deviations from $W(.,.)$ in the sense of Gateaux, i.e.

$$\lim \frac{F(u+tv)-F(v)}{t} = F_u(v) \text{ for all } v \in E \ .$$

Putting $W(u',u) := \frac{1}{2}(u',u') - F(u)$ the minimization problem

$$(*) \qquad J(u) := (Tu,Tu) + 2F(u) \to \min u \in U \subset E$$

leads to $Tu = u'$ and $(T^* u',.) = -F_u(.)$ and therefore to

Lemma A.2 (method of Noble): If $F(.)$ is a convex functional it follows that $W(u',u)$ is convex concerning u' and concave concerning u . The minimization problem (*) is equivalent to the variational equation

$$(v',T\varphi) + F_u(\varphi) = 0 \text{ for all } \varphi \in U \quad \text{resp.} \quad (T^* v',\varphi) = -F_u(\varphi) \text{ for all } \varphi \in U \ .$$

i.e. there is a characterization of the solution of (*) as a saddle point.

With lorentz-invariant field theory one tries to derivate field equation out of variation principle. Then Noether‘s theorem is valid, which ensure conservation laws. For Einstein’s vacuum equation in the gauges this leads to expressions in the form

$$W_{metric} = \frac{1}{2\kappa} \int R\sqrt{-g}\, d^4x$$

described by the choice of spherical wave coordinates x^α

$$\square_g = \frac{1}{\sqrt{|g|}} \partial_\mu g^{\mu\nu} \sqrt{|g|} \partial_\nu x^\alpha .$$

The gravitation field theory says, that the action of

$$W[g,\Phi] = W_{metric}[g] + W_{matter}[g,\Phi]$$

is stationary for all physical fields under all variations of the metric, which vanishes outside of bounded domains.

Remark B

Planck's black body spectral specific radiation function

$$\frac{dR(\lambda,T)}{d\lambda}=\frac{c_1}{\lambda^5}\frac{1}{e^{c_2/\lambda T}-1}=\frac{c_1}{\lambda^5}\sum_1^\infty e^{-nc_2/\lambda T}$$

with $c_1=2\pi hc^2$ and $c_2=hc/k$ is related to the Müntz formula

$$\zeta(s)\Gamma(s)=\int_0^\infty x^s\left[R(x)-\frac{1}{x}\int_0^\infty e^{-t}dt\right]\frac{dx}{x} \qquad \text{for} \quad 0<\operatorname{Re}(s)<1 \ .$$

This relation is given by

$$\zeta(s)\Gamma(s)=\int_0^{+\infty}\frac{x^s}{e^x-1}\frac{dx}{x} \quad \text{with} \quad \frac{1}{e^x-1}=\sum_1^\infty e^{-nx}$$

and

$$\frac{\pi^4}{90}=\zeta(4)\Gamma(4)=\int_0^{+\infty}x^4(\sum_1^\infty e^{-nx})\frac{dx}{x}=\int_0^{+\infty}x^{-4}(\sum_1^\infty e^{-\frac{n}{x}})\frac{dx}{x} \ ,$$

which gives the total radiation density and its corresponding spectral representation

$$g(x)dx=\frac{x^{-4}}{e^{1/x}-1}\frac{dx}{x}=\frac{x^4}{e^x-1}\frac{dx}{x}=g(\frac{1}{x})dx \ .$$

The weak counterpart (2.17) to (0.12), i.e. applying Müntz formula (lemma 2.1) using (0.8) alternatively to $\omega(x)=e^{-x}$ might provide an alternative model to (0.13),(0.14).

Remark C

From lemma 1.1 it follows, that the skew-symmetric bilinear form $\beta: HxH \to R$, defined by

$$\beta(u,v):=(Hu,v)=-(u,Hv) \ ,$$

is nondegenerated (or symplectic), as the associated dual map is bijective. The pair (H^*,β) is called a symplectic vector space with the symplectic structure β.

Hamiltonian mechanics and corresponding quantum field theory can be formulated in a geometric, coordinate invariant manner on a general class of manifolds, which are the symplectic manifolds ([23] R. Penrose, 13.10, 14.8, 20.4). Its relation to Pseudo differential operators are given in ([24] B. E. Petersen, 3.6).The two main classes of examples of symplectic manifolds are cotangent bundles and Kähler manifolds. For more detailed views, also in relation to gravitation theory we refer to the specific literature; a first google search by names like "G. Segal", "da Silva", R. Bryant" might be helpful.

Remark D (Hyperbolic Geometry)

A non euklidean geometry, also called Lobachevsky-Bolyai-Gauss Geometry, has constant sectional curvature -1. This geometry satisfies all of Euklid's postulates except the "parallel postulate". In hyperbolic geometry, the sum of angles of a triangle less than 180°, and triangles with the same angles have the same areas. Furthermore, not all triangles have the same angle sum. The best-known example of a hyperbolic space are spheres in Lorentzian 4-space. The Poincaré Hyperbolic Disk is a hyperbolic 2-space. Hyperbolic geometry is well understood in 2-D, but not in 3-D.

Felix Klein constructed an analytic hyperbolic geometry in 1870 in which a point is represented by a pair of real number (x_1, x_2) with

$$x_1^2 + x_2^2 < 1$$

i.e., points of an open disk in the complex plane, and the distance between two points is given by

$$d(x,y) := a * \operatorname{arccos}h\left[\frac{1 - x_1 y_1 - x_2 y_2}{\sqrt{1 - x_1^2 - x_2^2}\sqrt{1 - y_1^2 - y_2^2}}\right] .$$

The geometry generated by this formula satisfies all of Euklid's postulates except the fifth. The metric of this geometry is given by the Cayley-Klein-Hilbert metric,

$$g_{11} = \frac{a^2(1 - x_2^2)}{(1 - x_1^2 - x_2^2)^2}, \quad g_{12} = \frac{a^2 x_1 x_2}{(1 - x_1^2 - x_2^2)^2}, \quad g_{22} = \frac{a^2(1 - x_1^2)}{(1 - x_1^2 - x_2^2)^2} .$$

The Klein-Beltrami model of Hyperbolic Geometry consists of an Open Disk in the Euclidean plane whose open chords correspond to hyperbolic lines. Two lines l and m are then considered parallel if their chords fail to intersect and are perpendicular under the following conditions,

1.If at least one of l and m is a diameter of the disk, they are hyperbolically perpendicular iff they are perpendicular in the Euclidean sense.

2.If neither is a diameter, l is perpendicular to m iff m the Euclidean line extending l passes through the pole of m (defined as the point of intersection of the tangents to the disk at the „endpoints" of m).

There is an isomorphism between the Poincare hyperbolic model and the Klein-Beltrami model. Consider a Klein disk in Euclidean 3-space with a sphere of the same radius seated atop it, tangent at the origin. If we now project chords on the disk orthogonally upward onto the sphere's lower hemisphere, they become arcs of circles orthogonal to the equator. If we then stereographically project the sphere's lower hemisphere back onto the plane of the Klein disk from the north pole, the equator will map onto a disk somewhat larger than the Klein disk, and the chords of the original Klein disk will now be arcs of Circles orthogonal to this larger disk. That is, they will be Poincaré lines. Now we can say that two Klein lines or angles are congruent Iff their corresponding Poincaré lines and angles under this isomorphism are congruent in the sense of the Poincaré model.

8. Equivalent criteria to the Riemann Hypothesis

We sketch a possible link to the Gauss' Li-function and the Riemann's function, which approximate the number of primes less than a given value $x>2$:

Let $\Lambda(n)$ denote the weight assigned to the integer n by the measure $d\psi$, that is $\Lambda(n)$ is zero unless n is a prime power, in which case $\Lambda(n)$ is the log of the prime of which n is a power. It holds ([11] H.M. Edwards, 3.1ff)

$$-\frac{\zeta'(s)}{\zeta(s)}=s\int_0^\infty x^{-s}\psi(x)\frac{dx}{x}=\int_0^\infty x^{-s}d\psi(x)$$

with $\psi(x):=\sum_{n<x}\Lambda(n)=\sum_{p^n<x}\log p$ and $d\psi=(\log x)dJ$.

For Gauss' Li-function

$$Li(x):=\int_2^x\frac{dt}{\log t}$$

an equivalence criteria of the Riemann Hypothesis is given by

(*) RH holds if and only if $\pi(x)=Li(x)+O(x^{1/2}\log x)$.

Two equivalence conditions to (3.3) are given by

for $\varepsilon>0$ it holds either $R_1(x):=\pi(x)-Li(x)=O(x^{1/2+\varepsilon})$ or $R_2(x):=\psi(x)-x=O(x^{1/2+\varepsilon})$.

The standard "measure" in current Zeta function theory is

$$\sigma(x):=Ei(-x)=-\int_x^\infty e^{-t}\frac{dt}{t} \quad \text{with} \quad d\sigma(x)=\frac{e^{-x}dx}{x} \quad \text{resp.} \quad d\sigma(\log x)=\frac{dx}{x\log x} ,$$

enabling a representation of Gauss' Li-function in the form

(**) $$Li(x)=Ei(\log x):=-\int_{-\log x}^\infty\frac{e^{-t}}{\mathrm{t}}dt=\int_0^x td\sigma(\log t)=O(\frac{x}{\ln x}) \quad , \quad x>1.$$

(2.13) suggests as a first attempt a distribution valued Li^*-function analogue to (**) in the form

(3.6) $$Li^*(x)=Ei^*(\log x):\approx c\int_{-\log x^2}^\infty\frac{H(f(t))}{\mathrm{t}^2}\frac{dt}{\mathrm{t}} \quad , \quad x>1 \quad .$$

$Li^*(x)$ then might enable an alternative criteria to (*), which can now be verified, as the jeopardizing singularity is somehow under better control within the distribution concept.

The equivalence of the "$R_1(x)$ – criteria" to the RH is using Riemann's famous estimate, based on his analysis of the expression ([11] H.M. Edwards, 1.13ff)

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \log\zeta(s) x^s \frac{ds}{s} = -\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\log\zeta(s)}{s}\right] x^s ds \qquad \text{for } a > 1.$$

Riemann first applied an integration by part step to get to the second integral in order to guarantee the convergence of integrals, when applying the relations

$$\log\zeta(s) = \log\xi(s) - \log\Pi(s/2) - s\log\pi/2 - \log(s-1)$$

and

$$\log\xi(s) = \log\xi(0) + \sum_\rho \log(1 - s/\rho) \ .$$

Basically Riemann applied ([11] H.M. Edwards 1.14 ff.) the following lemma to each of the resulting integrals of (3.7) to get to

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \log\zeta(s) x^s \frac{ds}{s} =: J(x) = Li(x) - \sum_{\mathrm{Im}(\rho)>0} Li(x^\rho) + Li(x^{1-\rho}) + \sum_n \int_x^\infty \frac{t^{-2n-1}}{\log t} dt + \log\xi(0)$$

and his famous estimate

$$\pi(x) - R(x) \approx \int_x^\infty \frac{d\sigma(\log t)}{t^2-1} = \int_x^\infty \frac{dt}{t(t^2-1)\log t} = \int_x^\infty \sum_1^\infty t^{-2n} \frac{d\log t}{\log t} \ .$$

***Lemma**: For* $H(\beta) := \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{1}{s}\log(1-\frac{s}{\beta})\right] x^s ds$ *with* $\beta \in C$

it holds

$$H(\beta) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{s}\log(1-\frac{s}{\beta}) x^s ds = \begin{cases} -\int_x^\infty t^\beta d\sigma(\log t) & \text{.... } for\ \mathrm{Re}(\beta) < 0 \\ \int_o^x t^\beta d\sigma(\log t) & \text{..... } for\ \mathrm{Re}(\beta) > 0 \end{cases} .$$

The asymptotic convergence behavior at infinity of the term

$$\sum_{\mathrm{Im}(\rho)>0} Li(x^\rho) + Li(x^{1-\rho})$$

jeopardizes current attempts to verify the RH. The above a weak representation in the form

$$\log\zeta(s) = \log\xi^*(s) - \log\tilde{\Pi}^*(s) - \log(s-1)$$

might overcome this asymptotic convergence issues.

The equivalence of the “$R_2(x)-$criteria” to the RH is basically shown in that way, that $-\frac{\zeta'(s)}{\zeta(s)}-\frac{1}{1-s}$ is holomorphic for $\mathrm{Re}(s)>1/2$, which is derived by the relation ($\mathrm{Re}(s)>1$)

$$-\frac{\zeta'(s)}{\zeta(s)}=\sum_n\frac{\Lambda(n)}{n^s}=\int_1^\infty x^{-s}d\psi(x)=\frac{\psi(x)}{x^s}\Big|_1^\infty+s\int_1^\infty x^{-1-s}\psi(x)dx=-\frac{1}{s-1}-1+s\int_1^\infty x^{-1-s}R_2(x)dx\ ,$$

where the last integral is compact convergent for $R(s)>1/2$ and therefore holomorphic.

References

[1] B. C. Berndt, Ramanujan´s Notebooks, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985

[2] M. V. Berry, J. P. Keating, ," $H = xp$ and the Riemann zeros", in Supersymmetry and Trace Formulae: Chaos and Disorder (Ed. I.V. Lerner, J.P. Keating, D.E.Khmelnitski), Kluver, New York (1999) pp. 355–367

[3] Ph. Biane, J. Pitman, M. Yor, Probability laws related to the Jacobi Theta and Riemann Zeta functions, and Brownian excursions, Bull. Amer. Math. Soc, 38, 4 (2001) pp. 435-465

[4] E. Bombieri, Remarks on Weil's quadratic functional in the theory of prime numbers, I, Rend. Mat. Acc. Linceri, s. 9 11 (2000) pp. 183-233

[5] R. P. Brent, An asymptotic expansion inspired by Ramanujan, Appeared as technical report CMA-MR02-93/SMS-1-93, CMA, ANU, February 1993

[6] D. Bump, K.-K. Choi, P. Kurlberg, J. Vaaler, A Local Riemann Hypothesis I , Math. Zeit. 233 (2000) pp. 1-19

[7] D. A. Cardon, Convolution Operators and zeros of entire functions, Proc. Amer. Math. Soc. 130, 6 (2002) pp. 1725-1734

[8] D. A. Cardon, Fourier transforms having only real zeros, Proc. Amer. Math. Soc. 133 (2005) pp. 1349-1356

[9] G. Doetsch, Das Euler Prinzip, Randwertprobleme der Wärmeleitungstheorie und physikalische Deutung der Integralgleichung der Thetafunktion, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 2 serie, tome 2, no 3 (1933) pp. 325-342

[10] R.J. Duffin, H.F. Weinberger, Dualizing the Poisson summation formula, Proc. Natl. Acad. Sci. 88 (1991) pp. 7348-7350

[11] H. M. Edwards, Riemann´s Zeta Function, Dover Publications, Inc., Mineola, New York, 1974

[12] D. Gaier, Konstruktive Methoden der konformalen Abbildung, Springer-Verlag, Berlin, 1964

[13] G. Gasper, Using sums of squares to prove that certain entire functions have only real zeros, in *Fourier Analysis: Analytic and Geometric Aspects*, W. O. Bray, P. S. Milojevic and C. V. Stanojevic, eds., Marcel Dekker, Inc. (1994) pp. 171-186

[14] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals Series and Products, Fourth Edition, Academic Press, New York, San Francisco, London, 1965

[15] a. Haar, Zur Theorie der orthogonalen Funktionensysteme, Math. Ann. 69 (1910) pp. 331-371

[16] S.L. Hahn, Hilbert transforms in signal processing, Artech House, Inc., Boston, 1996

[17] H. Hamburger, Über einige Beziehungen, die mit der Funktionalgleichung der Riemannschen ζ -Funktion äquivalent sind, Math. Ann. 85 (1922) pp. 129-140

[18] J.M. Hill, On some integrals involving functions such that $\Phi(\frac{1}{x}) = \sqrt{x}\Phi(x)$, J. Math. Appl. 309 (2005) pp. 256-270

[19] Li, X.-J., The Positivity of a Sequence of Numbers and the Riemann Hypothesis, J. Number Th. 65 (1997) pp. 325-333

[20] H. Mellin, Abriss einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen, Math. Ann. 68 (1910) pp. 305-347

[21] N.I. Muskhelishvili, Singular Integral Equations, P.Noordhoff N.V., Groningen, 1953

[22] N. Nielsen, Handbuch der Theorie der Gammafunktion, Leipzig, B.G. Teubner Verlag, 1906

[23] R. Penrose, The Road to Reality, Alfred A. Knopf, New York, 2005

[24] B. E. Petersen, Introduction to the Fourier Transform & Pseudo-differential Operators, Pitman Publishing Limited, Boston, London, Melbourne

[25] G. Polya, On the zeros of an integral function represented by Fourier's integral, Messenger of Math. 52 (1923), pp. 185-188

[26] G. Polyá, Bemerkungen über die Integraldarstellung der Riemannschen $\zeta-$Funktion, Act. Math. 48 (1926) pp. 305-317

[27] G. Polya, Ueber trigonometrische Integrale mit nur reellen Nullstellen, J. rein. and angew. Math. 158 (1927) pp. 6-18

[28] G. Polya, Ueber die Nullstellen gewisser ganzer Funktionen, Math. Zeit. 2 (1918), pp. 352-383, also Collected Papers, Vol II, 166-197

[29] B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsberichte der Berliner Akademie (1859) pp. 671-680

[30] M. Riesz, Sur l'hypothèse de Riemann, Acta Mathematica, 40 (1916) pp.185-90

[31] P. Sarnak, Spectra of hyperbolic surfaces, Bull. Amer. Math. Soc. 40, 4 (2003) pp. 441-478

[32] M. du Sautoy, Why an unsolved Problem in Mathematical Matters, Fourth Estate, London, 2000

[33] E. Schroedinger, Space-Time Structure, Press Syndicate of University of Cambridge, Cambridge-New York, Melbourne, reprinted 1954, 1960, 1985

[34] da Silva, A. C., Lectures on Symplectic Geometry, published by Springer-Verlag as number 1764 of the series Lecture Notes in Mathematics, 2006

[35] E.M. Stein, Singular integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970

[36] R.S. Strichartz, A Guide to Distribution Theory and Fourier Transforms, World Scientific, New Jersey,, London, Singapore, Hongkong, reprinted 2008

[37] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Oxford University Press Inc., New York, first published 1951, Second Edition 1986

[38] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, Second Edition first published 1944, reprinted 1996, 2003, 2004, 2006

[39] W. Velte, Direkte Methoden der Variationsrechnung, Teubner Studienbücher, 1976

[40] A. Zygmund, Trigonometric series, Vol. I, Cambridge University Press, 1968