Elements of Gamma, Bessel and Zeta function theory

1. Definitions

The reference to the below are modified Bessel-Hankel functions

$$\Psi_\nu(x) := \arctan\left[Y_\nu(x) / J_\nu(x)\right]$$

and

$$\varphi_{2\nu}(2x) := 2\pi\left[J_\nu^2(x) + Y_\nu^2(x)\right] = 2\pi\left|H_\nu^{(1)}(x)\right|^2 \ ,$$

which are linked by the relation

$$\frac{1}{4}\varphi_{2\nu}(2x)d\Psi_\nu = \frac{dx}{x} \quad .$$

The functions are used to define and investigate the function

$$\Phi_{2\nu}(x) := \frac{1}{\sqrt{x}}\varphi_{2\nu}(2x)\varphi_{2\nu}(\frac{2}{x}) = \frac{1}{x}\Phi_{2\nu}(\frac{1}{x})$$

for $-1 < \mathrm{Re}(2\nu) < 1$ and $\mathrm{Re}(s \pm 2\nu) > 0$ with its Mellin transform

$$\frac{\pi}{2}\int_0^\infty x^{1/2-s}\Phi_{2\nu}(x)d\Psi_{2\nu} = 2^s \frac{\cos\frac{\pi}{2}2\nu}{\cos\frac{\pi}{2}s}\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+\nu)\Gamma(\frac{s}{2}-\nu) \cdot$$

2. Gamma function theory

An equivalent formulation to Jacobi's theta function for $\psi(x)=\sum_{1}^{+\infty}e^{-\pi n^2 x}$ (see [12] H. Hamburger, [13] N. Nielsen, chapter 13, §71) is the series

(2.1) $$i\cot i\pi x = 1+2\sum_{1}^{\infty}e^{-2\pi n x} = \frac{1}{\pi x}+\frac{2x}{\pi}\sum_{1}^{\infty}\frac{1}{x^2+n^2}$$

It further holds (see [13] N. Nielsen, chapter 1,11,13, §4, 60, 68, 71, 73, [1] B. C. Berndt, chapter 6, example 2, chapter 8, entry 4/5/17(v))

(2.2) $$\pi\cot\pi x = \frac{\Gamma'(1-x)}{\Gamma(1-x)}-\frac{\Gamma'(x)}{\Gamma(x)} = \int_0^1 \frac{x^s-x^{1-s}}{1-x}\frac{dx}{x} \qquad \text{for } 0<\mathrm{Re}(s)<1$$

(2.3) and with $$\frac{\Gamma'(1+x)}{\Gamma(1+x)}+\gamma = \sum_{n\le x}\frac{1}{n} = \sum_{1}^{\infty}\frac{1}{n}-\frac{1}{n+x})$$

(2.4) $$\pi\cot\pi x = \sum_{1}^{\infty}\frac{1}{n-1+x}-\frac{1}{n-x}) = 2\pi\sum_{1}^{\infty}\sin 2\pi k x = \frac{1}{x}(1+2\sum_{1}^{\infty}\frac{x^2}{x^2-n^2}) \ .$$

The symmetric Beta function (see [13] N. Nielsen, chapter 11, §60, chapter X, §53)

(2.5) $$B(s,1-s)=\frac{\pi}{\sin\pi s} = \int_0^1\frac{x^s+x^{1-s}}{1+x}\frac{dx}{x} = \int_0^{\infty}\frac{x^s}{1+x}\frac{dx}{x} \qquad \text{for } 0<\mathrm{Re}(s)<1$$

with its relations to [25] E. C. Titchmarsh, 2.11)

(2.6) $$\frac{(1-s)\pi}{\sin\pi s} = \int_0^{\infty}\frac{x^s}{(1+x)^2}\frac{dx}{x} \qquad \text{for } 0<\mathrm{Re}(s)<1$$

plays a key role in the Mellin's inverse theorem, with its relation to Ramanujan series ([1] B. C. Berndt, chapter 4 ff.) in the form

(2.7) $$\frac{\pi}{\sin\pi s}\Phi(-s) = \int_0^1 x^s\left[\sum_{0}^{\infty}(-x)^n\Phi(n)\right]\frac{dx}{x}$$

and its link to the Zeta function (see [8] H.M. Edwards 10.10) representations (integration by parts $\frac{d}{dx}(x^{1-s})=(1-s)x^{-s}$)

$$\frac{(1-s)\pi}{\sin\pi s}\zeta(s) = \int_0^{\infty}\left[\frac{d^2}{dx^2}\frac{\Gamma'(1+x)}{\Gamma(1+x)}-\frac{1}{x}\right]x^{1-s}dx \quad , \quad 0<\mathrm{Re}(s)<1$$

$$\frac{\pi}{\sin\pi s}\zeta(s) = \int_0^{\infty}\frac{d}{dx}\left[\log x-\frac{\Gamma'(1+x)}{\Gamma(1+x)}\right]x^{1-s}\frac{dx}{x} \quad , \quad 0<\mathrm{Re}(s)<1$$

and its relation to Ramanujan's infinite series analysis in the form

(2.8) $$\frac{\pi}{\sin\pi s}\zeta(s)=\int_0^\infty\left[\log x+\gamma+\sum_0^\infty\zeta(n+1)(-x)^n\right]x^{1-s}\frac{dx}{x} \quad , \quad 0<\operatorname{Re}(s)<1 \ ,$$

leading to the functional equation in the form ([25] E. C. Titchmarsh, 2.9)

$$\frac{\pi}{\sin\frac{\pi}{2}s}\zeta(s)=(2\pi)^s\Gamma(1-s)\zeta(1-s) \ .$$

Remark From [13] N. Nielsen, chapter 13, 14 § 68, 78) we recall

$$\log\frac{1}{2}B(\frac{1-s}{2},\frac{s}{2})=\log\Gamma(\frac{1-s}{2})+\log\Gamma(\frac{s}{2})+\frac{1}{2}\int_0^\infty\log\sin\pi x dx-\int_0^1\log\Gamma(x)dx$$

$$\log\left[\frac{1}{2}B(\frac{1}{2},\frac{s}{2})\right]=\log\frac{1}{2}+\int_0^\infty\frac{e^{-t}-e^{-2t}-e^{-3t}+e^{-tx}}{t(1+e^{-t})}dt$$

as starting point for Kummer's series representation of

$$\log\Gamma(x)=(\frac{1}{2}-x)(\gamma+\log 2)+(1-x)\log\pi-\frac{1}{2}\log\sin\pi x+\sum_1^\infty\frac{\log n}{n\pi}\sin(2\pi nx) \ .$$

Asymptotically it holds (see [8] H.M. Edwards 10.10, [13] N. Nielsen, chapter 13, §73, [25] E. C. Titchmarsh, 2.15.8)

$$\frac{\Gamma'(1+x)}{\Gamma(1+x)}-\log x=\frac{\Gamma'(x)}{\Gamma(x)}+\frac{1}{x}-\log x=\log\Gamma(1+x)-\log x\approx\begin{cases}\frac{1}{2x} & x\to\infty\\ -\log x & x\to 0\end{cases} \ .$$

$$\to\log'\Gamma(x)+\gamma=\frac{\Gamma'(x)}{\Gamma(x)}+\gamma=\int_0^1\frac{1-t^{x-1}}{1-t}dt$$

With reference to (1.20) we further mention (see [13] N. Nielsen, chapter 9, §51)

$$\pi\cot\pi s=\int_0^1\frac{x^s-x^{1-s}}{1-x}\frac{dx}{x} \qquad \text{for } 0<\operatorname{Re}(s)<1$$

$$\log\frac{s}{1-s}=\int_0^1\frac{x^s-x^{1-s}}{\log x}\frac{dx}{x} \qquad \text{for } 0<\operatorname{Re}(s)<1$$

$$\frac{\pi}{s\cos\frac{\pi}{2}s}=\int_0^1(arc\cot x)x^s\frac{dx}{x} \qquad \text{for } 0<\operatorname{Re}(s)<1$$

$$\frac{\pi}{s\sin\frac{\pi}{2}s} = \int_0^1 \log(1+\frac{1}{x^2})x^s\frac{dx}{x} \qquad \text{for } 0<\operatorname{Re}(s)<2$$

$$\log\tan(\frac{\pi}{2}s) = \int_0^1 \frac{x^s - x^{1-s}}{1+x}\frac{dx}{x\log x} \qquad \text{p. 153}$$

$$\log\sin(\pi s) = \int_0^1 \frac{x^s + x^{1-s} - 2x^{3/2}}{(1-t)\log t}\frac{dt}{t} = -\log 2 - \sum_1^\infty \frac{\cos(2\pi n s)}{n} \qquad 0<s<1$$

$$\log 2\sin(\frac{\pi}{2}s) = \sum_1^\infty \frac{\cos(ns)}{n}$$ see [13] N. Nielsen, Integrallogarithmus, chapter 1, §7 .

Using the abbreviation

(2.9) $$\Delta f(x) := f(x+1) - f(x)$$

$$\binom{s-1}{n} := \frac{(s-1)(s-2)....(s-n)}{1*2*........n}, \quad \binom{s-1}{0} := 1$$

we recall Mellin's inverse formula from [13] N. Nielsen, chapter 9, §45-53, chapter 16, §86-92, with its relation to Ramanujan's Master Theorem ([1] B. C. Berndt, The first quarterly report, 1.2 Theorem I (Ramanujan's Master Theorem) and Theorem (Hardy)) in the form

$$\int_0^\infty F(x)x^{n-1}dx = \Gamma(n)\varphi(-n) \quad \text{for } F(x) = \sum_0^\infty \frac{\varphi(k)}{k!}(-x)^k$$ in the neighborhood of $x=0$.

Lemma 2.1 (**Mellin inverse formula**): For a function

(2.10) $$W(s) := \sum_0^\infty \Delta^n W(1)\binom{s-1}{n} \text{ for } \operatorname{Re}(s)>0$$ fulfilling $\lim \frac{W(n)}{n^k} = 0$ for $n\to\infty$

the following series define analytical functions for $\operatorname{Re}(s)<1/2$ resp. $\operatorname{Re}(s)>-1/2$

(2.11) $$f(s) := \sum_0^\infty (-1)^n \Delta^n W(1)x^n \quad \text{and} \quad g(s) := \sum_0^\infty (-1)^n W(n+1)x^n .$$

If $W(s)$ can be represented in the form $W(s) = \int_0^1 \omega(x)x^{s-1}dx$ it further holds

(2.12) $$f(s) = \int_0^1 \frac{\omega(x)dx}{1-(1-x)s} \quad \text{and} \quad g(s) = \int_0^1 \frac{\omega(x)dx}{1+xs} ,$$

whereby both functions (1.28) are analytical (probably with the exception on the x-axis for $x \geq 1$ resp. $x \leq -1$). With respect to (1.27) it further holds (with variable substitution $t = \frac{x}{1+x} \leftrightarrow x = \frac{t}{1-t}$)

$$\frac{\pi}{\sin \pi s} W(s) = \int_0^1 \frac{\omega(1-x)x^s}{(1-x)^s} \frac{dx}{x} = \int_0^\infty \frac{f(\frac{1}{1+t})}{1+t} x^{t-1} dt = \int_0^\infty \frac{1}{x} g(\frac{1}{x}) x^{s-1} dx \quad \text{for} \quad 0 < \mathrm{Re}(s) < 1$$

(2.13) $$\frac{1}{x} g(\frac{1}{x}) = \frac{f(\frac{1}{1+x})}{1+x} = \frac{1}{2i} \int_{a-i\infty}^{a+i\infty} \frac{W(s)}{\sin \pi s} x^{-s} ds$$

for $0 < a < 1$ and $-(\pi + \delta) < \theta < \pi + \delta$ with $x = |x| e^{i\theta}$

Corollary 2.2 For $g(x) := \sum_0^\infty \Gamma(n + \frac{1}{2})(-x)^n$ it holds

(2.14) $$\frac{\pi}{\sin \pi s} \Gamma(s - \frac{1}{2}) = \int_0^\infty \frac{1}{x} g(\frac{1}{x}) x^s \frac{dx}{x} = \int_0^\infty g(x) x^{1-s} \frac{dx}{x}$$

and it holds $\int_0^\infty g(t) dt = divergent?$ and $\varphi(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{\sqrt{\sinh t}} \frac{1}{(\sinh t)^2 + x^2} dt$)

(2.15) $$\zeta(1-s) \frac{\pi}{\sin \pi s} \Gamma(s - \frac{1}{2}) = \int_0^\infty x^{1-s} \left[\sum_1^\infty g(nx) + \frac{1}{x} \int_0^\infty g(t) dt \right] \frac{dx}{x}$$

Remark 2.3 From [26] G.N. Watson 13-2 we recall

(2.16) $$\frac{(2b)^{s-1} \Gamma(s - \frac{1}{2})}{(a^2 + b^2)^{s-\frac{1}{2}}} = \sqrt{\pi} \int_0^\infty x^s J_{s-1}(bx) e^{-ax} \frac{dx}{x}$$

From (2.16) it follows

i) $$\Gamma(s - \frac{1}{2}) = \sqrt{2\pi} \int_0^\infty x^s J_{s-1}(x) e^{-x} \frac{dx}{x} \quad \text{for} \quad \mathrm{Re}(s) > 0$$

ii) Riemann's duality equation can be proven by a self-reciprocal function argument (see [25] E. C. Titchmarsh, 2.7), i.e. $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin(xy) dy$ which holds for

$$f(x)=\frac{1}{e^{x\sqrt{2\pi}}-1}-\frac{1}{x\sqrt{2\pi}} \quad .$$

For $\varphi(x)=\int_0^\infty \frac{e^{-x\sinh t}}{\sqrt{x\sinh t}}dt$ it holds

$$\varphi(x)\neq\int_0^\infty \varphi(y)\sin(xy)dy=\int_0^\infty\int_0^\infty \sqrt{y}\frac{e^{-y\sinh t}}{\sqrt{\sinh t}}dt\sqrt{\frac{2}{\pi}}\frac{\sin(xy)}{\sqrt{xy}}dy=\int_0^\infty \frac{1}{\sqrt{\sinh t}}\int_0^\infty \sqrt{y}e^{-y\sinh t}J_{1/2}(xy)dydt$$

and therefore $\int_0^\infty \varphi(y)\sin(xy)dy=\sqrt{x}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\sqrt{\sinh t}}\frac{1}{(\sinh t)^2+x^2}dt$

with $\frac{(2x)^{1/2}}{(\sinh t)^2+x^2}=\sqrt{\pi}\int_0^\infty y^{1/2}J_{1/2}(xy)e^{-y\sinh t}dy$ and $J_{1/2}(x)=\sqrt{\frac{2}{\pi}}\frac{\sin x}{\sqrt{x}}$.

iii) for $h_s(x):=\sum_1^\infty J_{s-1}(nx)e^{-nx}$ a Dirichlet type series can be defined in the form

(2.17) $$\Gamma(s-\frac{1}{2})\zeta(s)=\sqrt{2\pi}\int_0^\infty x^s h_s(x)\frac{dx}{x} \qquad \text{for} \quad \mathrm{Re}(s)>1$$

Proof of corollary 2.2 It holds

$$W_1(s):=\Gamma(s-\frac{1}{2})=\int_0^\infty e^{-x}x^{s-1/2}\frac{dx}{x}=\int_0^1 e^{-\frac{t}{1-t}}\left[\frac{t}{1-t}\right]^{s-1/2}\frac{dt}{t(1-t)}$$

leads to $g(x)=\sum_0^\infty \Gamma(n+\frac{1}{2})(-x)^n$ with $\frac{\pi}{\sin\pi s}\Gamma(s-\frac{1}{2})=\int_0^\infty \frac{1}{x}g(\frac{1}{x})x^s\frac{dx}{x}$

Examples 2.4

i) $$W_1(s):=\frac{1}{s}=\int_0^1 x^{s-1}dx=\int_0^1 \omega(x)x^{s-1}dx=\sum_0^\infty \frac{(-1)^n}{n+1}\binom{s-1}{n}, \quad \mathrm{Re}(s)>0\,,$$

ii) $$W_2(s):=B(s,1-s)=\frac{\pi}{\sin\pi s}=\int_0^\infty \frac{x^{s-1}}{1+x}dx=\int_0^1 \frac{x^s+x^{1-s}}{1+x}\frac{dx}{x}=\int_0^1 \omega(x)x^{s-1}dx$$

ii) $$W_3(s):=\pi\cot\pi x=\pi\frac{\cos\pi x}{\sin\pi x}=\int_0^1 \frac{x^s-x^{1-s}}{1-x}\frac{dx}{x}\neq\int_0^1 \omega(x)x^{s-1}dx \qquad \text{for } 0<\mathrm{Re}(s)<1$$

iii) $$W_4(s):=-\frac{\Gamma'(s)}{\Gamma(s)}+\gamma=\int_0^1 \frac{1-x^{s-1}}{1-x}dx=\sum_0^\infty \frac{(-1)^n}{n+1}\binom{s-1}{n+1}\neq\int_0^1 \omega(x)x^{s-1}dx\,, \mathrm{Re}(s)>0\,,$$

$$\rightarrow \frac{\pi}{\sin \pi s}\left[\frac{\Gamma'(s)}{\Gamma(s)}+\gamma\right]=\int_0^1 \frac{\log x}{(1-x)^s}x^{s-1}dx$$

iv) $W_5(s):=\xi(s)\neq\int_0^1 \omega(x)x^{s-1}dx??$ $\rightarrow \frac{\pi}{\sin \pi s}W_5(s)\neq ?\int_0^\infty \frac{1}{x}g(\frac{1}{x})x^s\frac{dx}{x}=\int_0^\infty g(x)x^{1-s}\frac{dx}{x}$

with $g(x)=?\sum_0^\infty W_4(n+1)(-x)^n=\zeta(1)+\sum_1^\infty \zeta(n+1)(-x)^n$.

3. Bessel function theory

The Bessel functions (see [26] G.N. Watson 13-74) in the form

(3.1) $$N(x):=N_{\frac{\sigma}{2}}(x)=2\pi x\left[J_{\frac{\sigma}{2}}^2(\frac{x}{2})+Y_{\frac{\sigma}{2}}^2(\frac{x}{2})\right]$$ with $\sigma:=\frac{1}{2}$

can be used to define an appropriate new measure $dN(x):=N'(x)dx$.

From (3.2) below it's being deduced (see [26] G.N. Watson 13-74) that $N_{\frac{\sigma}{2}}(x)$ is an increasing function when $\sigma<1$, using the fact that $\lambda\tanh\lambda t$ is an increasing function of λ, when $\lambda>0$ and therefore the last term in (2.3) below is negative or positive according to $\sigma>1$or $0<\sigma<1$. The analysis below is applicable for $0<\sigma<1$; we restrict ourselves to the critical value $\sigma:=\frac{1}{2}$.

$N'_{\frac{\sigma}{2}}(x)$ has a representation in the form (see [26] G.N. Watson 6-22, 6-3, 7-31, 13-74, 13-75)

(3.2) $$N'_{\frac{\sigma}{2}}(x)=\frac{32}{\pi}\int_0^\infty K_0(x\sinh t)\tanh t\cosh\sigma t\left[\tanh t-\sigma\tanh\sigma t\right]dt$$

with

(3.3) $$K_0(x)=\int_0^\infty e^{-x\cosh t}dt=\int_0^\infty \frac{\cos(xt)}{\sqrt{t^2+1}}dt=\frac{1}{2}\int_{-\infty}^\infty e^{-ix\sinh t}dt=\frac{e^{-x}}{\sqrt{2}}\int_{-\infty}^\infty \frac{e^{-tx}}{\sqrt{1+\frac{t}{2}}}\frac{dt}{\sqrt{t}}$$

from which it's being deduced (see [26] G.N. Watson 13-74) that $N_{\frac{\sigma}{2}}(x)$ is an increasing function when $\sigma<1$, using the fact that $\lambda\tanh\lambda t$ is an increasing function of λ, when $\lambda>0$ and therefore the last term in (2.3) below is negative or positive according to $\sigma>1$or

$0<\sigma<1$. The analysis below is applicable for $0<\sigma<1$; we restrict ourselves to the critical value $\sigma := \frac{1}{2}$.

Lemma 3.1 The function $N(x)$ can be represented as infinite integral in the form

$$N(x)=\sqrt{\frac{x}{\pi}}\int_0^\infty e^{-x\sinh t}\frac{dt}{\sqrt{\sinh t}}$$

Proof of lemma 3.1 We recall from [11] I.S. Gradshteyn, I.M. Ryzhik (6.518) the formula

(3.4) $$\frac{\pi^2}{\cos\nu\pi}\left[J_\nu^2(x)+Y_\nu^2(x)\right]=\int_0^\infty K_{2\nu}(2x\sinh t)dt \qquad \text{for} \quad -\frac{1}{2}<\mathrm{Re}(\nu)<\frac{1}{2} \ .$$

Putting $\nu := \frac{1}{4}$, using $\cos\frac{\pi}{4}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ and $K_{1/2}(x)=\sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}$ leads to

$$N(x)=2\pi x\frac{\cos\frac{\pi}{4}}{\pi^2}\int_0^\infty K_{\frac{1}{2}}(x\sinh t)dt=2\pi x\frac{1}{\sqrt{2}\pi^2}\int_0^\infty\sqrt{\frac{\pi}{2}}\frac{e^{-x\sinh t}}{\sqrt{x\sinh t}}dt$$

(3.5) i.e. $$N(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty\sqrt{x}e^{-x\sinh t}\frac{dt}{\sqrt{\sinh t}} \quad \bullet$$

Corollary 3.1 The function

(3.6) $$\rho(x):=2\pi\left[\left((\pi x)^{\frac{1}{4}}J_{\frac{1}{4}}(\frac{x}{2})\right)^2+\left((\pi x)^{\frac{1}{4}}Y_{\frac{1}{4}}(\frac{x}{2})\right)^2\right]=\int_0^\infty\frac{e^{-x\sinh t}dt}{\sqrt{\sinh t}}$$

and its Fourier transform can be represented as infinite integral

(3.7) $$\rho(x)=\int_0^\infty\frac{e^{-x\sinh t}}{\sqrt{\sinh t}}dt \quad , \quad \rho'(x)=-\int_0^\infty\sqrt{\sinh t}\,e^{-x\sinh t}dt$$

(3.8) $$\frac{1}{\sqrt{2\pi}}\hat{\rho}(x)=\frac{1}{\sqrt{4\pi}}\int_0^\infty\frac{1}{\sinh t}e^{-\frac{x}{4\sinh t}}dt \quad , \quad \frac{1}{\sqrt{2\pi}}\hat{\rho}'(x)=-\frac{1}{4\sqrt{2}\sqrt{2\pi}}\int_0^\infty e^{-\frac{x}{4\sinh t}}\frac{dt}{\sinh^2 t} \ .$$

Proof of corollary 3.1: It holds

(3.9) $$\frac{1}{\sqrt{2\pi}}\rho(x^2)=\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-x^2\sinh t}}{\sqrt{\sinh t}}dt=\int_0^\infty f(x,\varepsilon(t))dt$$

whereby $f(y,\varepsilon(t)):=\frac{1}{\sqrt{2\pi\varepsilon(t)}}e^{-\varepsilon(t)y^2}$ with $\varepsilon(t):=\sinh t$ and its Fourier transform

(3.10) $$\hat{f}(\xi,\varepsilon)=\frac{1}{\varepsilon(t)\sqrt{4\pi}}e^{-\frac{\xi^2}{4\varepsilon(t)}} .$$

It follows

(3.11) $$\frac{1}{\sqrt{2\pi}}\hat{\rho}(x^2)=\frac{1}{\sqrt{4\pi}}\int_0^\infty \frac{1}{\sinh t}e^{-\frac{\xi^2}{4\sinh t}}dt \qquad \bullet$$

From [26] G.N. Watson 6-22, 13-72 and 13-75 we note

$$\cos\frac{\sigma\pi}{4}K_{\frac{\sigma}{2}}(x)=\frac{1}{2}\int_{-\infty}^\infty e^{-ix\sinh t}\cosh\frac{\sigma}{2}tdt \qquad \text{resp.} \qquad K_{\frac{\sigma}{2}}(x)=\int_0^\infty e^{-x\cosh t}\cosh\frac{\sigma}{2}tdt$$

$$K^2_{\frac{\sigma}{2}}(x):=2\int_0^\infty K_\sigma(2x\cosh t)dt \ .$$

From [11] I.S. Gradshteyn, I.M. Ryzhik (6.518), (6.544) we note

$$\frac{\pi^2}{\cos\frac{\sigma}{2}\pi}\left[J^2_{\frac{\sigma}{2}}(x)+Y^2_{\frac{\sigma}{2}}(x)\right]=\int_0^\infty K_\sigma(2x\sinh t)dt \qquad \text{for} \quad -1<\mathrm{Re}(\sigma)<1$$

$$K_\sigma(2x\sinh t)=\frac{1}{\pi}\int_0^\infty \frac{x^2}{\tau}K_{\frac{\sigma}{2}}(\frac{x^2}{\tau})K_{\frac{\sigma}{2}}(\tau\sinh^2 t)\frac{d\tau}{\tau}=\frac{1}{2}\left[\frac{\sinh t}{x}\right]^{\frac{\sigma}{2}}\int_0^\infty \tau^{\frac{\sigma}{2}}e^{-\frac{x^2}{\tau}-\tau\sinh^2 t}\frac{d\tau}{\tau}$$

From [26] G.N. Watson 15-61 we note for $\nu:=\frac{\sigma}{2}$

$$\left|K_\nu(re^{-\frac{+}{}\frac{\pi i}{2}})\right|^2=\frac{\pi^2}{4}\left[J^2_\nu(r)+Y^2_\nu(r)\right] .$$

From [26] G.N. Watson 7-15 we further note for $\sigma:=\frac{1}{2}$

$$J^2_\nu(\frac{x}{2})+Y^2_\nu(\frac{x}{2})\approx\frac{4}{\pi x}\sum_0^\infty 1*3*...(2m-1)\frac{(\sigma^2-1^2)(\sigma^2-3^2)...(\sigma^2-(2m-1)^2)}{m!}x^{-2m} .$$

Remark 3.4 An alternative representation of the function $N(x)$ (see [26] G.N. Watson 7-31, 15-5, 15-53) is given by

(3.12) $$\frac{\pi}{2}x\left[J_\nu^2(x)+Y_\nu^2(x)\right]=\frac{1}{\frac{d\Psi_\nu}{dx}} \quad \text{resp.} \quad N(2x)=\frac{8}{\Psi'(x)}=\frac{8}{1-\psi'(x)}$$

whereby $$\Psi_\nu(x):=\arctan\left[\frac{Y_\nu(x)}{J_\nu(x)}\right] \quad \text{with} \quad \frac{d\Psi_\nu}{dx}=\frac{2/(\pi x)}{\left[J_\nu^2(x)+Y_\nu^2(x)\right]}>0$$

and $$\psi(x):=\arctan\left[-\frac{Q_0(x)}{P_0(x)}\right] \quad \text{with} \quad \frac{d\psi}{dx}=\frac{2/(\pi x)}{\left[P_0^2(x)+Q_0^2(x)\right]}$$

with $$\psi(x)\approx\tan\psi(x)\approx\frac{1}{8x}-\frac{33}{512x^3}+\frac{3417}{16384x^5}.......$$

and $P_0(x)$ and $Q_0(x)$ appropriate polynomials related to $J_0(x)$, $Y_0(x)$ defined by

(3.13) $$J_0(x)=\sqrt{\frac{2}{\pi x}}\left[\cos(x-\frac{\pi}{4})P(x)-\sin(x-\frac{\pi}{4})Q(x)\right] \qquad Y_0(x)=\sqrt{\frac{2}{\pi x}}\left[\sin(x-\frac{\pi}{4})P(x)+\cos(x-\frac{\pi}{4})Q(x)\right]$$

It holds (see [26] G.N. Watson 15-52)

(3.14) $$\Psi_\nu(x)=x-\frac{\nu\pi}{2}-\frac{\pi}{4}-\psi(x) \quad \text{and} \quad \frac{d}{dx}\left[\frac{Q_0(x)}{P_0(x)}\right]=\frac{1-\left[P_0^2(x)+Q_0^2(x)\right]}{P_0^2(x)}>0$$

with link to e.g. Euler's investigations of the zeros of

(3.15) $$J_0(2\sqrt{x})=J_0(2\sqrt{0})\prod_{n=1}^{\infty}(1-\frac{x}{\alpha_n})$$

and to e.g. the measure

(3.16) $$d\varsigma(x):=\left[P^2(x)+Q^2(x)\right]d\log x \quad .$$

With reference to (1.31) and (2.2) above we mention (see [26] G.N. Watson 13-21)

(3.17) $$\int_x^\infty\frac{e^{-t}dt}{t}\approx\int_x^\infty\frac{K_0(t)dt}{\sqrt{t}} \quad .$$

With reference to remark 1.1 we mention Hadamard's formula ([8] H.M. Edwards 2.1)

(3.18) $$\xi(s)=\xi(0)\prod_\rho(1-\frac{s}{\rho}) \quad .$$

We note that for $\Psi(x) := \arctan\left[\frac{Y_{1/4}(x)}{J_{1/4}(x)}\right]$ it holds

(3.19) $$\frac{1}{8}N(x)\frac{d\Psi}{x} = \frac{dx}{x} \qquad (\rightarrow \frac{1}{8}N(x)\frac{d\psi}{x} \approx \frac{dx}{x})$$

which leads with proposition 1.3 to the following formula

(3.20) $$\frac{\pi}{2}\int_0^\infty x^{1-s}N(x)N(\frac{1}{x})\frac{d\Psi}{x} = \Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2})\Gamma(s-\frac{1}{2}) \qquad (Q(x) := \frac{d\Psi}{dx} = \frac{8}{N(x)} \text{ !??!!!!}).$$

Remark A

From [26] G.N. Watson 14-4, 14-42 we recall Hankel's repeated integral and its inversion, i.e.

Let $F(R)$ *be an arbitrary function of the real variable* R *subject to the condition that* $\int_0^\infty F(R)\sqrt{R}dR$ *exists and is absolutely convergent; and let the order of* ν *of the Bessel functions be not less than* $-1/2$. *Then*

$$\int_0^\infty udu\int_0^\infty F(R)J_\nu(uR)J_\nu(ur)RdR = \int_0^\infty F(R)\left\{\int_0^\infty J_\nu(uR)J_\nu(ur)udu\right\}RdR = \frac{1}{2}[F(r+0)-F(r-0)]$$

provided that the positive number r lies inside an interval in which $F(R)$ *has limited total fluctuation.*

From [26] G.N. Watson 18-24 we recall the sum of the Fourier-Bessel expansion for a given function, i.e.

Let $f(t)$ *be a function defined arbitrarily in the interval* $(0,1)$*; and let* $\int_0^\infty f(t)\sqrt{t}dt$ *exists and let it be absolutely convergent. Let*

$$a_m = \frac{2}{J^2_{\nu+1}(j_m)}\int_0^\infty tf(t)J_\nu(j_m t)dt$$

where $\nu + \frac{1}{2} \geq 0$ *. Let x be any interval point of an interval* (a,b) *such that* $0 < a < b < 1$ *and such that* $f(t)$ *has limited total fluctuation in* (a,b)*. Then the series* $\sum_1^\infty a_m J_\nu(j_m x)$ *is convergent and its sum is given by*

$$\sum_1^\infty a_m J_\nu(j_m x) = \frac{1}{2}[f(r+0)-f(r-0)] \quad .$$

4. Zeta function theory

The Zeta function $\zeta(s)$ can be defined in the critical stripe $0<\mathrm{Re}(s)<1$ as a complex-valued transform of an integral operator with normal distribution measure, i.e. for

(4.1) $$R(x):=\sum_{1}^{\infty}e^{-nx}$$

it holds (see [25] E.C. Titchmarsh, 2.11)

(4.2) $$\zeta(s)\Gamma(s)=\int_0^\infty x^s\left[R(x)-\frac{1}{x}\int_0^\infty e^{-t}dt\right]\frac{dx}{x} \qquad \text{for} \quad 0<\mathrm{Re}(s)<1 \ .$$

Müntz' formula (see [25] E.C. Titchmarsh, 2.11) gives the Zeta function $\zeta(s)$ as Mellin transform of an integral operator in a more general form, i.e. it holds

Lemma 4.1 (**Müntz' formula**) For $\omega(x),\omega'(x)$ continuous and bounded in any finite interval with $\omega(x)=o(x^{-\alpha})$ and $\omega(x)=o(x^{-\beta})$ for $x\to\infty$ and $\alpha,\beta>1$ it holds

(4.3) $$\zeta(s)\int_0^\infty x^s\frac{\omega(x)dx}{x}=\int_0^\infty x^s\left[\sum_{1}^{\infty}\omega(nx)-\frac{1}{x}\int_0^\infty\omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0<\mathrm{Re}(s)<1.$$

Proof:

i) because $\omega(x)$ is continuous and bounded in any finite interval with $\omega(x)=o(x^{-\alpha})$ it holds

$$\sum_{1}^{\infty}\frac{1}{n^s}\left|\int_0^\infty x^{s-1}\omega(x)\right|dx \text{ exists for } \quad 1<\sigma<\alpha$$

i.e. the inversion leading to the left hand side of (4.3) is justified.

ii) $$\sum_{1}^{\infty}\omega(nx)-\int_0^\infty\omega(xt)dt=x\int_0^\infty\omega'(t)(t-[t])dt=x\int_0^{1/x}O(1)dt+x\int_{1/x}^\infty O((xt)^{-\beta})dt=O(1)$$

The first summand is justified, because $\omega(x)$ is continuous and bounded in any finite interval the second summand is justified, because $\omega(x)=o(x^{-\alpha})$, i.e. it holds

$$\sum_{1}^{\infty}\omega(nx)=O(1)+\frac{c}{x} \text{ with } c=\int_0^\infty\omega(t)dt\,.$$

Hence

$$\int_0^\infty x^s \sum_1^\infty \omega(nx) + \frac{dx}{x} = \int_0^1 x^s \left[\sum_1^\infty \omega(nx) - \frac{c}{x} \right] \frac{dx}{x} + \int_1^\infty x^s \sum_1^\infty \omega(nx) \frac{dx}{x} + \frac{c}{s-1}$$

for $\sigma > 0$ except $s = 1$. Also

$$-c \int_1^\infty x^{s-2} dx = \frac{c}{s-1} \quad \text{for } \sigma < 1$$

and therefore (4.3) for $0 < \sigma = \mathrm{Re}(s) < 1$ •

Remark 4.2

concerning an application of Polya's criterion in combination with an application of Müntz formula it holds from [19] G. Polya, p. 365:

Polya's criterion cannot be applied to Muentz's formula.

Polya's criterion is for an integral over a finite interval and to extend it to an infinite interval it needs certain conditions, see the notes by R.P. Boas in the second volume of Polya's collected works.

In order to apply Polya's criterion to Muentz's formula one needs to show that the function

$$G^*(x) := \sum_1^\infty \omega(nx) - \frac{1}{x}$$

is positive and increasing for $x > 0$. *It does not suffice to show this only for* $x > 1$, *because* $G^*(x)$ *is not the same as* $G^*(\frac{1}{x})$. *However,* $G^*(x)$ *cannot be positive and increasing in the whole range for x, because otherwise its value at infinity would be positive and not 0, as is the case. Muentz's formula requires* $\omega(x)$ *to vanish at infinity to order* $x^{-\alpha}$ *with* $\alpha > 1$, *hence the corresponding function*

$$\sum_1^\infty \omega(nx) - \frac{1}{x} \int_0^\infty \omega(t) dt$$

has the value 0 at infinity. Therefore, this expression cannot be both positive and increasing near infinity and Polya's criterion never applies to a formula of Muentz's type.

Remark 4.3 The standard "measure" in current Zeta function theory is

(4.4) $$\sigma(x) := Ei(-x) = -\int_x^\infty e^{-t} \frac{dt}{t}$$

with

(4.5) $$d\sigma(x)=\frac{e^{-x}dx}{x} \quad , \quad d\sigma(\log x)=\frac{dx}{x\log x} \quad , \quad \Gamma(s)=\int_0^\infty x^s d\sigma(x) \ .$$

(3.5) plays a key role in the analysis of Euler, Gauss and Riemann, e.g. Gauss' Li-function is defined by

(4.6) $$Li(x):=\int_0^x \frac{dt}{\log t}=Ei(\log x)=-\int_{-\log x}^{\infty}\frac{e^{-t}}{\mathrm{t}}dt=\int_0^x t d\sigma(\log t)=O(\frac{x}{\ln x}) \quad , \quad x>1.$$

Euler's $\log(\log x)$ divergence (see [8] H.M. Edwards 1.1) can be stated by

(4.7) $$\sum_1^\infty \frac{1}{p}\approx \log(\log x)=\int_1^{\log x}\frac{du}{u}=\int_e^x \frac{dt}{t\log t}=\int_e^x d\sigma(\log t)$$

and Riemann's estimate of $\pi(x)-R(x)$ (see remark 3.xx below) is given by

(4.8) $$\pi(x)-R(x)\approx\int_x^\infty \frac{d\sigma(\log t)}{t^2-1} \qquad (\pi(x)-R(x)\approx\int_x^\infty \frac{d\mu(\log t)}{t^2-1}).$$

Remark 4.4

This remark is about putting the new measure with its underlying Hilbert space in a Hilbert scale context with reference to Riemann's duality equation:

Basically the isometric property $f(x)=\hat{f}(x)$ of the Gauss-Weierstrass density function is used to prove Riemann's duality (see [8] H.M. Edwards)

(4.9) $$\xi(s):=\zeta(s)(s-1)\frac{s}{2}\Gamma(\frac{s}{2})\pi^{-s/2}=\xi(1-s) \ ,$$

which can be written in the form

(4.10) $$\xi(s)=\zeta(s)(1-s)\left[\int_0^\infty x^s f'(x)dx\right]=\zeta(1-s)s\left[\int_0^\infty x^{1-s} f'(x)dx\right] \ .$$

Jacobi's $\vartheta-$relation (see also [8] H.M. Edwards 10.3)

(4.11) $$G(x):=\sum_{-\infty}^{+\infty}e^{-\pi n^2x^2}=\sum_{-\infty}^{\infty}\hat{f}(nx)=1+2\sum_1^{+\infty}e^{-\pi n^2 x}=:1+2\psi(x^2)=\frac{1}{x}G(\frac{1}{x})$$

implies, that the invariant operator is formally self-adjoint (see also [12] H. Hamburger and [9] D. Gaier for relations to conformal mappings and singular integral operators). But the operator has no transform at all, because the integral

(4.12) $$\int_0^\infty x^{1-s}\frac{G(x)dx}{x}=\int_0^\infty x^{-s}\sum_{-\infty}^{\infty}dF(nx)$$

does not converge for any s. The integral *would converge at* ∞ if the constant term of (1.27) above, which is basically $f(0)$ *,* is absent. Roughly speaking the measure $f'(x)dx$ solves the issue, but the prize to be paid is a scale of higher regularly, jeopardizing adequate self-adjoint properties.

Riemann's duality can be derived from the representation

$$\zeta(s)\Gamma(\frac{s}{2})\pi^{-s/2}=\zeta(1-s)\Gamma(\frac{1-s}{2})\pi^{-(1-s)/2}=\int_1^\infty\psi(x)\left[x^{s/2}+x^{(1-s)/2}\right]\frac{dx}{x}-\frac{1}{s(1-s)}=\int_0^\infty x^{s/2}\Phi(x)\frac{dx}{x}$$

with $\Phi(x):=\sum_1^{+\infty}e^{-\pi n^2x}-\frac{1}{2\sqrt{x}}$ whereby $x^{1/4}\Phi(x)=\left[\frac{1}{x}\right]^{1/4}\Phi(\frac{1}{x})$.

Remark 4.5

Referring to Tauberian Theorems the integrals $\int_0^\infty x^s d\sigma(x)$ and $\int_0^\infty x^s d\mu(x)$ show the same divergence behavior for $s\to 0$ as $\approx\zeta(1)$, which can be seen from

(4.13) $$\int_0^\infty x^s\frac{e^{-x}-\varphi(x)}{x}dx=\Gamma(s)\left[1-\frac{1}{2\pi}\Gamma(\frac{1/2-s}{2})\Gamma(\frac{s+1/2}{2})\right]$$

We note the relations ([26] G.N. Watson 13.6)

$$\int_x^\infty\frac{J_0(t)}{t}dt=-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(1-\frac{s}{2})}{s\Gamma(1+\frac{s}{2})}x^s\frac{ds}{s}$$

and $$\mu(x):=\int_x^\infty\frac{e^{-t}-J_o(2\sqrt{\mathrm{t}})}{\mathrm{t}}dt=\gamma+\sum_2^\infty\frac{(-1)^{k-1}}{k}\frac{x^k}{k!}(1-\frac{1}{k!})$$

with $$\gamma=\mu(0)=-\Gamma'(1)=-\int_0^\infty e^{-x}\log t dt=\int_0^1\log(\log(\frac{1}{t})dt=0.57721566\ldots\ .$$

The Mellin transforms of $J_0(x)$, $Y_0(x)$, $K_o(x)$ are given in [26] G.N. Watson 13.21, 13-24, 13-3). We mention the formulas

$$\int_0^\infty J_0(t)\log t dt=-\gamma-\log 2\ ,\ \log 2=\int_0^\infty\frac{e^{-t}-e^{-2t}}{t}dt\quad,\ \int_0^\infty Y_0(t)dt=0\ .$$

Remark 4.6

We mention the following equivalent formulation for the Riemann conjecture (see [8] Edwards, chapter 5)

(4.14) $$\pi(x) = Li(x) + O(\sqrt{x}\ln x) = Li(x) + O(x^{\frac{1}{2}+\varepsilon})$$

and Euler's formula

(4.15) $$\sigma(x) = \int_x^\infty e^{-t}\frac{dt}{t} = \sum_1^\infty \frac{(-1)^{k-1}}{k}\frac{x^k}{k!} - \log x - \gamma = O(\frac{e^{-x}}{x})$$

resp. $$Li(x) = O(\frac{x}{\ln x}),$$

see [1] B. C. Berndt, Ramanujan's corollary 2, chapter 4 . Riemann's estimate is essentially based on the analysis of the function:

$$H(\beta) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{s}\log(1-\frac{s}{\beta})x^s ds = \begin{cases} -\int_x^\infty t^\beta d\sigma(\log t) & \text{.... } for \ \mathrm{Re}(\beta) < 0 \\ \int_o^x t^\beta d\sigma(\log t) & \text{..... } for \ \mathrm{Re}(\beta) > 0 \end{cases}$$

The new "measure" (x.y) motivates the alternative function

$$H^*(\beta) := \begin{cases} -\int_x^\infty t^\beta d\mu(\log t) & \text{.... } for \ \mathrm{Re}(\beta) < 0 \\ \int_o^x t^\beta d\mu(\log t) & \text{..... } for \ \mathrm{Re}(\beta) > 0 \end{cases}$$

to be put into the context of Gauss' Li-function (1.22) and Riemann's formula (see [8] H.M. Edwards 1.14)

(4.16) $$J(x) = Li(x) - \sum_{\mathrm{Im}(\rho)>0} Li(x^\rho) + Li(x^{1-\rho}) + \sum_n \int_x^\infty \frac{t^{-2n-1}}{\log t}dt + \log\xi(0) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^s\frac{ds}{s}$$

Analyzing analogue convergence behavior to

(4.17) $$\int_0^\infty \frac{x^{1-r}}{1-r}dr = \int_0^\infty \frac{e^{(1-r)\log x}}{1-r}dr \to Li(x)$$

(4.18) $$\int_0^\infty \frac{x^{\rho-r}}{\rho-r}dr = -\int_{-\infty}^{\rho\log x}\frac{e^t}{t}dt \to \sum_{\mathrm{Im}(\rho)>0} Li(x^\rho) + Li(x^{1-\rho}) \ ,$$

and to the density

(4.19) $$\frac{1}{\log x} = \int_0^\infty x^{-r} dr \quad .$$

(3.19) is roughly speaking the density of the primes. Note that due to Euler it holds

$$\log x = \int_0^\infty (e^{-t} - e^{-tx}) \frac{dt}{t} \quad \text{for} \quad \mathrm{Re}(x) > 0 \ .$$

Riemann analyzed the expression

(4.20) $$J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log\zeta(s) x^s \frac{ds}{s} = -\frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\log\zeta(s)}{s}\right] x^s ds \qquad \text{for} \quad a > 1$$

to prove the convergence estimate

(4.21) $$\pi(x) - R(x) \approx \int_x^\infty \frac{dt}{t(t^2-1)\log t} = \int_x^\infty \sum_1^\infty t^{-2n} \frac{d\log t}{\log t}$$

applying the Fourier inverse technique (see [8] H.M. Edwards 1.14 ff.). He used the following

***Lemma**: For* $H(\beta) := \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{1}{s}\log(1-\frac{s}{\beta})\right] x^s ds$ *with* $\beta \in C$

it holds

$$H(\beta) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{s}\log(1-\frac{s}{\beta}) x^s ds = \begin{cases} -\int_x^\infty t^\beta d\sigma(\log t) & \text{.... } for \ \mathrm{Re}(\beta) < 0 \\ \int_o^x t^\beta d\sigma(\log t) & \text{..... } for \ \mathrm{Re}(\beta) > 0 \end{cases}$$

For $\beta = (\rho, 1-\rho)$ *it follows*

(4.22) $$\sum_{\mathrm{Im}(\rho)>0} Li(x^\rho) + Li(x^{1-\rho}) = \sum_\rho \log = \sum_\rho \log(1-\frac{s}{\rho})$$

(4.22) is only conditionally convergent, it must be summed in order of increasing $\mathrm{Im}(\rho)$.

(4.23) is the critical term concerning an appropriate convergence behavior like (x.y), due to its oscillating behavior. Using the new measure with the alternative Li-function (x.y) the (convergence) damping behavior of Bessel functions first and second kind to infinite and to zero should provide significant contribution to overcome this issue.

Remark 4.7

In [5] D. Bump et.al. it's shown that the zeros of the transforms of the Hermite polynomials lie all on the critical line. The Hermite polynomials are the orthogonal polynomial system related to the normal distribution, building the eigenfunctions $\psi_n(x)$ of the quantum harmonic oscillator with its ground state $\psi_0(x) = c_1 f(c_2 x)$.

The relation of the Hermite polynomials to the density $dF(x)$ in relation to the concept of convolution operators is given in [6] D.A. Cardon.

Both analysis' could be applied replacing $dF(x)$ by $dN(x)$.

The Bessel function $K_0(x)$ plays a key role in the analysis of the next section. The analysis technique of [10] G. Gasper might be applicable using the Mellin transform of $K_0(x)$ (see [25] G. N. Watson 13-21), which is

(4.24) $$\int_0^\infty x^{s-1} K_0(2x)dx = \frac{1}{4}\Gamma^2(\frac{s}{2}) \qquad \text{for} \qquad 0 < \mathrm{Re}(s) \; .$$

or the relation (2.2) ff. below.

Remark 4.8

A famous usage of Dirichlet's series is in the context of Planck's black-body radiation function

(4.25) $$\frac{dR(\lambda,T)}{d\lambda} = \frac{c_1}{\lambda^5}\frac{1}{e^{c_2/\lambda T}-1} = \frac{c_1}{\lambda^5}\sum_1^\infty e^{-nc_2/\lambda T}$$

with $c_1 = 2\pi hc^2$ and $c_2 = hc/k$. The relation to the Zeta function

(4.26) $$\zeta(s)\Gamma(s) = \int_0^{+\infty} \frac{x^s}{e^x - 1}\frac{dx}{x}$$

is given by

(4.27) $$\frac{\pi^4}{90} = \zeta(4)\Gamma(4) = \int_0^{+\infty} x^4 (\sum_1^\infty e^{-nx})\frac{dx}{x} = \int_0^{+\infty} x^{-4}(\sum_1^\infty e^{-\frac{n}{x}})\frac{dx}{x} \; .$$

(4.42) describes the total radiation and its spectral density at the same time, i.e.

(4.28) $$g(x)dx = \frac{x^{-4}}{e^{1/x}-1}\frac{dx}{x} = \frac{x^4}{e^x - 1}\frac{dx}{x} = g(\frac{1}{x})dx \; .$$

The new measure $N'(x)dx$ resp. the "measure" $d\mu(x)$ allows a modified definition of this radiation function.

Referring to the probability model of the location of an electron we recall Parseval's equation (see [8] Edwards 10.7), which G.H. Hardy used (see [8] H.M. Edwards 11.1), to prove that there are infinitely many roots ρ of $\xi(\rho)=0$ on the line $\mathrm{Re}(s)=1/2$. With lemma 1.2 and lemma 2.6 it holds

$$2\pi\int_0^\infty x\left|N'(x)\right|^2 dx=\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\left[(\frac{1}{2}-s)\Gamma(1-s)\Gamma(\frac{3/2-s)}{2})\Gamma(\frac{s-1/2}{2})\right]^2 ds$$

(4.29)

$$2\pi\int_0^\infty x\left|\hat{N}'(x)\right|^2 dx=\frac{2^{3-2s}}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\left[(\frac{1}{2}-s)(\frac{3}{2}-s)\Gamma(\frac{1}{2}-s)\Gamma(\frac{3/2-s)}{2})\Gamma(\frac{s-1/2}{2})\right]^2 ds \ .$$

Remark 4.9

A modified norm to the standard inner product

$$\|f\|^2=(f,f)=\int_0^\infty f(x)\overline{f(x)}dx$$

can be defined in the form

(4.30) $$\|\|f\|\|^2:=\int_0^\infty f^2(x)N'(x)dx=\frac{16}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{x^2}{4\tau}-\tau\sinh^2 t}\frac{d\tau}{\tau}\tanh t\cosh\frac{t}{2}\left[\tanh t-\frac{1}{2}\tanh\frac{t}{2}\right]dtdx$$

where $\tanh t-\frac{1}{2}\tanh\frac{t}{2}>0$.

The above might be seen as a step forwards "The Road to Reality", as it's about complex number systems in combination with "real" duality (see [15] R. Penrose 34.8) in the context of location and frequency probability.

The Helmholtz equation with space dimension n is given by

$$\nabla u+\lambda^2 u=\begin{cases}-\Delta_i & in\ \ \Omega\\ 0\end{cases}$$

where Δ_i represents the Dirac delta function at the source point i corresponding to the fundamental solution. The domain Ω can be unbounded or bounded with or without boundary conditions; x denotes the n-dimensional coordinate variable and $r_k:=\|x-x_k\|$

The kernel wavelet basis functions are

$$h_n(\lambda r_k):=-\frac{i\lambda^{n-1/2}}{4}(2\pi\lambda r_k)^{1-n/2}H^{(2)}_{n/2-1}(\lambda r_k)\quad ,\ n\ge 2$$

where $\overline{h}_n$ comply with the divergence (conservation) theorem

$$\lim r_k^{n-1} S_n(1) \frac{\partial g_n}{\partial r_k} = -1 \quad , \quad r_k \to 0$$

and h_n satisfy the Sommerfeld radiation condition at infinity

$$\lim r \left[\frac{\partial g_n}{\partial r_k} + i\lambda g_n \right] = 0 \quad , \quad r_k \to \infty$$

The link to the density function (2.1) below is given by

$$r\left| H_{n/2-1}^{(1)}(r) \right|^2 = rH_{n/2-1}^{(1)}(r) H_{n/2-1}^{(2)}(r) = r\left[J_{\frac{n-2}{2}}^2(r) + Y_{\frac{n-2}{2}}^2(r) \right]$$

With reference to fractional mathematics we note that $\nu := \frac{1}{4}$ in (2.1) would correspond to a fractional dimension of space of $n = 2.5$.

Using the Hankel functions

$$H_\nu^{(1)}(x) := J_\nu(x) + iY_\nu(x) \text{ resp. } H_\nu^{(2)}(x) := J_\nu(x) - iY_\nu(x)$$

and

$$R_\nu^2(x) := \left[J_\nu^2(x) + Y_\nu^2(x) \right] = H_\nu^{(1)}(x) \overline{H_\nu^{(2)}(x)} = \left| H_\nu^{(1)}(x) \right|^2$$

it follows

$$H_\nu^{(1)}(x) = R_\nu(x) e^{i\Psi_\nu} \qquad \text{resp.} \qquad H_\nu^{(2)}(x) = R_\nu(x) e^{-i\Psi_\nu} \; .$$

Putting $\Psi(x) := \Psi_{\frac{1}{4}}(x)$ ...... can be re-formulated to

$$\frac{\pi}{2} R_\nu^2(x) d\Psi_\nu = \frac{dx}{x} \qquad \text{resp.} \qquad \frac{N(2x)}{8x} d\Psi = \frac{2}{\pi} \left| K_{\frac{1}{4}}(\pm ix) \right|^2 d\Psi = \frac{dx}{x}$$

With reference to (x.y) and remark x.y below we mention

$$H_\nu^{(1)}(r) = J_\nu(r) + iY_\nu(r) = \sqrt{J_\nu^2(r) + Y_\nu^2(r)} e^{i\Psi_\nu(r)}$$

resp. $$H_\nu^{(1)}(r) = N(2r) \frac{e^{i\Psi_\nu(r)}}{4\pi} = \frac{2}{\pi} \frac{e^{i\Psi_\nu(r)}}{\Psi_\nu'(r)} = \frac{2}{\pi} \frac{e^{i\Psi_\nu(r)}}{1 - \psi'(r)} \qquad \text{with } \nu := \frac{1}{4}$$

which might motivate an alternative or additional “polar” coordinate transformation in the context of Riemann manifolds.

The simplest version of the harmonic oscillator is the Hamiltonian system with Hamiltonian

$$H(p,q)=\frac{1}{2}(p^2+\omega^2 q^2) \text{ and } \dot{q}=p\,,\ \ \dot{p}=-\omega q\,,\ddot{q}=-\omega^2 q$$

Identifying $R^2 \cong C$ by putting $z=p+i\omega q$ a solution to $H(p,q)=\frac{1}{2}|z|^2$ is given in the form

$$z(t)=Ce^{i\omega t}\ .$$

Remark 4.10 (just to kick off a next level of brainstorming --> linkage hyperbolic functions and strings)

With respect to Lemma 1.2 we mention the somehow "birthday" of the Superstring theory. In 1968 Gabriel Veneziano and Mahiko Suzuki came across using the Euler beta function to describe interactions of elementary particles:

Consider an elastic scattering process with 2 incoming spinless particles of transverse momenta p_1, p_2, outgoing particles of momenta $-p_3,-p_4$. With a metric with signature $\{-,+,+,+,...+\}$ the mass squared of a particle is $m^2=-p^2$. The conventional Mandelstam variables are defined as

$$s=-(p_1+p_2)^2\ ,\ t=-(p_2+p_3)^2,\quad u=-(p_1+p_3)^2\ .$$

which obey the one identity

$$s+t+u=\sum m_i\ .$$

The largest $J=\alpha(s)$ value at given s with $s=m^2=(2p)^2$ the square of the energy in the center of mass frame and the angular momentum $J=2p\frac{r}{2}=pr$ formed the so-called "leading trajectory". Experimentally, it was discovered that the leading trajectories were almost linear in s.

In the field theory of the weak interactions the simplest model amplitude $A(s,t)$ is constructed as a sum of s-channel & t-channel input diagrams in the form

$$A(s,t)=A(t,s)=\frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}=B(-\alpha(s),-\alpha(t))=\sum_{j=0}^{\infty}\binom{\alpha(t)+j}{j}\frac{1}{j-\alpha(s)}\quad ,$$

that shows poles, where the resonance of the leading (Regge) trajectories $\alpha(s)$ is necessarily linear in s, i.e. $\alpha(x)=\alpha' x-\alpha(0)$ with the «daughter trajectories" $\alpha(s)=\alpha' s-\alpha(0)-n$, (postulated by Veneziano), to achieve, that the formula is physically acceptable. $\alpha(0)$ depends on the quantum numbers such as strangeness and baryon number, but α' appeared to be universal, approximately, i.e.

$$\alpha' \approx 1\left[\frac{1}{GeV}\right]^2 = constant$$ Regge slope

$$\alpha(x) = \alpha' x - \alpha(0)$$ linear Regge trajectory.

A resonance occurs at those s values where $\alpha(s)$ is a *nonnegative integer* (***mesons***) or a *nonnegative integer plus ½* (***baryons***).

$\alpha(s) \in N$ mesons

$\alpha(s) + 1/2 \in N$ baryons

which gives some relation to our

$$\Gamma^*(s) = B(1-s,s)\Gamma(s+\frac{1}{2}) = 2\int_0^\infty x^s \rho'(x)dx \ .$$

Remark 4.11

A wavelet transform is similar as a Fourier transform, which delivers the frequency spectrum of a timely signal f(t) without any loss of information, although the Fourier transform itself gives the frequencies without any information about the points in time, when the frequencies occur. The wavelet transform delivers this sort of information in a better distinguishing form: one gets both the frequency analysis and the points in time, when those frequencies happen, similar like the written notes, which results into the music of an orchestra, which are described in form of a wavelet transform on a 2-dimensional paper ([23] M. du Sautoy: "the primes have music in them")

A wavelet is a function $\psi(x) \in L_2(R)$ with a Fourier transform which fulfills

$$0 < c_\psi := 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \ .$$

The wavelet transform of a function $f(x) \in L_2(R)$ with the wavelet $\psi(x) \in L_2(R)$ is the function

$$W_\psi[f](a,b) := \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty} f(t)\overline{\psi}_{b,a}(t)dt = \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{a}}\overline{\psi}(\frac{t-b}{a})dt \ , \quad a \in R - \{0\}, b \in R$$

For a wavelet $\psi(x) \in L_1(R)$ its Fourier transform is continuous and fulfills

$$0 = \hat{\psi}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t)d\omega$$

The wavelet transform to the wavelet $\psi(x) \in L_2(R)$

$$W_\psi : L_2(R) \to L_2(R^2, \frac{dadb}{a^2}),$$

is isometric and for the adjoint operator

$$W_\psi^* : L_2(R^2, \frac{dadb}{a^2}) \to L_2(R)$$

$$W_\psi^*[g](a,b) := \frac{1}{\sqrt{c_\psi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t) \frac{1}{\sqrt{a}} \psi(\frac{t-b}{a}) g(a,b) \frac{dadb}{a^2}$$

it holds $W_\psi^* W_\psi = Id$ and $W_\psi W_\psi^* = P_{range(W_\psi)}$.

The continuous wavelet transform is known in pure mathematics as Calderón's reproducing formula, i.e. for $\psi(x) \in L_1(R^n)$ real and radial with vanishing mean, i.e.

$$\int_0^\infty \frac{|\hat{\psi}(a\omega)|^2}{a} da \equiv 1$$

It holds for $\psi_a(x) := \frac{1}{a^n} \psi(\frac{x}{a})$ Calderón's formula, i.e.

$$f = \int_0^\infty \psi_a * \psi_a * f \frac{da}{a} \quad .$$

Classical Hilbert spaces in complex analysis are examples of wavelets, like Hardy space of L_2 functions on the unit circle with analytical continuation inside the unit disk.

We note that $\varphi'(x^2)$ has a similar structure than the Mexican hut, which is a continuous wavelet function (see remark 1.16 below) $\psi(x) = -\frac{d^2}{dx} e^{-x^2/2} = (1-x^2)e^{-x^2/2} \in L_2(R)$ fulfilling

$$0 < c_\psi := 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \quad .$$