FUNCTIONAL TOPOLOGY AND ABSTRACT VARIATIONAL THEORY

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This paper is concerned with a summary of results which will appear at length in a fasicule in the series "Mémorial des sciences mathématiques," Gauthier-Villars, Paris. Part of the theory is based on certain conceptions and theorems in group theory. These will be summarized first in order to make clear their independence from the rest of the theory.

1. *The Group Theory.*—Let G be an additive abelian operator group with coefficients δ in a field Δ. With certain of the elements u of G we associate an element $\rho(u)$ in a simply ordered set of elements $[\rho]$. The set $[\rho]$ may in particular be the set of real numbers. We term $\rho(u)$ the *rank* of u. The rank $\rho(0)$ shall not be defined. The elements of G with rank (with 0 added) will not in general form a group. A subgroup g of G will be termed an *operator* subgroup if when u is in g, δu is in g. The property A is termed an *operator property* if whenever u has the property A and $\delta \neq 0$, δu has the property A. By a subgroup g of G with property A is meant an operator subgroup every element of which with the possible exception of 0 has the property A. The group g will be termed *maximal* if it is a proper subgroup of no subgroup of G with property A. The ranks $\rho(u)$ shall satisfy the following three conditions:

I—If u has a rank and $\delta \neq 0$, $\rho(u) = \rho(\delta u)$

II—If u, v, and $u + v$ have ranks, then

$$\rho(u + v) \leqq \max\,[\rho(u), \rho(v)]$$

III—If u and v have unequal ranks $\rho(u + v)$ exists.

The proof of the following theorem is due to R. Baer.

THEOREM 1.1. *Let* g *be an operator subgroup of* G *whose dimension is at most alef-null. If each element of* g *save the null element has a rank,* g *is a direct sum of suitably chosen maximal subgroups* g (ρ) *of elements of* g *with the respective ranks* ρ.

The author has proved a similar theorem in the case where the ranks taken in their natural order are well ordered. Without some restriction on g a theorem of this sort cannot be proved. A fourth condition on the ranks is met in practice.

IV—If $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ are elements of G with ranks at most a_0 while the sums $u = \Sigma u_i$ and $v = \Sigma v_j$ have no rank and $u + v$ has a rank, then $\rho(u + v) < a_0$.

We shall say that two elements u and v of G are in the same *rank class* if u and v have the same rank while $u - v$ has no rank or a lesser rank. An isomorphism between two subgroups of G of elements with rank will be termed a *rank isomorphism* if corresponding non-null elements are in the same rank class.

THEOREM 1.2. *When conditions I to IV are satisfied any two maximal subgroups of elements of* G *with the same rank are rank isomorphic.*

This theorem enables us to assign type groups and type numbers to critical sets.

2. *The Space* M *and Function* F.—Let M be a metric space of points p, q, etc. Let $F(p)$ be a real single-valued function of the point p on M, with $0 \leqq F \leqq 1$. By the set $F \leqq b$ is meant the subset of points of M at which $F(p) \leqq b$. Let U be an homology class with elements which are non-bounding k-cycles u. If u is on $F \leqq b$, b will be called a *cycle bound* of u and of U. The greatest lower bound of the cycle bounds of U will be called the *cycle limit* $s(u)$ of U and of the elements u of U. If U is the class of bounding k-cycles $s(u)$ will not be defined. Let G be the group of all k-cycles. With some but not all of the elements u of G we have thus associated a number $s(u)$. We term $s(u)$ the rank of u. These ranks satisfy the four rank conditions of §1. Hence the theorems of §1 hold with the present interpretation. Theorem 1.1 has here the following corollary, considerably weaker than the theorem.

COROLLARY. *The sum of the dimensions of maximal groups,* g(s), *of non-bounding* k*-cycles with the respective cycle limits* s *is at least the smaller of the two numbers alef-null and the* k*th connectivity* R_k *of* M.

Up to the present point it has been immaterial whether ordinary singular cycles are understood or Vietoris cycles. From this point on we shall refer to Vietoris cycles. See M., §2.[1] We shall now state the first of two fundamental hypotheses, that of *F-accessibility*. If Vietoris cycles are used this hypothesis is fulfilled in the ordinary variational theory. It is not in general fulfilled if ordinary singular cycles are used.[2]

Under the hypothesis of F*-accessibility any non-bounding* k*-cycle which is* ~ 0 *mod* $F < c + e$ *for each positive* e, *is homologous to a* k*-cycle on* $F \leqq c$.

A non-bounding k-cycle v whose rank is $s(v)$ and which is on $F \leqq s(v)$, will be termed *canonical*. Under the hypothesis of F-accessibility there is at least one canonical k-cycle in each non-null homology class. If the sets $F \leqq c$ are compact for each $c < 1$ the hypothesis of F-accessibility is satisfied, as we prove.

k-*Caps*. A point set A will be said to be *definitely below* a (written d-below a) if A lies on $F < a - e$ for some positive e. The phase d-mod $F < a$ shall be understood to mean mod some compact set d-below a. If u is a k-cycle on $F \leqq a$, d-mod $F < a$, an homology

$$u \sim 0 \qquad\qquad (\text{on } F \leqq a, \text{ d-mod } F < a)$$

will be called an a-*homology*. A k-cycle u, d-mod $F < a$ on $F \leqq a$, not a-homologous to zero, will be called a k-cap with cap limit a. We write $a = a(u)$. These cap-limits satisfy the four rank conditions of §1.

THEOREM 2.1. *Under the hypothesis of* F-*accessibility a canonical non-bounding* k-*cycle* u *with cycle limit* s(u) *is a* k-*cap with cap limit* s(u).

Let p be a point of M at which $F(p) = c$. The set M will be said to be *locally* F-*connected of order* m > 0 at p if corresponding to each positive constant e there exists a positive constant δ such that each singular $(m-1)$-sphere on the δ-neighborhood of p and on $F \leqq c + \delta$ bounds an m-cell of diameter less than e on $F \leqq c + e$. If each subset $F \leqq c < 1$ of M is compact and if M is locally F-connected of all orders from 1 to $m + 1$ at points of $F < 1$, then the dimension of the mth homology group of $F < 1$ is at most alef-null, and the cycle limits $s(u)$ have at most the cluster value 1.

3. *Homotopic Critical Points.*—We shall say that a continuous deformation D of a subset A of M admits a *displacement function* $\delta(e)$ on A, if whenever q precedes r on a trajectory of D and $qr > e > 0$, then $F(q) - F(r) > \delta(e)$, where $\delta(e)$ is a positive single-valued function of e. A continuous deformation of E which possesses a displacement function on each compact subset of E is termed an F-*deformation* of E. A point p will be termed *homotopically* ordinary if some neighborhood of p relative to $F \leqq F(p)$ admits an F-deformation which displaces p. A point p which is not homotopically ordinary is termed *homotopically critical.*

The function F will be said to be *upper-reducible* at p if corresponding to each constant $c > F(p)$ some neighborhood of p relative to $F \leqq c$ admits on F-deformation onto a set d-below c. A function F which is lower semi-continuous is not necessarily upper-reducible, and conversely. We have the following principal theorem.

THEOREM 3.1. *If* F *is upper-reducible at each point, each cap limit is assumed by* F *in at least one homotopic critical point.*

If then the hypothesis of F-accessibility is satisfied and F is upper-reducible each cycle limit $s(u)$ is assumed by F at some homotopic critical point. This should be contrasted with the following theorem: When the space M is compact and F is lower semi-continuous, the absolute minimum of F is assumed at some critical point. As ever $F \geqq 0$.

By the *complete critical* set ω at the level c is meant the set of all homotopic critical points at which $F = c$. Any subset σ of ω which is closed in ω and at a positive distance from ω–σ will be termed a *critical set.* A k-cap u with cap limit c will be said to be *associated* with σ if u is c-homologous to a k-cap on an arbitrarily small neighborhood of σ. A maximal group of k-caps associated with σ will be called the kth *type group* of σ. Any two kth type groups of σ are rank isomorphic (with the ranks the cap limits). The dimension of a kth type group of σ is termed the kth *type number* of σ. A kth type group of ω can be obtained as a direct sum of the kth type groups of any finite set of disjunct critical sets summing to ω. We have the following theorem:

THEOREM 3.2. *If* M *is* F*-accessible and* F *is upper-reducible on* F < 1, *the sum of the* k*th type numbers of the respective critical sets on* F < 1 *is at least the smaller of the two cardinal numbers, alef-null and the* k*th connectivity of* F < 1.

In the special case where F is locally a function of class C'' of n coördinates, and p is a critical point at which the Hessian of F is not zero, the jth type number of p equals the Kronecker δ_k^j, where k is the number of negative characteristic roots of the Hessian of F at p.

4. *Variational Theory.*—We apply the preceding theory to the problem of finding extremals joining two points a and b of a connected space Σ with a symmetric metric pq. The space of all sensed curves joining a to b on Σ with a Frêchet distance between curves will be denoted by $\Omega(a, b)$. The space $\Omega(a, b)$ here replaces M. We begin by showing that the kth homology group of $\Omega(a, b)$ is isomorphic with that of $\Omega(a', b')$, provided Σ is arcwise connected. To define F on Ω we suppose that we have a second metric $[pq]$ defined for p and q on Σ. We do not assume that $[pq] = [qp]$. Otherwise $[pq]$ shall satisfy the usual axioms. We assume that $[pq]$ is continuous in p and q in terms of the first metric pq. The function $J(\lambda)$ shall be the length of the curve λ of $\Omega(a, b)$ defined in the usual way in terms of the second metric $[pq]$. We set

$$F(\lambda) = \frac{J(\lambda)}{1 + J(\lambda)}$$

with $F(\lambda) = 1$ when $J(\lambda)$ is infinite.

We assume that Σ is *finitely* J*-compact* in that for each fixed point p of Σ and finite constant c, the subset $[pq] \leqq c$ of Σ is compact. It follows that the hypothesis of F-accessibility is satisfied on $\Omega(a, b)$. A simple sensed curve λ joining two points p and q of Σ will be termed a *right arc* if a point r lies on λ when and only when $[pq] = [pr] + [rq]$. We assume that Σ is *locally* J*-convex* in the following sense. With each point p of Σ there shall be associated a positive number $\rho(p)$ continuous in p and such that when $q \neq p$ and $[pq] \leqq \rho(p)$, p can be joined to q on Σ by a right arc every subarc of which is a right arc. It follows that F is upper-reducible on the subspace $F < 1$. This is sufficient for our purposes. A curve h will be called a *metric extremal* provided every closed subarc of h whose J-length is sufficiently small is a right arc. Regarded as a curve each homotopic critical point of F will be called a *homotopic extremal.* We have the following fundamental theorem.

THEOREM 4.1. *Each homotopic extremal of* Ω(a, b) *is a metric extremal.*

We also show that Ω (a, b) is locally F-connected, that all cycle limits are less than 1, and that the subsets $F \leqq c < 1$ are compact. The preceding theory is readily applied to the calculus of variations with the usual positive, and positive regular integrand in parametric form.

[1] Morse, "Functional Topology and Abstract Variational Theory," *Annals of Mathematics*, **38**, 386–449 (1937). We refer to this paper by the letter M. Complete references are given in this paper and in the fasicule to appear later.

[2] The following book will appear shortly: Seifert und Threlfall, *Variationsrechnung im Grossen. Theorie von Marston Morse*. Teubner, Berlin. This book is highly recommended. The authors begin with two axioms similar to our accessibility hypothesis, but referring to singular cycles. These axioms are satisfied when the critical values cluster at most at infinity and when the critical points are isolated. In this way the most important cases are treated in the simplest way. To obtain greater generality Vietoris cycles seem to be useful. In fact the present author has shown in 3 (following) that the accessibility hypothesis is not in general satisfied when ordinary cycles are used, even when f is of class C^n on regular analytic manifolds and when the critical values are finite in number.

[3] Morse, "Sur le calcul des variations," *Bull. Societé Mathématique de France* (1938).

ON CRITERIA CONCERNING SINGULAR INTEGERS IN CYCLOTOMIC FIELDS

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As elsewhere[1] a singular integer is defined as an integer α in the field $k(\zeta)$; $\zeta = e^{2i\pi/l}$, l an odd prime, such that $\alpha = \mathfrak{a}^l$ where $\mathfrak{a}$ is an ideal in $k(\zeta)$ which is not principal. Necessary conditions that an integer in $k(\zeta)$ be singular were given by Takagi,[2] and when the field $k(\zeta)$ is properly irregular, that is to say, when the second factor of its class number is prime to l; a necessary and sufficient condition was given by the writer.[1] Here we shall give some other necessary conditions for singular integers in any irregular cyclotomic field. Based on a result of Kummer's the writer[3] obtained the relation

$$\prod_{\nu=1}^{k-1} \prod_{r=1}^{[\nu l/k]} \mathfrak{b}(\zeta^{r_1}) \sim 1, \tag{1}$$

that is, the ideal on the left is principal, where k is an integer $1 < k < l$; $rr_1 \equiv 1 \pmod{l}$; $[s]$ is greatest integer in s, and $\mathfrak{b}$ is any integer in $k(\zeta)$ and $\mathfrak{b}(\zeta^t)$ is obtained from $\mathfrak{b}(\zeta)$ by the substitution (ζ/ζ^t); and it follows if we assume that (Vandiver[3]) α is singular and semi-primary, then

$$\prod_{\nu=1}^{k-1} \prod_{r=1}^{[\nu l/k]} \alpha(\zeta^{r_1}) = \omega^l$$