INTERPOLATION BY ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND THEIR APPLICATION TO EXPANSION IN EXPONENTIAL SERIES

B. Ya. Levin and Yu. I. Lyubarskii

Let D be a closed convex polygon in **C** (a segment is considered to be a special case of D). We assume that the origin belongs to D. From the origin we drop normals to the sides $l_1, l_2, \ldots, l_n$ and let these normals form with the positive real axis the angles $\theta_1 < \theta_2 < \ldots < \theta_n$. The vertices of D are denoted by w_j ($w_j \in l_j, l_{j+1}$). By D* we denote the polygon which is symmetric to D with respect to the real axis; by l_j^* and w_j^* we denote the sides and vertices of this polygon. Let $h(\theta)$ be the support function of D and let $H(z) = |z|h(\arg z)$ be its circular support function.

When $k > 0$ we set $\Pi_j(K) = \{z : \mathrm{Re}\, ze^{-i\theta_j} > 0, |\mathrm{Im}\, ze^{-i\theta_j}| < K\}$. We call the union of all semistrips $\Pi_j(k)$ the D_K-star.

<u>Definition.</u> An entire function of exponential type (e.f.e.t.) S(z) is said to belong to the class S_D if for some constants c, C, K

$$0 < c < |S(z)| \exp[-H(z)] < C < \infty \quad (z \notin D_K). \tag{1}$$

An example of a function of class S_D is given by any function of the form

$$S(z) = \int_{\partial D^*} e^{\zeta z} d\sigma(\zeta), \tag{2}$$

where $\sigma(\zeta)$ has bounded variation on ∂D^* and jumps at the vertices w_j^*. It is possible to construct examples of functions in the class S_D which cannot be represented in the form (2).

From now on $\{z_k\}$ will denote the sequence of zeros ($|z_{k+1}| \geq |z_k|$) of some function $S(z) \in S_D$. For simplicity we assume that $\inf\{|z_k - z_j|, k \neq j\} > 0$.

Consider the space L_D^p, $p \in [1, \infty]$ consisting of those e.f.e.t. F(z) with norm

$$\|F\|_{L_D^p} = \sup_j \left\{\int_0^\infty |F(re^{i\theta_j})|^p e^{-prh(\theta_j)} dr\right\}^{\frac{1}{p}} < \infty.$$

We note that if D is the interval $[-i\pi, i\pi]$ of the imaginary axis, then the class S_D reduces to the well known (see [1, 2, and 3]) class of functions of sine type while L_D^p reduces to the space L_π^p of e.f.e.t. π belonging to $L^p(-\infty, \infty)$ on the real axis.

<u>THEOREM 1.</u> The operator T, defined by the relation $T[F] = \{F(z_k)\exp[-H(z_k)]\}$ maps L_D^p isomorphically into l^p when $p \in (1, \infty)$. The inverse operator is defined by the series

---

Physicotechnical Institute of Low Temperatures, Academy of Sciences of the Ukrainian SSR. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 8, No. 2, pp. 85-86, April-June, 1974. Original article submitted October 22, 1973.

$$F(z) = T^{-1}[\{c_k\}](z) = S(z) \sum \frac{c_k \exp H(z_k)}{S'(z_k)(z - z_k)}, \tag{3}$$

which converges in the L_D^p norm.

For the case $D = [-i\pi, i\pi]$ this theorem was proved in [3]. We note that when $p = 1$ or ∞ there need not be a function from L_D^p solving the corresponding interpolation problem. Holding true are some generalizations of theorems of M. Cartwright, G. Polya, and A. Plancherel (see for example [4] and also [3]).

THEOREM 2. Let $\omega_j = \arg w_j$ ($j = 1, 2, \ldots, n$) and let $F(z)$ be an e.f.e.t. such that $|F(re^{i\omega_j})| = O(\exp[r(h(\omega_j) - \varepsilon)])$ for some $\varepsilon > 0$ and $\{F(z_k)\exp[-H(z_k)]\} \in l^p$ $(1 < p \leqslant \infty)$. Then $F \in L_D^p$.

When $p < \infty$ the condition on the growth of $F(z)$ can be replaced by the weaker condition: $F(re^{i\omega_j})\exp[-rh(\omega_j)] \to 0$ as $r \to \infty$, $j = 1, 2, \ldots, n$.

A theorem on the expansion of functions from E_2 of the Smirnov space in the polygon D^* (for the definition see [5]) follows from Theorem 1 when $p = 2$. Functions from E_2 are homomorphic in D^* and have limit values almost everywhere on ∂D^*. The norm in E_2 is defined by $\|\varphi\|_{E_2}^2 = \int_{\partial D^*} |\varphi(\zeta)|^2 |d\zeta|$. When $D = [-i\pi, i\pi]$ the space E_2 becomes $L^2(-i\pi, i\pi)$.

THEOREM 3.† The system of functions

$$\{e^{-H(z_k)} e^{\zeta z_k}\} \tag{4}$$

is a Riesz basis in E_2; that is, every function $\varphi \in E_2$ can be expanded in a series

$$\varphi(\zeta) = \sum \alpha_k e^{-H(z_k)} e^{\zeta z_k}, \tag{5}$$

converging in the E_2 norm, where $\{\alpha_k\} \in l^2$ and $\|\{\alpha_k\}\|_{l^2} \asymp \|\varphi\|_{E_2}$.

Remark. A system biorthogonal to (4) is formed by the limit values on ∂D^* of the Borel transformations of the functions $S(z)[S'(z_k)(z - z_k)]^{-1}$. The existence and membership in $L^2(\partial D^*)$ of these limit values were proved in [4].

The following theorems prove that expansions with respect to system (4) lead one into the same series of cases as the usual Fourier series.

THEOREM 4.‡ Let $a \in \{z_k\}$ and $\psi_a(\zeta)$ be the Borel transform of the function $S(z)(z - a)^{-1}$. In order that the coefficients of the series (5) have the form $\alpha_k = \beta_k k^{-m}$ with $\{\beta_k\} \in l^2$ it is necessary and sufficient that there exist $\varphi^{(m)}(\zeta) \in E_2$ and that

$$\int_{\partial D^*} e^{a\zeta} [\varphi(\zeta) e^{-a\zeta}]^{(p)} \psi_a(\zeta)\, d\zeta = 0, \quad p = 1, 2, \ldots, m. \tag{6}$$

For $q < 1$ let $U_q = \{z : \pi \operatorname{Re} z e^{i\theta_j} < -d_j \ln q,\ j = 1, 2, \ldots, n\}$. In this case d_j is the length of the side l_j.

THEOREM 5. Let α_k be the coefficients of series (5). In order that $|\alpha_k| < C(q)q^k$, $q_0 < q < 1$, it is necessary and sufficient that for any $\mu \in U_{q_0}$ the function $\varphi(\zeta + \mu) \in E_2$ and $\int_{\partial D^*} e^{a\zeta} [\varphi(\zeta + \mu) e^{-a\zeta}] \psi_a(\zeta)\, d\zeta = 0.$

†When D is an interval the theorem was proved by B. Ya. Levin [1, 3] and V. D. Golovin [2]. Pavlov [6] gave a new proof based on the theory of differential operators. In an unpublished paper of V. D. Golovin expansion (5) was proved without the hypothesis $\|\{\alpha_k\}\|_{l^2} \asymp \|\varphi\|_{E_2}$.

‡This theorem and the corollary to Theorem 6 are closely connected with results of A. F. Leont'ev obtained in a series of papers on the convergence of a series of Dirichlet functions holomorphic in an arbitrary convex region G. In particular, in [7, 8] he proved: if G is a polygon, the function $S(z)$ has the form (2), the function $\varphi(\zeta)$ is holomorphic in G and is m times continuously differentiable up to the boundary, and $I = \int_{\partial G} \varphi^{(k)}(\zeta)\, d\sigma(\zeta) = 0$, $k = 0, 1, \ldots, m-1$, then $\alpha = Q(k^{-m})$. If $I \neq 0$ and $m = 1$, then series (5) converges uniformly in any compacta in G not containing a vertex of G.

By the partial sum $\sigma_R(S, \varphi, z)$ of series (5) we mean the sum of all of its terms for which $|z| < R$.

THEOREM 6. Let $S_1, S_2 \in S_D$, $\varphi \in E_2$ and $G \subset D^*$ be an arbitrary compacta not containing a vertex of D^*. Then $|\sigma_R(S_1, \varphi, z) - \sigma_R(S_2, \varphi, z)| \rightrightarrows 0$ when $R \to \infty$, $z \in G$.

Theorem 6 is proved by the variation method of N. Levinson (see [9]).

COROLLARY. Let the function $\varphi \in E_2$ be such that $|\varphi(z+h) - \varphi(z)| < \mathrm{const}\cdot h^{\alpha}$, $\alpha > 0.5$. Then series (5) converges uniformly in G.

Convergence tests for series (5) similar to the type of theorems of Jordan–Dirichlet, Dini, and others follow from the above results. When $D = [-i\pi, i\pi]$ these were proved by A. M. Sedletskii [10] by a simple application of N. Levinson's method.

LITERATURE CITED

1. B. Ya. Levin, Zap. Mekh.-Matem. Fak. KhGU i Khar'k. Matem. Ob-va, 27, 39-48 (1961).
2. V. D. Golovin, Zap. Mekh.-Matem. Fak. KhGU i Khar'k. Matem. Ob-va, 30, 18-29 (1964).
3. B. Ya. Levin, Matem. Fizika i Funkts. Analiz, FTINT Akad. Nauk USSR, Khar'kov, 1, 136-146 (1969).
4. B. Ya. Levin, Distribution of Zeros of Entire Functions [in Russian], Gostekhizdat, Moscow (1956).
5. I. I. Privalov, Boundary Properties of Analytic Functions, Gostekhizdat, Moscow–Leningrad (1950).
6. B. S. Pavlov, Problemy Matem. Fiziki, LGU, No. 6, 101-119 (1973).
7. A. F. Leont'ev, Izv. Akad. Nauk SSSR, Ser. Matem., 37, 577-592 (1973).
8. A. F. Leont'ev, Izv. Akad. Nauk SSSR, Ser. Matem., 38, 127-137 (1974).
9. N. Levinson, "Gap and density theorems," Amer. Math. Soc. Coll. Publ., New York (1940).
10. A. M. Sedletskii, Sib. Matem. Zh., 12, 1100-1114 (1971).