Bessel-integral functions

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§ 1. *Summary.* In a very remarkable work on the operational Calculus, Dr Balth. van der Pol[1] has introduced a new function, playing with respect to Bessel function of order zero the same part as the cosine- or sine-integral with respect to the ordinary cosine or sine. He showed that this function—which he called *Bessel-integral function*—can be used to express the differential coefficient of any Bessel function with respect to its index. But he did not investigate the further properties of his new function. I propose to give here some of them, which appear to be interesting, and to introduce and study the functions connected, in the same way, with Bessel functions of any order.

§ 2. *Definition of the function of order zero.*

It is well known that the *cosine-integral*[2] is defined by the formula

$$\mathrm{ci}\,(x) = -\int_x^\infty \frac{\cos u}{u}\,du,$$

and the *sine-integral* by

$$\mathrm{si}\,(x) = -\int_x^\infty \frac{\sin u}{u}\,du = \int_0^x \frac{\sin u}{u}\,du - \frac{\pi}{2}.$$

Following Dr Van der Pol,[3] we shall define the *Bessel-integral function of order zero* by

$$Ji_0\,(x) = -\int_x^\infty \frac{J_0\,(u)}{u}\,du,$$

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[1] *Philosophical Magazine*, 8 (1929), 861-898, (887).

[2] For properties of cosine- and sine-integrals, the reader is referred to Niels Nielsen, *Theorie des Integrallogarithmus* (Leipzig, 1906). Nielsen's notation will be used through this paper. As regards Bessel functions, we shall follow Watson's notation.

[3] Dr van der Pol uses the simple notation $Ji(x)$. As we shall deal with functions connected with the Bessel function of order n, we find it convenient to denote van der Pol's function by $Ji_0\,(x)$, thus introducing the *order* of Bessel-integral functions.

where J_0 is Bessel function of order zero. It is readily seen that this integral is convergent. We shall now investigate the properties of this function.

§ 3. *Integral representation and various expressions.*

Let us start from Parseval's integral for the Bessel function of order zero, namely

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin\theta)\, d\theta.$$

Writing u in place of x, dividing both members by u, and integrating with respect to u from x to ∞, we obtain

$$\int_x^\infty \frac{J_0(u)}{u}\, du = \frac{1}{\pi}\int_x^\infty \frac{du}{u}\int_0^\pi \cos(u \sin\theta)\, d\theta$$

or

$$Ji_0(x) = -\frac{1}{\pi}\int_0^\pi d\theta \int_x^\infty \frac{\cos(u \sin\theta)}{u}\, du.$$

But we have

$$-\int_x^\infty \frac{\cos(u\sin\theta)}{u}\, du = -\int_x^\infty \frac{\cos(u\sin\theta)}{u\sin\theta}\, d(u\sin\theta)$$
$$= \operatorname{ci}(x\sin\theta).$$

Hence the interesting formula, a generalization of Parseval's integral,

$$Ji_0(x) = \frac{1}{\pi}\int_0^\pi \operatorname{ci}(x\sin\theta)\, d\theta. \tag{1}$$

Various results can be deduced from it. It is known that

$$\operatorname{ci}(x) = \gamma + \log x + \int_0^x \frac{\cos t - 1}{t}\, dt, \tag{2}$$

γ being Euler's constant, so that

$$\operatorname{ci}(x\sin\theta) = \gamma + \log(x\sin\theta) + \int_0^{x\sin\theta} \frac{\cos t - 1}{t}\, dt,$$
$$= \gamma + \log(x\sin\theta) + \int_0^x \frac{\cos(u\sin\theta) - 1}{u}\, du.$$

If we integrate from 0 to π with respect to θ, we obtain, after dividing by π,

$$Ji_0(x) = \gamma + \frac{1}{\pi}\int_0^\pi \log(x\sin\theta)\, d\theta + \int_0^x \frac{J_0(u) - 1}{u}\, du.$$

Now

$$\frac{1}{\pi}\int_0^\pi \log(x\sin\theta)\, d\theta = \log x + \frac{1}{\pi}\int_0^\pi \log\sin\theta\, d\theta$$

and a classical result is

$$\frac{1}{\pi}\int_0^\pi \log \sin\theta \, d\theta = -\log 2.$$

So we have the formula

$$Ji_0(x) = \gamma + \log \frac{x}{2} + \int_0^x \frac{J_0(u) - 1}{u} du. \qquad (3)$$

From (2) and (3) we have

$$Ji_0(x) - \mathrm{ci}(x) = \int_0^x \frac{J_0(u) - \cos u}{u} du - \log 2, \qquad (4)$$

so that

$$Ji_0(0) - \mathrm{ci}(0) = -\log 2,$$

a result found by Dr van der Pol, and written by him in the following form

$$\int_0^\infty \frac{J_0(x) - \cos x}{x} dx = \log 2.$$

We can easily generalize it by writing

$$\int_0^\infty \frac{J_0(ax) - \cos bx}{x} dx = \log \frac{2b}{a}.$$

From formula (3), we can obtain an expansion for Ji_0, by expanding the Bessel function J_0 in ascending powers of u, and integrating: we find

$$Ji_0(x) = \gamma + \log \frac{x}{2} + \sum_{s=1}^{\infty} \frac{(-1)^s (\frac{1}{2}x)^{2s}}{s!\, s!\, 2s}$$

or, introducing a hypergeometric function,

$$Ji_0(x) = \gamma + \log \tfrac{1}{2}x - \tfrac{1}{8}x^2 \, {}_2F_3(1,\ 1;\ 2,\ 2,\ 2;\ -x^2/4).$$

Using some results stated by Nielsen on the cosine-integral, we can also obtain, through an integrating process the following results

$$Ji_0(x) - \log x = \frac{2x \sin \pi x}{\pi}(\gamma - \log 2) + \frac{2x \sin \pi x}{\pi} \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s^2 - x^2} [Ji_0(s) - \log s],$$

$$\lim_{n \to \infty} \sum_{s=n+1}^{\infty} \frac{J_0(sx/n)}{s} = -Ji_0(x).$$

§ 4. *Function of second kind.* If we apply the same integrating process to the formula

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin(x \cosh t)\, dt,$$

we find readily

$$Ji_0(x) = \frac{2}{\pi} \int_0^\infty \operatorname{si}(x \cosh t)\, dt.$$

Now, Y_0 being Bessel function of second kind, we have

$$Y_0(x) = -\frac{2}{\pi} \int_0^\infty \cos(x \cosh t)\, dt$$

so that if we introduce the new function

$$Yi_0(x) = -\int_x^\infty \frac{Y_0(u)}{u}\, du,$$

we shall be able to write

$$Yi_0(x) = -\frac{2}{\pi} \int_0^\infty \operatorname{ci}(x \cosh t)\, dt.$$

This function $Yi_0(x)$ can be called *Bessel-integral function of second kind* (and order zero).

We have the two important formulae, the first of which can be immediately deduced from (3),

$$\frac{d\, Ji_0(x)}{dx} = \frac{J_0(x)}{x},$$

$$\frac{d\, Yi_0(x)}{dx} = \frac{Y_0(x)}{x}.$$

Using them, it is easy to write the differential equation of the third order satisfied by Ji_0: we find

$$x^2 y''' + 3xy'' + (x^2 + 1)\, y' = 0,$$

and this equation has the three independent solutions $Ji_0(x)$, $Yi_0(x)$ and constant, just as the equation

$$xy''' + 2y'' + xy' = 0$$

is satisfied by $\operatorname{ci}(x)$, $\operatorname{si}(x)$ and constant.

§ 5. *Properties of definite integrals.* We can use the Ji_0 function to express some integrals connected with Bessel functions. For instance, let us consider the integral

$$\int_0^x J_1(t) \log t\, dt,$$

where J_1 is Bessel function of order unity. As we have

$$J_1(t) = -J'_0(t),$$

we can integrate by parts, and use formula (3) to obtain

$$\int_0^x J_1(t) \log t \, dt = Ji_0(x) - J_0(x) \log x - \gamma + \log 2.$$

The same process of integrating by parts enables us to find the value of the integral

$$\int_0^x Ji_0(t) \, dt$$

which can be written

$$x \, Ji_0(x) - \int_0^x J_0(t) \, dt$$

so that we have

$$\int_0^x Ji_0(t) \, dt = x \, Ji_0(x) - 2 \left[J_1(x) + J_3(x) + J_5(x) + \ldots . \right].$$

§ 6. *Functions of order n.* We shall now introduce *Bessel-integral functions of order n,* by the formula

$$Ji_n(x) = -\int_x^\infty \frac{J_n(u)}{u} du. \tag{5}$$

As Weber showed, we have

$$\int_0^\infty \frac{J_n(u)}{u} du = \frac{1}{n},$$

so that we can write

$$Ji_n(x) = \int_0^x \frac{J_n(u)}{u} du - \frac{1}{n}. \tag{6}$$

This integral is finite, $J_n(u)/u$ containing only integer powers of u: and we can infer from it that the integral in (5) is convergent.

These two definitions lead us to a number of formulae introducing the Ji functions: we shall write some of them without giving the demonstrations, which are easy.

For instance, using one of the recurrence-formulæ of Bessel functions, we obtain

$$(n-1) \, Ji_{n-1}(x) - (n+1) \, Ji_{n+1}(x) = \frac{2n}{x} J_n(x);$$

and, as

$$\frac{d \, Ji_n(x)}{dx} = \frac{J_n(x)}{x},$$

we can write

$$(n-1)\,Ji_{n-1}(x)-(n+1)\,Ji_{n+1}(x)=2n\,Ji'_n(x),$$

which is a recurrence-formula between Bessel-integral functions and their derivates.

In the particular case $n=1$, it gives

$$Ji_2(x)=-\frac{J_1(x)}{x}. \tag{7}$$

From the recurrence-formula and the definition, we can obtain the following expansions

$$\begin{aligned}
nJi_n(x) &= -1+J_n(x)+2\,[J_{n+2}(x)+J_{n+4}(x)+\ldots.],\\
mJi_{2m} &= -\frac{1}{x}\,[J_1+3J_3+\ldots.+(2m-1)\,J_{2m-1}],\\
mJi_{2m} &= -\tfrac{1}{2}J_0-J_2-J_4-\ldots.-J_{2m-2}-\tfrac{1}{2}J_{2m},\\
(2m+1)\,Ji_{2m+1} &= -Ji_1+\frac{4}{x}\,(J_2+2J_4+\ldots.+mJ_{2m}),
\end{aligned}$$

and the two results

$$\begin{aligned}
&Ji_{2p}(x)=Ji_{2p}(-x),\\
&Ji_{2p+1}(x)+Ji_{2p+1}(-x)=-\frac{2}{2p+1}.
\end{aligned}$$

Formula (5) can be extended to Bessel functions whose order is not an integer. Specially interesting is the case $n=\frac{1}{2}$. We have then

$$Ji_{\frac{1}{2}}(x)=-\int_x^\infty \frac{J_{\frac{1}{2}}(u)}{u}\,du=-\sqrt{\frac{2}{\pi}}\int_x^\infty \frac{\sin u}{u\sqrt{u}}\,du.$$

If we integrate by parts, we obtain

$$Ji_{\frac{1}{2}}(x)=-2\sqrt{\frac{2}{\pi}}\;\frac{\sin x}{\sqrt{x}}-2\sqrt{\frac{2}{\pi}}\int_x^\infty \frac{\cos u}{\sqrt{u}}\,du.$$

In this last integral, let us make $u=t^2$: it becomes

$$2\int_{\sqrt{x}}^\infty \cos t^2\,dt.$$

But this is one of Fresnel integrals,

$$C(x)=\int_x^\infty \cos t^2\,dt$$

so that we have

$$Ji_{\frac{1}{2}}(x) = -2\sqrt{\frac{2}{\pi}}\,\frac{\sin x}{\sqrt{x}} - 4\sqrt{\frac{2}{\pi}}\,C(\sqrt{x})$$

a formula which can also be written

$$Ji_{\frac{1}{2}}(x) = -2J_{\frac{1}{2}}(x) - 4\sqrt{\frac{2}{\pi}}\,C(\sqrt{x}).$$

§ 7. *Integral representations.* If we start from the well known formulae for Bessel functions

$$J_{2p}(x) = \frac{1}{\pi}\int_0^\pi \cos 2p\theta \cos(x\sin\theta)\,d\theta,$$

$$J_{2p+1}(x) = \frac{1}{\pi}\int_0^\pi \sin(2p+1)\theta \sin(x\sin\theta)\,d\theta,$$

we obtain, following the integrating process already used, the two expressions

$$Ji_{2p}(x) = \frac{1}{\pi}\int_0^\pi \cos 2p\theta \,\mathrm{ci}(x\sin\theta)\,d\theta,$$

$$Ji_{2p+1}(x) = \frac{1}{\pi}\int_0^\pi \sin(2p+1)\theta \,\mathrm{si}(x\sin\theta)\,d\theta.$$

If, in the first of these formulae, we take $p=1$, we obtain an expression for Ji_2, *i.e.*, through (7), for J_1, namely

$$J_1(x) = -\frac{x}{\pi}\int_0^\pi \cos 2\theta \,\mathrm{ci}(x\sin\theta)\,d\theta.$$

This formula, expressing Bessel function of order unity by use of the cosine-integral, seems to be new.

§ 8. *Expansions in series of Bessel-integral functions.* A number of interesting expansions containing Bessel-integral functions can be found by transforming some known properties of Bessel functions.

Let us start from the Schlömilch formula

$$e^{\frac{1}{2}u(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(u).$$

Dividing by u and integrating with respect to u, we have

$$\int_x^\infty \frac{e^{\frac{1}{2}u(t-1/t)}}{u}\,du = \Sigma\, t^n \int_x^\infty \frac{J_n(u)}{u}\,du$$
$$= -\Sigma\, t^n Ji_n(x).$$

But we know that the *logarithmic-integral function* is defined by

$$\mathrm{li}\,(e^{ax}) = -\int_x^\infty \frac{e^{au}}{u}\,du,$$

and we can write the expansion

$$\mathrm{li}\,\{e^{\frac{1}{2}x(t-1/t)}\} = \sum_{n=-\infty}^{+\infty} t^n\, Ji_n(x),$$

a curious generalization of the Schlömilch formula.

Considering now Neumann's expansions

$$\cos x = J_0(x) - 2\,J_2(x) + 2\,J_4(x) - \ldots.,$$
$$\sin x = 2\,J_1(x) - 2\,J_3(x) + 2J_5(x) - \ldots.,$$

we obtain, using again the integrating process, the two following expansions for the cosine- and sine-integrals

$$\mathrm{ci}\,(x) = Ji_0(x) - 2Ji_2(x) + 2Ji_4(x) - \ldots.,$$
$$\mathrm{si}\,(x) = 2Ji_1(x) - 2Ji_3(x) + 2Ji_5(x) - \ldots.$$

Similarly, using another Neumann's expansion for the function $\cos(x \sin\theta)$, we are able to write

$$\mathrm{ci}\,(x\sin\theta) = Ji_0(x) + 2\,Ji_2(x)\cos 2\theta + 2\,Ji_4(x)\cos 4\theta + \ldots.,$$

or

$$\mathrm{ci}\,(x\cos\theta) = Ji_0(x) - 2\,Ji_2(x)\cos 2\theta + 2\,Ji_4(x)\cos 4\theta - \ldots.,$$

and analogous expansions for the sine-integral.

If, in the last formula, we take $\theta = \frac{\pi}{3}$, we have

$$\mathrm{ci}\,\frac{x}{2} = Ji_0(x) + Ji_2(x) - Ji_4(x) + Ji_6(x) - Ji_8(x) + \ldots.,$$

and if we take $\theta = \frac{\pi}{4}$,

$$\mathrm{ci}\,\frac{x}{\sqrt{2}} = Ji_0(x) - 2\,Ji_4(x) + 2\,Ji_8(x) - 2\,Ji_{12}(x) + \ldots..$$

Now, considering the expansion for $\mathrm{ci}\,(x\sin\theta)$, and differentiating both members with respect to θ, we obtain an expansion for

$$\cos(x\sin\theta)\cot\theta,$$

which, making $\theta = \frac{\pi}{4}$, gives

$$\cos\frac{x}{\sqrt{2}} = -4\,[Ji_2(x) - 3\,Ji_6(x) + 5\,Ji_{10}(x) - \ldots.].$$

If in the expansion obtained above for ci $(x \cos\theta)$ we write $\cos\theta = a$, and if we introduce Gegenbauer's polynomial of order zero, by

$$\cos n\theta = nC_n^0(a),$$

we have

$$\mathrm{ci}\,(ax) = Ji_0(x) - 4\,C_2^0(a)\,Ji_2(x) + 8\,C_4^0(a)\,Ji_4(x) - \ldots.$$

§ 9. *Integral properties.* Nielsen has established the following integral formulae[1]

$$\int_0^\infty \mathrm{si}\,(t)\,J_0\,(2\sqrt{tx})\,dt = -\,\frac{\sin x}{x}$$

$$\int_0^\infty \mathrm{ci}\,(t)\,J_0\,(2\sqrt{tx})\,dt = \frac{\cos x - 1}{x}$$

$$\int_0^\infty \mathrm{si}\,(t)\,J_1\,(2\sqrt{tx})\,\frac{dt}{\sqrt{t}} = -\,\frac{\frac{\pi}{2} + \mathrm{si}\,(x)}{\sqrt{x}}$$

$$\int_0^\infty \mathrm{ci}\,(t)\,J_1\,(2\sqrt{tx})\,\frac{dt}{\sqrt{t}} = \frac{\mathrm{ci}\,(x) - \gamma - \log x}{\sqrt{x}}$$

Dividing by x and integrating, we obtain

$$\int_0^\infty \mathrm{si}\,(t)\,Ji_0\,(2\sqrt{tx})\,dt = \frac{\sin x}{2x} - \frac{\mathrm{ci}\,(x)}{2}$$

$$\int_0^\infty \mathrm{ci}\,(t)\,Ji_0\,(2\sqrt{tx})\,dt = \frac{1 - \cos x}{2x} - \frac{\mathrm{si}\,(x)}{2}$$

$$\int_0^\infty \mathrm{si}\,(t) Ji_1(2\sqrt{tx})\,\frac{dt}{\sqrt{t}} = \frac{\frac{\pi}{2} + \mathrm{si}\,(x) + 2\,\sin x}{\sqrt{x}} + 4\,C\,(\sqrt{x})$$

$$\int_0^\infty \mathrm{ci}\,(t) Ji_1(2\sqrt{tx})\,\frac{dt}{\sqrt{t}} = \frac{\gamma + 2 + \log x - \mathrm{ci}\,(x) - 2\,\cos x}{\sqrt{x}} + 4\,S\,(\sqrt{x})$$

C and S being Fresnel integrals,

$$C\,(x) = \int_x^\infty \cos t^2\,dt, \quad S\,(x) = \int_x^\infty \sin t^2\,dt.$$

[1] The third of these formulae is given in Nielsen's book with a slight mistake: he wrote $\frac{\pi}{2} - \mathrm{si}\,(x)$ instead of $\frac{\pi}{2} + \mathrm{si}\,(x)$, which is the real value.

Let us also write the following integral, which can be derived from a similar one containing J_0,

$$\int_0^\infty e^{-at}\, Ji_0(bt)\, dt = \frac{1}{2a} \log \frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}+a}.$$

And, as a conclusion, we add the result obtained by Dr van der Pol

$$\frac{\partial J_n(x)}{\partial n} = \int_0^x Ji_0(x-t)\, J'_n(t)\, dt$$

which holds for $n \geqq 1$.