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Exact Nonlinear Plasma Oscillations

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The problem of a one-dimensional stationary nonlinear electrostatic wave in a plasma free from interparticle collisions is solved exactly by elementary means. It is demonstrated that, by adding appropriate numbers of particles trapped in the potential-energy troughs, essentially arbitrary traveling wave solutions can be constructed.

When one passes to the limit of small-amplitude waves it turns out that the distribution function does not possess an expansion whose first term is linear in the amplitude, as is conventionally assumed. This disparity is associated with the trapped particles. It is possible, however, to salvage the usual linearized theory by admitting singular distribution functions. These, of course, do not exhibit Landau damping, which is associated with the restriction to well-behaved distribution functions.

The possible existence of such waves in an actual plasma will depend on factors ignored in this paper, as in most previous works, namely interparticle collisions, and the stability of the solutions against various types of perturbations.

I. INTRODUCTION

PAST treatments[1–3] of the problem of electrostatic oscillations in a collision-free, completely ionized plasma have customarily resorted to linearization of the governing equations in order to obtain a mathematically tractable problem. This procedure has led, for the case of sinusoidal waves, to mathematical difficulties associated with those particles which travel with the wave velocity. These problems are discussed and resolved within the framework of the linearized theory, on formal mathematical grounds, by Van Kampen.[4] Bohm and Gross[3] have suggested that these difficulties are associated with those particles which are trapped in the potential-energy troughs of the wave, which implies a breakdown of the usual linearized theory. This is indeed the case, as we here demonstrate by solving the nonlinear problem.

In one space dimension it is possible to derive simply exact general solutions of the coupled Boltzmann and Poisson equations, which solutions correspond to stationary travelling waves. These solutions are not completely determined by the aforementioned equations and it is possible to construct waves of quite arbitrary shape, for instance isolated pulses, and sinusoidal waves. The method of derivation emphasizes the special nature of those particles trapped in the troughs of the electrostatic potential.

In particular, when one passes to the limit of small amplitude waves it transpires that the distribution function describing the trapped particles does not possess an expansion in integral powers of the amplitude of the electrostatic potential, but rather one in half-integral powers. This feature would appear to indicate that the usual linearized theory is inadequate to describe stationary traveling waves. It is shown, however, that the conventional linearized theory can be salvaged if one admits singular first-order distribution functions, in particular Dirac delta functions.[5] These singular solutions do not exhibit the phenomenon of Landau damping,[2] which is associated with distribution functions which are required to be initially analytic.

II. FORMULATION OF THE NONLINEAR PROBLEM

We seek stationary nonlinear electrostatic waves. Consider for simplicity a plasma composed of electrons and ions. The extension of the following considerations to more complicated systems is immediate. It is convenient to work in a coordinate system in which the wave is at rest, the so-called wave frame, so that all quantities are time-independent. The equations governing the phenomenon are

$$v\frac{\partial f_\pm(x,v)}{\partial x}\mp\frac{e}{m_\pm}\frac{\partial\phi(x)}{\partial x}\frac{\partial f_\pm(x,v)}{\partial v}=0, \tag{1}$$

$$\frac{\partial^2\phi(x)}{\partial x^2}=4\pi e\int_{-\infty}^{\infty}dvf_-(x,v)-4\pi e\int_{-\infty}^{\infty}dvf_+(x,v). \tag{2}$$

Equation (1) is the Boltzmann equation in one space dimension in which collisions have been ignored, written for ions or electrons, accordingly as one chooses the upper or lower sign. Equation (2) is the Poisson equation. In the above expressions $f_\pm$ represents the distribution function in joint configuration and velocity space, $m_\pm$ the mass, e the magnitude of the charge of the electron, ϕ the electrostatic potential, and v the velocity.

If one introduces the energy

$$E_\pm=\tfrac{1}{2}m_\pm v^2\pm e\phi, \tag{3}$$

the general solution of Eq. (1) may be written

$$f_\pm=f_\pm(E_\pm). \tag{4}$$

Note that Eq. (4) satisfies Eq. (1) independently of the

[1] A. Vlasov, J. Phys. U.S.S.R. 9, 25 (1945).
[2] L. D. Landau, J. Phys. U.S.S.R. 10, 25 (1946).
[3] D. Bohm and E. P. Gross, Phys. Rev. 75, 1851 (1949).
[4] N. G. Van Kampen, Physica 21, 949 (1955).
[5] P. A. M. Dirac, *Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition, p. 58.

partition between the two directions of the velocity of particles with a given energy.

In addition to Eqs. (1) and (2), there are certain other conditions which have to be satisfied. Consider a potential of the form shown in Fig. 1. Ions with an energy E_+ such that $e\phi_{\min} \leq E_+ \leq e\phi_{\max}$ are trapped by the potential, i.e., restricted to regions where $e\phi < E_+$. Thus, an ion with energy $E_+ = e\phi_1$ is restricted to regions C and F of Fig. 1, but once in C cannot move into F and *vice versa*. That is, x_1, x_2, and x_3 are turning points at which the ion reverses its velocity. Since the distribution function must be independent of the time, ions of energy $E_+ = e\phi_1$ must be equally distributed between the two directions of the velocity, and similarly for all trapped ions. Of course, ions with energies $E_+ > e\phi_{\max}$ can move freely in either direction, and hence their partition between the two directions of the velocity is arbitrary. Note that the distributions of the trapped ions in regions C and F are independent since the two regions are isolated from each other.

Similarly electrons of energy $E_- < -e\phi_{\min}$ are trapped. For instance an electron of energy $E_- = -e\phi_1$ can be only in regions B and D of Fig. 1, and the trapped electrons must be equally distributed between the two directions of velocity. Electrons of energy $E_- > -e\phi_{\min}$ can move freely, and their partition between the two directions of velocity is arbitrary.

Equation (4), the general solution of Eq. (1), can be substituted in Eq. (2), which on introduction of the energy of Eq. (3) in place of the velocity reads

$$\frac{d^2\phi(x)}{dx^2} = 4\pi e\left\{\int_{-e\phi}^{\infty} \frac{dEf_-(E)}{[2m_-(E+e\phi(x))]^{\frac{1}{2}}} - \int_{e\phi}^{\infty} \frac{dEf_+(E)}{[2m_+(E-e\phi(x))]^{\frac{1}{2}}}\right\}. \quad (5)$$

In Eq. (5) we have suppressed the subscript plus or minus where its deletion will cause no confusion.

III. DIFFERENTIAL EQUATION[6]

If one prescribes f_+ and f_- Eq. (5) is a second-order nonlinear ordinary differential equation for the potential ϕ. It can be integrated once on multiplying it by $d\phi/dx$ and integrating with respect to x. The result is

$$(d\phi/dx)^2 + V(\phi) = \text{const}, \quad (6)$$

where

$$V(\phi) = 8\pi\int_{e\phi}^{\infty} dEf_+(E)[2(E-e\phi)/m_+]^{\frac{1}{2}} + 8\pi\int_{-e\phi}^{\infty} dEf_-(E)[2(E+e\phi)/m_-]^{\frac{1}{2}}. \quad (7)$$

[6] Equation (5) has been considered independently, but in somewhat less detail by E. G. Harris, Bull. Am. Phys. Soc. Ser. II, **2**, 67 (1957). An equation similar to Eq. (5) has also been treated by D. Bohm and E. P. Gross, (see reference 3), who, however, did not exploit its full power and generality. [See K. H. Prendergast, Astron. J. **59**, 260 (1954); H. K. Sen, Phys. Rev. **97**, 849 (1955).]

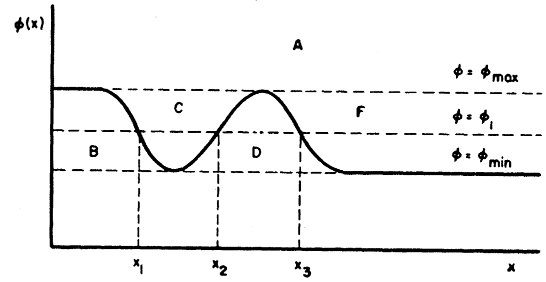

FIG. 1. Potential energy trough illustrating the trapping of particles.

Equation (6) can be solved by a quadrature, namely

$$x - x_0 = \pm\int_{\phi_0}^{\phi} d\phi\,|V(\phi_0) - V(\phi)|^{-\frac{1}{2}}, \quad (8)$$

where x_0 is a point at which $d\phi/dx$ vanishes and at which the potential is ϕ_0. Clearly one can construct periodic solutions of Eq. (6) if $V(\phi)$ has the form of a well [e.g., $V(\phi) = \text{const}\phi^2$]. Aperiodic solutions are also possible, but these will be treated by an alternate way of viewing Eq. (5).

IV. INTEGRAL EQUATION

One can look upon Eq. (5) as an integral equation for the distribution function of trapped electrons. Namely, if one writes $d^2\phi/dx^2 = -N(e\phi)$, where $N(e\phi)$ is the net charge density, Eq. (5) can be written in the form

$$\int_{-e\phi}^{-e\phi_{\min}} dEf_-(E)[2m_-(E+e\phi)]^{-\frac{1}{2}} = g(e\phi), \quad (9)$$

where

$$g(e\phi) = -N(e\phi)/4\pi e + \int_{e\phi}^{\infty} dEf_+(E)[2m_+(E-e\phi)]^{-\frac{1}{2}} - \int_{-e\phi_{\min}}^{\infty} dEf_-(E)[2m_-(E+e\phi)]^{-\frac{1}{2}} \quad (10)$$

is the density of trapped electrons at the point x corresponding to the potential $\phi(x)$. Thus, if one prescribes the potential $\phi(x)$, which determines $d^2\phi/dx^2$ and hence $N(e\phi)$; the entire ion distribution $f_+(E)$; and the distribution of untrapped electrons, $f_-(E)$ for $E > -e\phi_{\min}$; then $g(e\phi)$ is a known function, and Eq. (7) is an integral equation of the convolution type for the distribution function of the trapped electrons. It can be readily solved by the Laplace transformation, yielding

$$f_-(E) = \frac{(2m_-)^{\frac{1}{2}}}{\pi}\int_{e\phi_{\min}}^{-E} dV\frac{dg(V)}{dV}[-E-V]^{-\frac{1}{2}}, \quad E < -e\phi_{\min} \quad (11)$$

which result can be verified directly by substituting Eq. (11) in Eq. (9).

The choice of the arbitrary quantities in Eq. (10) is restricted only by the weak requirements that

$g(e\phi_{\min})=0$, which follows from Eq. (9), and that $f_-(E)$ as given by Eq. (11) be non-negative in order that $f_-(E)$ be a legitimate distribution function.

If one substitutes Eq. (10) in Eq. (11), it is possible to perform certain of the indicated integrations. The result is

$$f_-(E)=\frac{(2m_-)^{\frac{1}{2}}}{4\pi^2 e}\int_{e\phi_{\min}}^{-E}\frac{dV}{[-E-V]^{\frac{1}{2}}}\frac{dN(V)}{dV}+\frac{1}{\pi}\int_{-e\phi_{\min}}^{\infty}\frac{dVf_-(V)}{V-E}\left[\frac{e\phi_{\min}-E}{V-e\phi_{\min}}\right]^{\frac{1}{2}}$$
$$+\frac{1}{\pi}\left[\frac{m_-}{m_+}\right]^{\frac{1}{2}}\int_{e\phi_{\min}}^{\infty}dV\frac{df_+(V)}{dV}\frac{1}{2}\ln\frac{(E+V)^2}{[(V-e\phi_{\min})^{\frac{1}{2}}-(-E-e\phi_{\min})^{\frac{1}{2}}]^4},\quad E<-e\phi_{\min}. \tag{12}$$

Clearly one can equally well solve for the distribution function for the trapped positive ions.

We shall now show how to achieve almost any potential wave form which has a continuous second derivative. To this end prescribe a situation such as that represented in Fig. 2. Divide the x coordinate axis into intervals AB, BC, CD, $\cdots$, which are bounded by values of x for which $d\phi/dx=0$, and which contain no such points. Within any such region, x is a monotonic function of ϕ. Suppose that the potential and distribution functions are consistent to the left of point B. Note that the distributions of trapped electrons in the intervals AB and BC are not necessarily related since they are isolated by the potential minimum at B. The distribution in energy of untrapped electrons, and the distribution in energy of all ions, must be taken be to the same in the interval BC as in AB. The distribution in energy of trapped electrons, however, can be determined via Eq. (12) to be compatible with the desired potential in the interval BC (subject only to the condition that it turn out non-negative). The process can clearly be continued, by determining the distribution in energy of the trapped ions in the interval CD, the distribution in energy of the trapped electrons in the interval DE, etc. Continuity of $d^2\phi/dx^2$ at the extrema of ϕ guarantees that the charge density is continuous.

One corollary of the above result is that it is possible to construct isolated potential pulses. It is only necessary to prescribe that both $d\phi/dx$ and $d^2\phi/dx^2$ vanish at the points where the pulse joins on to regions of constant potential. For a positive pulse one determines the distribution in energy of the trapped electrons, for a negative pulse the distribution of trapped ions, so that Poisson's equation is satisfied. Such a pulse solution can, of course, connect regions of different constant potential, in which case one might call the resultant traveling front a "shock wave," since the width of the transition region can be chosen as small as one wishes. Of course, there is no dissipation and the "shock wave" can travel in either direction.

Clearly on the basis of the preceding considerations, one can prescribe a periodic potential wave form of arbitrary wavelength. The wave velocity, however, is also arbitrary since it is given by the arbitrary Galilean transformation from the wave frame to the laboratory frame. Moreover one can always avail oneself of the freedom in the partition between the two directions of motion of the untrapped particles of a given energy, so as to arrange that the mass velocity of the plasma is zero in the laboratory system. Thus, there is no dispersion relation in the usual sense of a one-to-one correspondence between wavelength and wave velocity, or alternatively between frequency and wave number.

V. SMALL AMPLITUDE WAVES

Let us consider now the case of small-amplitude waves, that is, waves for which $e(\phi_{\max}-\phi_{\min})$ is very much less than the mean particle energy. This suggests that one expand the particle distribution functions in powers of $e(\phi_{\max}-\phi_{\min})$. For the trapped electrons this can be effected by integrating Eq. (12) by parts.[7] There results

$$f_-(E)=[(2m_-)^{\frac{1}{2}}/4\pi^2e]\left\{2[-e\phi_{\min}-E]^{\frac{1}{2}}N'(e\phi_{\min})+\tfrac{4}{3}[-e\phi_{\min}-E]^{\frac{3}{2}}N''(e\phi_{\min})+O[(-e\phi_{\min}-E)^{\frac{5}{2}}]\right\}$$
$$+(1/\pi)\left\{\pi f_-(-e\phi_{\min})+2[-e\phi_{\min}-E]^{\frac{1}{2}}\int_{-e\phi_{\min}}^{\infty}dV V^{-\frac{1}{2}}f_-'(V)-\pi[-e\phi_{\min}-E]f_-'(-e\phi_{\min})\right.$$
$$\left.+O[(-e\phi_{\min}-E)^{\frac{3}{2}}]\right\}+(1/\pi)[m_-/m_+]^{\frac{1}{2}}\left\{2[-e\phi_{\min}-E]^{\frac{1}{2}}\int_{+e\phi_{\min}}^{\infty}dV V^{-\frac{1}{2}}f_+'(V)+O[(-e\phi_{\min}-E)^{\frac{3}{2}}]\right\}, \tag{13}$$

where a prime indicates a derivative of the associated function with respect to its argument. Since the minimum value of E is $-e\phi_{\max}$, Eq. (13) above is the desired expansion. Note that the expansion is in half-integral powers and hence while it follows from Eq. (13) that f_- is continuous at $E=-e\phi_{\min}$, df_-/dE is not necessarily. However, in the conventional linearized

[7] The details of the integration by parts are presented in the appendix. In all of these it is assumed that $f_-(E)$ is an analytic function of E for $E=\mathrm{Re}E>-e\phi_{\min}$.

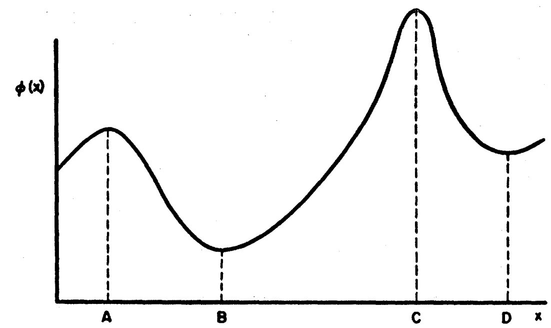

FIG. 2. Potential wave form.

theory of plasma oscillations,[1–3] it is assumed that one can write the distribution function as the sum of two terms, the first of which is space independent, and the second of which is proportional to the amplitude of the potential. Thus, it would seem that the conventional linearized theory is inadequate for representing the stationary waves here considered. It is possible, however, to salvage the conventional theory.

This is accomplished by observing that it is necessary and sufficient that the "first-order" distribution function reproduce to first order in the amplitude of the potential all the moments of the distribution function. These are given in general by

$$N\langle v^\nu\rangle=\int_{-\infty}^{\infty} dv v^\nu f(x,v) \quad \nu=0,1,2,\cdots. \tag{14}$$

Thus, if we henceforth delete the subscript minus on the electron distribution function, indicate particles with $v>0$ by a superscript plus, and particles with $v<0$ by a superscript minus, one can write, observing that for the trapped particles $f^+=f^-=\frac{1}{2}f$,

$$N\langle v^\nu\rangle=\int_{-\infty}^{-u} dv f^-[\tfrac{1}{2}m(v^2-u^2)]v^\nu+\int_{-u}^{u} dv \tfrac{1}{2}f[\tfrac{1}{2}m(v^2-u^2)]v^\nu+\int_{u}^{\infty} dv f^+[\tfrac{1}{2}m(v^2-u^2)]v^\nu, \tag{15}$$

where $u=[2e(\phi-\phi_{\min})/m]^{\frac{1}{2}}$. On appropriate expansion, one obtains

$$N\langle v^\nu\rangle=\int_{-\infty}^{-u} dv v^\nu\left\{f^-(\tfrac{1}{2}mv^2)-\frac{e(\phi-\phi_{\min})}{mv}\frac{\partial f^-(\frac{1}{2}mv^2)}{\partial v}+\cdots\right\}+\int_{-u}^{u} dv \tfrac{1}{2}v^\nu\{f(0)+a[u^2-v^2]^{\frac{1}{2}}+b[u^2-v^2]+\cdots\}$$
$$+\int_{u}^{\infty} dv v^\nu\left\{f^+(\tfrac{1}{2}mv^2)-\frac{e(\phi-\phi_{\min})}{mv}\frac{\partial f^+(\frac{1}{2}mv^2)}{\partial v}+\cdots\right\}, \tag{16}$$

where the coefficients a, b, $\cdots$ are given by Eq. (13.) The zeroth-order distribution function $f_0(v)$ characteristic of the conventional linearized theory is

$$f_0(v)=f^+(\tfrac{1}{2}mv^2)s(v)+f^-(\tfrac{1}{2}mv^2)s(-v), \tag{17}$$

while in general

$$f(E)=f^+(E)s(v)+f^-(E)s(-v), \tag{18}$$

where the step function $s(v)$ vanishes if its argument is negative, and is unity otherwise. It is usually assumed that $f_0(v)$ is analytic in v for real values of v, a condition which should permit one to represent all physical situations. This implies, however, that $f^+(E)$ and $f^-(E)$ in general have an expansion in half-integral powers of E, and that $\partial f_0(v)/\partial v|_{v=0}\neq 0$. Thus, if one rewrites Eq. (16) in the form

$$N\langle v^\nu\rangle=\int_{-\infty}^{\infty} dv f_0(v)+\frac{e(\phi-\phi_{\min})}{m}\int_{-\infty}^{-u} dv v^{\nu-1}\frac{\partial f_0(v)}{\partial v}+\cdots+\frac{e(\phi-\phi_{\min})}{m}\int_{-u}^{\infty} dv v^{\nu-1}\frac{\partial f_0(v)}{\partial v}+\cdots$$
$$+\int_{-u}^{u} dv v^\nu \tfrac{1}{2}\left\{-v\left.\frac{\partial f_0(v)}{\partial v}\right|_{v=0}-\cdots+a[u^2-v^2]^{\frac{1}{2}}+\cdots\right\}, \tag{19}$$

it is clear that the only moment which is affected to first order in $\phi-\phi_{\min}=mu^2/2e$ by the integral from $-u$ to $+u$, which manifests the effects of the trapped particles, as well as the analytic character of $f_0(v)$ for $v\sim 0$, is the density ($\nu=0$). Moreover, by definition of the principal value (indicated by Pr),

$$\mathrm{Pr}\int_{-\infty}^{\infty} dv \frac{1}{v}\frac{\partial f_0(v)}{\partial v}=\lim_{u\to 0}\left\{\int_{-\infty}^{-u} dv \frac{1}{v}\frac{\partial f_0(v)}{\partial v}+\int_{u}^{\infty} dv \frac{1}{v}\frac{\partial f_0(v)}{\partial v}\right\}. \tag{20}$$

Thus, if one introduces the Dirac delta function[5] $\delta(v)$, and interprets integrals in the sense of a principal value, a legitimate "first-order" distribution function is

$$f(x,v)=f_0(v)-\frac{e(\phi-\phi_{\min})}{mv}\frac{\partial f_0(v)}{\partial v}+c\delta(v), \tag{21}$$

in the sense that Eq. (21) will reproduce all the moments of the distribution correct to first order in the amplitude of the potential. The constant c is determined by Eq. (19).

The first two terms on the right of Eq. (21) are just those which enter in the usual Landau theory[3] of plasma oscillations. The singular term involving the δ function has been introduced previously by Van Kampen[4] on formal mathematical grounds for the case of periodic linear waves. The derivation presented here has the advantage of yielding the physical interpretation of the source of the singularity, namely the necessity to account within the framework of the linearized theory for the effects of the trapped particles, on the basis of more elementary mathematical considerations. It also covers more general situations than periodic waves. The phenomenon of Landau damping, the universal damping in time of all waves excited in a plasma close to thermal equilibrium is, of course, absent from Eq. (21) when f_0 is taken to be the appropriate Maxwell distribution. This is because Landau damping is associated with the restriction to analytic first-order distribution functions. It is to be emphasized, however, that while Eq. (21) is singular, the exact solution from which it was derived was perfectly well behaved.

Note that in order to transform any of the preceding expressions to the laboratory frame, it is only necessary to replace x everywhere by $x-ut$, and v by $v-u$, where u is the wave velocity. For periodic waves $u=\omega/k$, where ω is the frequency, and k the wave number.

In closing it is to be emphasized that whether such waves can exist in an actual plasma will depend on factors ignored in this paper, as in most previous works, namely inter-particle collisions, and the stability of the solutions against various kinds of perturbations.

ACKNOWLEDGMENT

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APPENDIX. THE ASYMPTOTIC EVALUATION OF CERTAIN INTEGRALS

Consider the second integral on the right side of Eq. (10). If one introduces the variables $u=-E$, $x=v/u$, and for convenience chooses $\phi_{\min}=0$, it can be written

$$I=\frac{1}{\pi}\int_0^\infty \frac{dx f_-(xu)}{x^{\frac{1}{2}}(1+x)}. \qquad \text{(A-1)}$$

The problem is to obtain a representation of I valid for small u. To this end observe that one can write, on integration by parts,

$$\pi I=-[\pi-2\arctan x^{\frac{1}{2}}]f_-(xu)\Big|_0^\infty + u\int_0^\infty dx[\pi-2\arctan x^{\frac{1}{2}}]f_-'(xu) \qquad \text{(A-2)}$$

$$=\pi f_-(0)+u\int_0^\infty dx[\pi-2\arctan x^{\frac{1}{2}}]f_-'(xu),$$

where a prime indicates a derivative with respect to the argument of the associated function. Note, however, that for $x>1$,

$$\pi-2\arctan x^{\frac{1}{2}}=2x^{\frac{1}{2}}-\tfrac{2}{3}x^{-\frac{3}{2}}+\tfrac{2}{5}x^{-\frac{5}{2}}-\cdots. \qquad \text{(A-3)}$$

This indicates that in order to proceed with the process of integration by parts so as to develop a power series in ascending powers of u, one writes

$$\pi I=\pi f_-(0)+u\int_0^\infty dx\,[\pi-2\arctan x^{\frac{1}{2}}-2x^{-\frac{1}{2}}]f_-'(xu)$$

$$+u\int_0^\infty dx\,2x^{-\frac{1}{2}}f_-'(xu)$$

$$=\pi f_-(0)+2u^{\frac{1}{2}}\int_0^\infty dV\,V^{-\frac{1}{2}}f_-'(V)$$

$$+uf_-'(xu)\Big[(x+1)(\pi-2\arctan x^{\frac{1}{2}})-2x^{\frac{1}{2}}\Big]_0^\infty$$

$$-u^2\int_0^\infty dx f_-''(xu)[(x+1)(\pi-2\arctan x^{\frac{1}{2}})-2x^{\frac{1}{2}}]$$

$$=\pi f(0)+2u^{\frac{1}{2}}\int_0^\infty dV\,V^{-\frac{1}{2}}f'_-(V)-\pi u f_-'(u)$$

$$-\tfrac{4}{3}u^{\frac{3}{2}}\int_0^\infty dV\,V^{-\frac{1}{2}}f_-''(V)$$

$$-u^2\int_0^\infty dx\,f_-''(xu)[(x+1)(\pi-2\arctan x^{\frac{1}{2}})$$

$$-2x^{\frac{1}{2}}-\tfrac{4}{3}x^{-\frac{1}{2}}]. \qquad \text{(A-4)}$$

The coefficient of u^2 in Eq. (A-4) can be bounded by the product of $\max|f_-''|$ and the integral from zero to infinity of the term in square brackets, which latter has been constructed to be convergent. Thus Eq. (A-4) is the desired expression.

The first integral on the right of Eq. (12) can be handled by straightforward integration by parts. The third integral, by judicious subtraction of properly chosen factors, can be written $(1/\pi)(m_-/m_+)^{\frac{1}{2}}J$, where

$$J=2u^{\frac{1}{2}}\int_0^\infty dV V^{-\frac{1}{2}}f_+'(V)+\tfrac{4}{3}u^{\frac{3}{2}}\int_0^\infty dV V^{-\frac{1}{2}}f_+''(V)$$

$$-u^2\int_0^\infty dx f_+''(xu)\Big[(x-1)\tfrac{1}{2}\ln\frac{(x-1)^2}{(x^{\frac{1}{2}}-1)^4}-2x^{\frac{1}{2}}+\tfrac{4}{3}x^{-\frac{1}{2}}\Big].$$

The coefficient of u^2 is bounded by a number independent of u.