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Quantization of the Sobolev Space of Half-Differentiable Functions, II

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To the memory of Mikhail Karasev

Abstract. We construct a quantization of the Sobolev space $V = H_0^{1/2}(S^1, \mathbb{R})$ of half-differentiable functions on the circle provided with a symplectic action of the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle by reparameterization. A quantum algebra of observables, associated with this system, is defined.

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Our objective is to quantize the system in which the role of phase manifold is played by the Sobolev space $V = H_0^{1/2}(S^1, \mathbb{R})$ of half-differentiable functions on the circle. On this space, there is an action of the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle, i.e., homeomorphisms of S^1 extendable to quasiconformal homeomorphisms of the disk. So, it is natural to take, for the group $\mathcal{G}$ associated with V, the semi-direct product of the Heisenberg group of the space V with the group $\mathrm{QS}(S^1)$ (the group $\mathcal{G}$ may be considered as an infinite-dimensional analog of the Poincaré group). If the group $\mathcal{G}$ were smooth, then we would take its Lie algebra for the algebra of observables on the phase manifold V. However, neither the group $\mathcal{G}$ nor its action on V are smooth. So, it is impossible to construct a classical system related to the phase manifold V provided with the action of the group $\mathcal{G}$. For this reason, we propose to construct directly a quantum algebra of observables associated with $\mathcal{G}$. The quantization of the first component of $\mathcal{G}$ was constructed in previous papers devoted to this topic (cf., e.g., [5–7]). In this paper, we define quantum observables corresponding to quasisymmetric homeomorphisms of the circle.

Let us briefly describe the content of the paper. In Sec. 1, the Sobolev space $V := H_0^{1/2}(S^1, \mathbb{R})$ of half-differentiable functions on the circle is defined. In Sec. 2, we introduce the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle which are boundary values of quasiconformal homeomorphisms of the disk. The group $\mathrm{QS}(S^1)$ acts on V by symplectic transformations given by reparameterizations. In Sec. 3, we recall the Connes definition of quantization and introduce the quantization space for our system that coincides with the Fock space associated with the Sobolev space V. The last section, Sec. 4, is devoted to the construction of the quantum algebra of observables corresponding to the group $\mathcal{G}$.

1. SOBOLEV SPACE OF HALF-DIFFERENTIABLE FUNCTIONS

The *Sobolev space of half-differentiable functions* on the circle S^1 is the Hilbert space $V := H_0^{1/2}(S^1, \mathbb{R})$ consisting of functions $f \in L^2(S^1, \mathbb{R})$ having Fourier series of the form

$$f(z) = \sum_{k \neq 0} f_k z^k \ , \quad f_k = \bar{f}_{-k}, \quad z = e^{i\theta},$$

with finite Sobolev norm of order 1/2:

$$\|f\|_{1/2}^2 = \sum_{k \neq 0} |k||f_k|^2 = 2 \sum_{k > 0} k|f_k|^2 < \infty.$$

The space V may be provided with a 2-form $\omega : V \times V \to \mathbb{R}$ given in terms of Fourier coefficients of vectors $\xi, \eta \in V$ by the formula

$$\omega(\xi,\eta) = 2\,\mathrm{Im}\sum_{k>0} k\,\xi_k\bar{\eta}_k.$$

This form is well defined and determines a symplectic structure on V.

Apart from the symplectic form, the Sobolev space V has a complex structure J which is given in terms of Fourier decompositions by the formula

$$\xi(z) = \sum_{k\neq 0}\xi_k z^k \longmapsto (J\xi)(z) = -i\sum_{k>0}\xi_k z^k + i\sum_{k<0}\xi_k z^k.$$

This complex structure is compatible with the symplectic form ω in the sense that they define together a positively definite inner product g on V given by the formula $g(\xi,\eta) := \omega(\xi, J\eta)$, or in terms of Fourier series

$$g(\xi,\eta) = 2\,\mathrm{Re}\sum_{k>0} k\,\xi_k\bar{\eta}_k.$$

The complexification $V^{\mathbb{C}} = H_0^{1/2}(S^1,\mathbb{C})$ of the space V is a complex Hilbert space and the inner product g on V extends to a Hermitian inner product on $V^{\mathbb{C}}$ given by

$$<\xi,\eta>= \sum_{k\neq 0}|k|\xi_k\bar{\eta}_k.$$

We extend the symplectic form ω and the complex structure operator J complex-linearly to $V^{\mathbb{C}}$.

The space $V^{\mathbb{C}}$ can be decomposed into the direct sum $V^{\mathbb{C}} = W_+ \oplus W_-$, where $W_\pm$ is the $(\mp i)$-eigenspace of the operator $J \in \mathrm{End}\,V^{\mathbb{C}}$. In other words,

$$W_+ = \{f \in V^{\mathbb{C}}:\ f(z) = \sum_{k>0} f_k z^k\},\quad W_- = \overline{W}_+ = \{f \in V^{\mathbb{C}}:\ f(z) = \sum_{k<0} f_k z^k\}.$$

2. QUASISYMMETRIC HOMEOMORPHISMS

Recall that a homeomorphism $w : D \to D$ of the unit disk D onto itself, preserving orientation and having locally integrable derivatives, is called *quasiconformal* if there exists a bounded measurable function $\mu \in L^\infty(D,\mathbb{C})$ with norm $\|\mu\|_\infty =: k < 1$ for which the following *Beltrami equation*

$$w_{\bar{z}} = \mu w_z \tag{2.1}$$

holds almost everywhere on D. The function μ is called the *Beltrami differential*.

In the particular case when $k = 0$, i.e., $\mu = 0$, equation (2.1) converts into the Cauchy–Riemann equation and the map w is conformal.

We list here some basic properties of quasiconformal maps (a detailed exposition of the theory of quasiconformal maps may be found in the book [1]).

(1) Quasiconformal homeomorphisms $w : D \to D$ extend continuously (even Hölder-continuously) to the boundary to homeomorphisms $w : S^1 \to S^1$ of the unit circle S^1 onto itself.
(2) The composition of quasiconformal maps is again a quasiconformal map. The same is true for the maps inverse to quasiconformal ones.
(3) Solutions of Beltrami equation are uniquely defined up to conformal maps. In more detail, if there are two solutions w_1, w_2 of this equation with the same Beltrami differential μ, then the maps $w_1 \circ w_2^{-1}$ and $w_2 \circ w_1^{-1}$ are conformal.
(4) For any function $\mu \in L^\infty(D)$ satisfying the condition $\|\mu\|_\infty < 1$, there exists a quasiconformal map w which is a solution of the Beltrami equation in D with Beltrami differential equal to μ almost everywhere.

The property (2) implies that quasiconformal automorphisms of the disk D form a group with respect to composition.

We shall call an orientation-preserving homeomorphism $f : S^1 \to S^1$ *quasisymmetric* if it extends to a quasiconformal homeomorphism w of D onto itself. Since quasiconformal automorphisms of the disk D form a group, the same is true also for quasisymmetric homeomorphisms of S^1. Denote by $\mathrm{QS}(S^1)$ the group of all quasisymmetric homeomorphisms of S^1 onto itself. This group may be included into the following chain of embeddings

$$\text{Möb}(S^1) \subset \text{Diff}_+(S^1) \subset \text{QS}(S^1) \subset \text{Homeo}_+(S^1)$$

where $\text{Homeo}_+(S^1)$ denotes the group of orientation-preserving homeomorphisms of the unit circle S^1 and $\text{Möb}(S^1)$ is the Möbius group of fractional-linear automorphisms of the unit disk D restricted to S^1.

We associate with an orientation-preserving homeomorphism h of the unit circle S^1 the operator $T_h(\xi) := \xi \circ h$ of change of variable acting on functions $\xi \in V$. This operator has the following remarkable property.

Theorem 2.1. [(*Nag–Sullivan*) [3]] *The operator T_h acts from the space V into itself if and only if $h \in QS(S^1)$. The operators T_h with $h \in QS(S^1)$ generate symplectic transformations of the space V.*

3. CONNES QUANTIZATION

A *classical system* is given by the pair $(M, \mathcal{A})$ consisting of the phase space M and algebra of observables $\mathcal{A}$. The *phase space* M is a smooth symplectic manifold. The *algebra of observables* $\mathcal{A}$ is an associative algebra of functions on M provided with involution and exterior differential, i.e. a linear map from $\mathcal{A}$ to the space $\Omega^1(\mathcal{A})$ of 1-forms on this algebra satisfying Leibniz rule (cf. [2]).

The *quantization* of such system is given by an irreducible linear representation π of observables from $\mathcal{A}$ by closed linear operators acting in a complex Hilbert space H called the *quantization space*. Under this representation, the involution in $\mathcal{A}$ transforms into Hermitian conjugation while the exterior derivative operator d maps to a *quantum derivation operator* given by the commutator with some *symmetry operator* S which is a self-adjoint operator in H with square $S^2 = I$. In other words,

$$\pi : df \longmapsto d^q f := [S, \pi(f)], \quad f \in \mathcal{A}.$$

We can reformulate this definition in terms of Lie algebras. Recall that a derivation of an algebra $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule $D(ab) = (Da)b + a(Db)$. Denote by $\mathrm{Der}(\mathcal{A})$ the Lie algebra of all derivations of the algebra $\mathcal{A}$. In terms of $\mathrm{Der}(\mathcal{A})$ the quantization is an irreducible representation of the Lie algebra $\mathrm{Der}(\mathcal{A})$ in the algebra of closed linear operators in the quantization space H provided with commutator as the Lie bracket.

The Lie algebra $\mathrm{Der}^q(\mathcal{A})$, generated by the quantum derivation operators $d^q f$ with $f \in \mathcal{A}$, is called the *quantum algebra of observables* corresponding to the algebra of observables $\mathcal{A}$ while the operators $d^q f$ are called the *quantum observables*.

In our case, the role of the quantization space H will be played by the Fock space associated with the Sobolev space V. Recall that the complex structure J on V generates the decomposition of the complexified space $V^{\mathbb{C}}$ into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

of $(\mp i)$-eigenspaces of operator J. This decomposition is orthogonal with respect to the Hermitian inner product $< z, w >= \omega(z, Jw)$ on $V^{\mathbb{C}}$ generated by J and ω.

The Fock space F is the completion of the algebra of symmetric polynomials in the variables $z \in W_+$ with respect to the norm generated by the inner product $< \cdot, \cdot >$.

In more detail, denote by $\mathfrak{S}(W_+)$ the algebra of symmetric polynomials in the variables $z \in W_+$ and introduce on it the inner product generated by the inner product $< \cdot, \cdot >$. On monomials of the same degree, it is given by the formula

$$\langle z_1 \otimes \cdots \otimes z_n, z'_1 \otimes \cdots \otimes z'_n >:= \sum_{\{i_1,\ldots,i_n\}} < z_1, z'_{i_1} > \cdot \ldots \cdot < z_n, z'_{i_n} >$$

where summation is taken over all permutations $\{i_1, \dots, i_n\}$ of the set $\{1, \dots, n\}$ (inner product of monomials of different degrees is set to zero). The inner product on monomials is extended by linearity to the whole algebra $\mathfrak{S}(W_+)$.

The *Fock space* $F \equiv F(V^{\mathbb{C}})$ is the completion of the algebra $\mathfrak{S}(W_+)$ with respect to the norm $< \cdot, \cdot >$.

If $\{w_n\}_{n=1}^{\infty}$ is an orthonormal basis of the space W_+, then, for the orthonormal basis of the Fock space F, we can take monomials of the form

$$P_K(z) = \frac{1}{\sqrt{k!}} < z, w_1 >^{k_1} \cdot \ldots \cdot < z, w_n >^{k_n}, \quad z \in W_+,$$

where $K = (k_1, \dots, k_n, 0, \dots)$ is a finite collection of natural numbers $k_i \in \mathbb{N}$, $k! = k_1! \cdot \ldots \cdot k_n!$.

4. QUANTIZATION OF THE SOBOLEV SPACE OF HALF-DIFFERENTIABLE FUNCTIONS

For the phase space of our system we take the Sobolev space V of half-differentiable functions.

On this space, we have a natural action of the group $\mathcal{G}$ consisting of the two following components. The first component of $\mathcal{G}$ is given by the *Heisenberg group* $\mathrm{Heis}(V)$ which coincides with the central extension of the Abelian group V. In other words, $\mathrm{Heis}(V)$ is the direct product $\mathrm{Heis}(V) = V \times S^1$ provided with the group operation given by the formula

$$(v_1, s_1) \cdot (v_2, s_2) = \big(v_1 + v_2, s_1 s_2 e^{i\omega(v_1, v_2)}\big).$$

For the second component of $\mathcal{G}$, we take the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle S^1 acting on V by reparameterization, i.e., by change of variable. By definition, $\mathcal{G}$ is the semidirect product of the group $\mathrm{Heis}(V)$ and the group of quasisymmetric homeomorphisms of the circle $\mathrm{QS}(S^1)$. We can regard it as an infinite-dimensional analog of the Poincaré group which is the semidirect product of the group of translations and the group of hyperbolic rotations.

If $\mathcal{G}$ were a Lie group acting on V by smooth symplectic transformations, then we could take the Lie algebra of this group for the algebra of observables $\mathcal{A}$. However, neither the group $\mathcal{G}$ nor its action on the Sobolev space V are smooth. For this reason, we cannot construct the classical system corresponding to the phase space V with the group $\mathcal{G}$ acting on it. Instead, we shall directly define the quantum system associated with V. In other words, we change our original point of view on quantization and first construct the quantum system associated with the space V and the group $\mathcal{G}$, passing by the stage of construction of the classical system.

We turn to the construction of the quantum algebra of observables associated with the Sobolev space V and the group $\mathcal{G}$.

We start from the first component corresponding to Heisenberg group $\mathrm{Heis}(V)$. Note that any bounded function f on the circle S^1 generates the bounded multiplication operator M_f in the Hilbert space $V^{\mathbb{C}}$ acting by the formula

$$M_f : h \in V^{\mathbb{C}} \longmapsto fh \in V^{\mathbb{C}}.$$

The symmetry operator S in this case coincides with the *Hilbert transform*:

$$(Sh)(\phi) = \frac{1}{2\pi} \int_0^{2\pi} K(\phi, \psi) h(\psi) d\psi, \quad h \in V^{\mathbb{C}},$$

where the integral is taken in the principal value sense and the Hilbert kernel K is equal to

$$K(\phi, \psi) = 1 + i \cot \frac{\phi - \psi}{2}\,.$$

For $\phi \to \psi$ it behaves like $1 + 2i/(\phi - \psi)$.

The differential of a general function $f \in V^{\mathbb{C}}$ is not defined in the classical sense, however, its quantum analog $d^q f := [S, M_f]$ is correctly defined as a bounded linear operator in the space $V^{\mathbb{C}}$. It is given by the formula

$$(d^q f)(h)(\phi) = \frac{1}{2\pi} \int_0^{2\pi} k_f(\phi, \psi) h(\psi) d\psi, \quad h \in V^{\mathbb{C}},$$

where $k_f(\phi, \psi) = K(\phi, \psi)(f(\phi) - f(\psi))$. For $\phi \to \psi$ the kernel $k_f(\phi, \psi)$ behaves like const$\cdot \frac{f(\phi)-f(\psi)}{\phi-\psi}$. The quasiclassical limit of this operator, obtained by its restriction to smooth functions and taking the trace on the diagonal $\phi = \psi$, coincides with the multiplication operator $h \mapsto f' \cdot h$. The operators $d^q f$ are the quantum observables corresponding to the elements $f \in V$.

In order to define the quantum observables corresponding to elements $g \in \mathrm{QS}(S^1)$, it is convenient to switch from the circle S^1 to the real line $\mathbb{R}$. Then the space V will be replaced by the Sobolev space $H^{1/2}(\mathbb{R})$ of real-valued half-differentiable vector-functions on the real line (still denoted by V), and $\mathrm{QS}(S^1)$ will be replaced by the group $\mathrm{QS}(\mathbb{R})$ of quasisymmetric homeomorphisms of the real line $\mathbb{R}$ extending to quasiconformal homeomorphisms of the upper half-plane.

According to a theorem of Reimann [4], the tangent space to $\mathrm{QS}(\mathbb{R})$ at the origin coincides with the *Zigmund space* $\Lambda(\mathbb{R})$ consisting of continuous functions $f(x)$ satisfying the condition:

$$|f(x+t) + f(x-t) - 2f(x)| \leqslant C|t|$$

uniformly with respect to $x \in \mathbb{R}$, $t > 0$.

This motivates the definition of the differentiation operator $d^q g$ for $g \in \mathrm{QS}(\mathbb{R})$ as

$$d^q g(v) = \int_{\mathbb{R}} \frac{g(x+t) + g(x-t) - 2g(x)}{t} v(t) dt, \quad v \in V^{\mathbb{C}}.$$

Using this operator, we can introduce the quantum observables corresponding to elements $g \in \mathrm{QS}(\mathbb{R})$ as the operators $T_g^q h := d^q h(g) \circ d^q g$. The quasiclassical limit of the operator T_g^q coincides with the multiplication operator $h \mapsto h'(g) g'$.

This operator can be extended to the whole Fock space F in the following way. We define it first on the elements of the orthonormal basis of F given by the monomials $P_K(z)$ (cf. Sec. 2) by the Leibniz rule. We then extend it to the whole algebra of symmetric polynomials in variables the W_+ by linearity. The closure of the obtained operator yields an operator $T_g^q h$ in the Fock space F. In the same way, the operator $d^q h$ extends to a closed operator $d^q h$ in the Fock space F.

The required *quantum algebra of observables*, associated with the Sobolev space V provided with the action of the group $\mathcal{G}$ is the Lie algebra Der^q generated by the operators $d^q h$ and $T_g^q h$ acting in the Fock space F with $g \in \mathrm{QS}(\mathbb{R})$, $h \in V$.

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