WIENER TAUBERIAN THEOREMS FOR DISTRIBUTIONS

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Abstract

Wiener Tauberian theorems are proved for integral transforms of Schwartz distributions in which the kernel of the transform belongs to a suitable testing function space.

1. *Introduction*

Recall the celebrated Wiener Tauberian theorem [15] (for the notation see §2 below). Suppose that $f \in \mathscr{L}^\infty(\mathbb{R})$, $k \in \mathscr{L}^1(\mathbb{R})$ and $\mathscr{F}[k](y) \neq 0$, for $y \in \mathbb{R}$, where $\mathscr{F}[k]$ is the Fourier transform of k. If

$$\lim_{x \to \infty} \int_{\mathbb{R}} f(y)\, k(x-y)\, dy = a \int_{\mathbb{R}} k(y)\, dy, \quad a \in \mathbb{R}, \tag{1}$$

then for every $G \in \mathscr{L}^1(\mathbb{R})$

$$\lim_{x \to \infty} \int_{\mathbb{R}} f(y)\, G(x-y)\, dy = a \int_{\mathbb{R}} G(y)\, dy. \tag{2}$$

This theorem has been much used in various branches of mathematics and so generalizations of it are, even now, important. Pitt's form of Wiener's theorem [10, 11] gives the behaviour of the function f when $x \to \infty$ from the relation (1), with some additional conditions on f. Many generalizations of these two basic results have been proved; cf. [2], especially in part 4.8 where there is a comprehensive analysis of classical Wiener and Pitt Tauberian type results via the Mellin convolution. For Tauberian remainder theorems one can consult Ganelius [5].

The Wiener–Pitt type theorems can be proved in a quick and elegant way by a Banach-algebra approach (see [12]) or by the method of generalized functions as in [7]. One can make use of the direct connection between spectral synthesis and Tauberian theorems (see [1]).

In the last thirty years the method of integral transforms has appeared as one of the most powerful tools, especially in mathematical physics. The book of Zemanian [16] is the first systematic monograph which gives different integral transforms of generalized functions. Abelian and Tauberian type results related to generalized asymptotic behaviour have been elaborated only for some special integral transforms of distributions (see for example [9, 14]). It was natural to expect theorems of Wiener's or Pitt's form. In [8] Peetre proved a Wiener Tauberian theorem when the kernel k belongs to a Banach space $\mathscr{W}$, with a translation invariant norm and with the properties that the space of rapidly decreasing functions is a dense subspace of $\mathscr{W}$ and $\mathscr{W} * \mathscr{W}' \subset \mathscr{L}^1$.

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In this paper we shall prove Wiener Tauberian theorems in a more general setting. The main result of this paper is Theorem 2. Applications to special integral transforms, especially for the Laplace transform of numerical functions and distributions in connection with S-asymptotics [9] and quasiasymptotics [14] of distributions, will appear elsewhere. We quote here only two propositions which are corollaries of Theorem 2 and for which proofs will be given after the proof of Theorem 2.

PROPOSITION 1. *Let $f \in \mathscr{D}'(\mathbb{R})$ be such that the set $\{f(\cdot + h)/(L(e^h)e^{\alpha h});\ h > 0\}$ is bounded in $\mathscr{D}'(\mathbb{R})$ and let $\phi \in \mathscr{D}(\mathbb{R})$ be such that the Fourier transform of $\phi \exp(-\alpha \cdot)$ is different from zero on $\mathbb{R}$. If*

$$\lim_{x\to\infty} \frac{(f*\phi)(x)}{L(\exp x)\exp(\alpha x)} = a \int_{\mathbb{R}} \phi(t)\exp(-\alpha t)\,dt, \quad a \in \mathbb{R},$$

then for every $\psi \in \mathscr{D}(\mathbb{R})$

$$\lim_{x\to\infty} \frac{(f*\psi)(x)}{L(\exp x)\exp(\alpha x)} = a \int_{\mathbb{R}} \psi(t)\exp(-\alpha t)\,dt.$$

(Note that L is a slowly varying function; see [2, 6] and the next paragraph.)

PROPOSITION 2. *Let f and k be from $\mathscr{L}^1_{\text{loc}}(\mathbb{R})$ and let $\alpha > \beta$ be such that:*

(i) *$f/c \in \mathscr{L}^\infty(\mathbb{R})$, where $c(x) = L(\exp x)\exp(\alpha x)$, if $x \geqslant 0$ and $c(x) = \exp(\beta x)$ if $x < 0$;*

(ii) *$\check{k}\exp((\alpha+\delta)\cdot) \in \mathscr{L}^1((a,\infty))$ and $\check{k}\exp(\beta\cdot) \in \mathscr{L}^1((-\infty,b))$ for some $a, b \in \mathbb{R}$, and some $\delta > 0$, where $\check{k}(x) = k(-x)$.*

Then $(f*k)(x)$, $x \in \mathbb{R}$, exists and $\check{k}\exp(\alpha\cdot) \in \mathscr{L}^1(\mathbb{R})$.

Moreover, if we assume that:

(iii) *$\mathscr{F}[k\exp(-\alpha\cdot)](y) \neq 0$, for $y \in \mathbb{R}$,*

(iv) $$\lim_{x\to\infty} \frac{(f*k)(x)}{L(\exp x)\exp(\alpha x)} = a \int_{\mathbb{R}} k(t)\exp(-\alpha t)\,dt, \quad a \in \mathbb{R},$$

then

$$\lim_{x\to\infty} \frac{(f*\psi)(x)}{L(\exp x)\exp(\alpha x)} = a \int_{\mathbb{R}} \psi(t)\exp(-\alpha t)\,dt \tag{2'}$$

for every $\psi \in \mathscr{L}^1_{\text{loc}}(\mathbb{R})$ such that $\check{\psi}\exp((\alpha+\delta)\cdot) \in \mathscr{L}^1((a,\infty))$ and $\check{\psi}\exp(\beta\cdot) \in \mathscr{L}^1((-\infty,b))$.

In particular, if $\beta + 1 > 0$, then

$$\int_0^x f(\ln t)\,dt \sim \frac{a}{\alpha+1} x^{\alpha+1} L(x), \quad x \longrightarrow \infty.$$

NOTE. Theorem 1.7.5 in [2] gives conditions on f under which

$$f(t) \sim aL(\exp t)\exp(\alpha t), \quad t \longrightarrow \infty.$$

Proposition 2 contains both forms—Wiener's and Pitt's—of a Tauberian theorem [2, 3, 10].

All our results are given in the one-dimensional case for simplicity. Also, they can be done in a many-dimensional case with the asymptotic behaviour in a cone.

2. *Notation*

We denote by $\mathbb{N}$ the set of natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We always denote by L a slowly varying function bounded away from 0 and ∞ on every compact subset of $(0, \infty)$ [2, 6]. We shall use the following properties of this function.

For every interval $[a, b] \subset (0, \infty)$

$$\frac{L(xh)}{L(h)} \longrightarrow 1, h \longrightarrow \infty, \text{ uniformly for } x \in [a, b]. \tag{3}$$

For every $\delta > 0$ there exists $c_\delta > 0$ such that for $x > 0, y > 0$, .

$$\frac{1}{c_\delta} \min\left\{\left(\frac{x}{y}\right)^\delta, \left(\frac{y}{x}\right)^\delta\right\} \leqslant \frac{L(x)}{L(y)} \leqslant c_\delta \max\left\{\left(\frac{x}{y}\right)^\delta, \left(\frac{y}{x}\right)^\delta\right\}.$$

We denote by $\mathscr{D}(\mathbb{R})$ and $\mathscr{D}'(\mathbb{R})$ the basic spaces of test functions and of Schwartz distributions; $\mathscr{D}_{\mathscr{L}^1}$ is the space of smooth functions ϕ on $\mathbb{R}$ ($\phi \in C^\infty(\mathbb{R})$) such that

$$p_l(\phi) \equiv \|\phi^{(l)}\|_{\mathscr{L}^1} < \infty \text{ for every } l \in \mathbb{N}_0.$$

Convergence in $\mathscr{D}_{\mathscr{L}^1}$ is defined as convergence under each seminorm p_l, $l \in \mathbb{N}_0$; $\mathscr{B}'$ is the strong dual of $\mathscr{D}_{\mathscr{L}^1}$. We recall that by the $p = \infty$ case of [13, théorème XXV, 1°] $f \in \mathscr{B}'$ if and only if

$$f = \sum_{i=0}^{m} F_i^{(i)}, \tag{4}$$

where F_i, $i = 0, 1, \ldots, m$, are bounded continuous functions on $\mathbb{R}$ and the derivatives are in the distributional sense.

For $f \in \mathscr{B}'$ and $\phi \in \mathscr{D}_{\mathscr{L}^1}$ we set $\langle f, \phi \rangle = \int_{\mathbb{R}} f\phi$. For the properties of distribution spaces we refer to [13].

By η we always denote a function in $C^\infty(\mathbb{R})$ which is equal to 1 in a neighbourhood of $+\infty$ and to 0 in a neighbourhood of $-\infty$.

The Fourier transform of an $f \in \mathscr{L}^1(\mathbb{R})$ is defined by

$$\mathscr{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi t} f(t)\, dt, \quad \xi \in \mathbb{R}.$$

3. *Wiener Tauberian theorems*

THEOREM 1. *Let $f \in \mathscr{B}'$ and $K \in \mathscr{D}_{\mathscr{L}^1}$ be such that $\mathscr{F}[K](\xi) \neq 0$, for $\xi \in \mathbb{R}$. If*

$$\lim_{x \to \infty} (f * K)(x) = a \int_{\mathbb{R}} K(t)\, dt, \quad a \in \mathbb{R},$$

then for every $\psi \in \mathscr{D}_{\mathscr{L}^1}$,

$$\lim_{x \to \infty} (f * \psi)(x) = a \int_{\mathbb{R}} \psi(t)\, dt.$$

Proof. We follow the proof of the Wiener Tauberian Theorem given in [4, p. 234–235]. Let $\mathscr{M}$ be the subspace of $\mathscr{D}_{\mathscr{L}^1}$ consisting of all finite linear combinations of translations of K. First, we show that $\mathscr{M}$ is dense in $\mathscr{D}_{\mathscr{L}^1}$. For, if not, there exists an element g of the dual space $\mathscr{B}'$ such that $\int_{\mathbb{R}} gJ = 0$ for every $J \in \mathscr{M}$. Therefore

$\int_{\mathbb{R}} g(x)K(x-h)\,dx = 0$ for all h, and hence $g * \check{K}$ vanishes identically. That implies that $\mathscr{F}[\check{K}](\xi) = 0$ for $\xi \in \operatorname{supp} \mathscr{F}[g]$, the proof of this fact being the same as for the corresponding result for $g \in \mathscr{L}^\infty(\mathbb{R})$ in [4, p. 232], based on Beurling's theorem.

Since we assume $\mathscr{F}[\check{K}](\xi) = \mathscr{F}[K](-\xi)$ is never zero, we conclude that $\mathscr{M}$ is dense in $\mathscr{D}_{\mathscr{L}^1}$.

Let $\psi \in \mathscr{D}_{\mathscr{L}^1}$ and $\varepsilon > 0$. There is $H \in \mathscr{M}$ such that

$$\|\psi^{(k)} - H^{(k)}\|_{\mathscr{L}^1} < \varepsilon, \quad k = 0, \ldots, m.$$

By (4) and Lebesgue's theorem we have

$$\begin{aligned} |(f*\psi)(x) - a\int_{\mathbb{R}} \psi(t)\,dt| &\leqslant |((\psi - H)*f)(x) - a\int_{\mathbb{R}} (\psi(t) - H(t))\,dt| \\ &\quad + |(H*f)(x) - a\int_{\mathbb{R}} H(t)\,dt| \\ &\leqslant \sum_{i=0}^{m} (|(\psi - H)^{(i)}| * |F_i|)(x) + a\varepsilon + \varepsilon \\ &\leqslant \left(\sum_{i=0}^{m} \|F_i\|_{\mathscr{L}^\infty} + a + 1\right)\varepsilon. \end{aligned}$$

This implies the assertion.

The main theorem of the paper is Theorem 2 below. In this theorem and in its proof we shall use the function

$$c(x) = \begin{cases} L(e^x)\,e^{\alpha x}, & x \geqslant 0, \\ e^{\beta x}, & x < 0, \end{cases}$$

where we shall always assume that $\alpha > \beta$.

THEOREM 2. *For $f \in \mathscr{D}'(\mathbb{R})$ and $K \in C^\infty(\mathbb{R})$ we assume that*

(i) *the set $\{f(\cdot + h)/c(h); h \in \mathbb{R}\}$ is bounded in $\mathscr{D}'(\mathbb{R})$,*
(ii) *there exists $\delta > 0$ such that $\eta\check{K}\exp((\alpha+\delta)\cdot), (1-\eta)\check{K}\exp(\beta\cdot) \in \mathscr{D}_{L^1}$.*

Then
(a) *$\mathscr{F}[K\exp(-\alpha\cdot)](\xi) = \mathscr{F}[K](\xi - i\alpha)$, for $\xi \in \mathbb{R}$, exists,*
(b) *the convolution $f * K$ exists.*
Moreover, if we assume that
(iii) *$\mathscr{F}[K](\xi - i\alpha) \neq 0$, for $\xi \in \mathbb{R}$,*
(iv) *$\displaystyle\lim_{x\to\infty} \frac{(f*K)(x)}{L(\exp x)\exp(\alpha x)} = a\int_{\mathbb{R}} K(t)\exp(-\alpha t)\,dt$, for $a \in \mathbb{R}$,*

then
(c) *for every $\psi \in C^\infty(\mathbb{R})$ for which $\eta\check{\psi}\exp((\alpha+\delta)\cdot)$, $(1-\eta)\check{\psi}\exp(\beta\cdot) \in \mathscr{D}_{\mathscr{L}^1}$,*

$$\lim_{x\to\infty} \frac{(f*\psi)(x)}{L(\exp x)\exp(\alpha x)} = a\int_{\mathbb{R}} \psi(t)\exp(-\alpha t)\,dt.$$

Proof. First step: we need some technical results concerning the function c and its smooth approximation c_0 which is defined as follows.

Let $\omega \in C^{\infty}(\mathbb{R})$, $\operatorname{supp}\omega \subset [-1,1]$, $\omega \geqslant 0$, and $\int_{-1}^{1}\omega(t)\,dt = 1$. Put

$$\tilde{L}(x) = (L(\exp t) * \omega(t))(x), \quad x > 0.$$

We define

$$c_0(x) = \begin{cases} \tilde{L}(x)\exp(\alpha x), & x > 1, \\ \exp(\beta x), & x < 0, \end{cases}$$

and extend this function on $[0,1]$ to be smooth and positive on $\mathbb{R}$. Denote this extension by c_0, again.

Properties (3) imply that for every $x > 0$,

$$\frac{\tilde{L}^{(k)}(x+h)}{L(\exp h)} = \int_{-1}^{1}\frac{L(\exp(x+h-t))}{L(\exp h)}\omega^{(k)}(t)\,dt \longrightarrow \begin{cases} 1, & k=0 \\ 0, & k \in \mathbb{N}, \end{cases} \quad h \longrightarrow \infty \tag{5}$$

and that for every $k \in \mathbb{N}_0$ there are $c_k > 0$ and $\tilde{c}_k > 0$ such that

$$|\tilde{L}^{(k)}(x)| \leqslant c_k L(\exp x) \leqslant \tilde{c}_k \tilde{L}(x), \quad x > 0. \tag{6}$$

By (6) we have that for every $k \in \mathbb{N}_0$ there exists $c_k > 0$ such that

$$\left|\left(\frac{1}{\tilde{L}(x)}\right)^{(k)}\right| \leqslant \frac{c_k}{\tilde{L}(x)}, \quad x > 0. \tag{7}$$

Now (7) implies that for every $k \in \mathbb{N}_0$ there exists $c_k > 0$ such that

$$|c_0^{(k)}(t)| \leqslant \begin{cases} c_k L(e^t)\,e^{\alpha t}, & t > 1, \\ c_k, & 0 \leqslant t \leqslant 1, \\ c_k e^{\beta t}, & t < 0. \end{cases} \tag{8}$$

For every compact set $B \subset \mathbb{R}$ and every $k \in \mathbb{N}_0$ there exists $c_{B,k} > 0$ such that

$$|c(h)/c_0(\cdot + h)^{(k)}(x)| \leqslant c_{B,k}, \quad h \in \mathbb{R}, \quad x \in B. \tag{9}$$

Let $\delta > 0$. We also need the following estimate: for every $k \in \mathbb{N}_0$ there exists $c_k > 0$ such that for every $h > 0$

$$\left|\left(\frac{c_0(\cdot + h)}{L(\exp h)\exp(\alpha h)} - \exp(\alpha\,\cdot)\right)^{(k)}(x)\right| \leqslant \begin{cases} c_k\exp(\alpha x + \delta|x|), & x+h \geqslant 0, \\ c_k\exp(\beta x), & x+h < 0. \end{cases} \tag{10}$$

Let us prove (10). If $x+h > 1$, by Leibnitz's formula, it is enough to prove that for every $j \in \mathbb{N}_0$ there exists $c_j > 0$ such that

$$\left|\left(\frac{c_0(\cdot + h)}{L(\exp h)\exp(\alpha(\cdot + h))}\right)^{(j)}(x)\right| \leqslant c_j\exp(\delta|x|). \tag{11}$$

This easily follows from (8) and (3).

Let $x+h \in [0,1]$. We shall prove (11) because it implies (10). Since

$$\frac{c_0(x+h)}{\exp(\alpha(x+h))}, \quad x+h \in [0,1],$$

is bounded, (11) follows from the estimate

$$\frac{1}{L(\exp h)} = \frac{1}{L(\exp(x-t))} \leqslant c_j e^{\delta|x|}, \quad x+h = t \in [0,1],$$

which is a consequence of (3).

If $x+h<0$, then (10) follows from (8) and the assumption $\alpha>\beta$.

Second step: the proof of (a) and (b). From assumption (ii) we have $\check{K}\exp(\alpha\,\cdot)\in\mathscr{D}_{\mathscr{L}^1}$ and this implies (a).

Let us prove that

$$f/c_0\in\mathscr{B}'. \tag{12}$$

By [13, Chapitre VI théorème XXV], we have to prove that for every $\phi\in\mathscr{D}(\mathbb{R})$

$$(f/c_0)*\phi\in\mathscr{L}^\infty(\mathbb{R}). \tag{13}$$

With $h\in\mathbb{R}$ and $x\in B=\operatorname{supp}\check{\phi}$ we have

$$((f/c_0)*\phi)(h)=\left\langle\frac{f(\cdot+h)}{c_0(\cdot+h)},\check{\phi}(\cdot)\right\rangle=\left\langle\frac{f(\cdot+h)}{c(h)},\frac{c(h)}{c_0(\cdot+h)}\check{\phi}(\cdot)\right\rangle.$$

By (9) we have that the set

$$\left\{\frac{c(h)}{c_0(\cdot+h)}\check{\phi};h\in\mathbb{R}\right\}\text{ is bounded in }\mathscr{D}(\mathbb{R}).$$

This and assumption (i) imply (13), and thus (12) is proved. Note that (12) implies

$$\frac{f(\cdot+h)}{c_0(\cdot+h)}\in\mathscr{B}'\text{ for every }h\in\mathbb{R}.$$

Now the proof of assertion (b) follows from the equality

$$(f*K)(h)=\langle f(x+h),\check{K}(x)\rangle=\left\langle\frac{f(x+h)}{c_0(x+h)},c_0(x+h)\check{K}(x)\right\rangle,\quad h\in\mathbb{R},$$

if we prove that for every $h\in\mathbb{R}$, $c_0(\cdot+h)\check{K}\in\mathscr{D}_{\mathscr{L}^1}$.

Let $k,j\in\mathbb{N}_0$, and $h\in\mathbb{R}$ be fixed. For $x_0=\max(1-h,h)$ we have the following

$$x+h>1,\text{ when }x>x_0\quad\text{and}\quad x+h<0,\text{ when }x<-x_0.$$

By assumption (ii) and the estimate

$$|c_0^{(k)}(x+h)\check{K}^{(j)}(x)|\leqslant\begin{cases}c_{k,j}\exp(\alpha x+\delta x)\,|\check{K}^{(j)}(x)|, & x+h>1,\\ c_{k,j}\exp(\beta x)\,|\check{K}^{(j)}(x)|, & x+h<0,\end{cases}$$

where $c_{k,j}>0$ is a suitable constant, it follows that $c_0^{(k)}(\cdot+h)|\check{K}^{(j)}|$ is bounded by a function from $\mathscr{L}^1(\mathbb{R})$ and this implies that $c_0(\cdot+h)\check{K}\in\mathscr{D}_{\mathscr{L}^1}$.

Third step: we prove that assumption (iv) (with (ii) and (i)) implies that

$$\left(\frac{f}{c_0}*(K\exp(-\alpha\,\cdot))\right)(h)=\left\langle\frac{f(x+h)}{c_0(x+h)},\check{K}(x)\exp(\alpha x)\right\rangle$$
$$\longrightarrow a\int_{\mathbb{R}}K(t)\exp(-\alpha t)\,dt,\text{ as }h\longrightarrow\infty. \tag{14}$$

Assuming (iv) it is enough to prove that

$$\left\langle\frac{f(x+h)}{c_0(x+h)},\left(\frac{c_0(x+h)}{L(\exp h)\exp(\alpha h)}-\exp(\alpha x)\right)\check{K}(x)\right\rangle\longrightarrow 0,\text{ as }h\longrightarrow\infty. \tag{15}$$

Since $f/c_0 \in \mathscr{B}'$, we have $f/c_0 = \sum_{j=0}^{m} F_j^{(j)}$, where the F_j, $j = 0, \dots, m$, are bounded and continuous functions. So the left side of (15) is equal to

$$\sum_{j=0}^{m}(-1)^j \int_{\mathbb{R}} F_j(x+h)\left(\left(\frac{c_0(\cdot+h)}{L(\exp h)\exp(\alpha h)} - \exp(\alpha\cdot)\right)\check{K}(\cdot)\right)^{(j)}(x)\,dx, \quad h > 0,$$

and if we prove that for every $j \in \mathbb{N}_0$

$$\left(\left(\frac{c_0(\cdot+h)}{L(\exp h)\exp(\alpha h)} - \exp(\alpha\cdot)\right)\check{K}(\cdot)\right)^{(j)} \longrightarrow 0, \text{ as } h \longrightarrow \infty, \text{ in } \mathscr{L}^1, \tag{16}$$

we get (15). Let us prove that (16) holds.

Assumption (ii) implies that $\exp(\alpha\cdot)\check{K}$ belongs to $\mathscr{D}_{\mathscr{L}^1}$. In proving (b) we established that $c_0(\cdot+h)\check{K}$ belongs to $\mathscr{D}_{\mathscr{L}^1}$ for every $h \in \mathbb{R}$. By (5), for any $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$

$$\left(\exp(\alpha\cdot)\left(\frac{\tilde{L}(\exp(\cdot+h))}{L(\exp h)} - 1\right)^{(k)}\right)(x) \longrightarrow 0, \text{ as } h \longrightarrow \infty.$$

For the use of Lebesgue's theorem it only remains to prove that

$$\left(\left(\frac{c_0(\cdot+h)}{L(\exp h)\exp(\alpha h)} - \exp(\alpha\cdot)\right)\check{K}\right)^{(n)}, \quad n \in \mathbb{N}_0, \quad h > 0,$$

is bounded by some $\mathscr{L}^1$-function, which does not depend on h. For this purpose it is enough to prove that the set of functions

$$\left\{\left|\left(\frac{c_0(\cdot+h)}{L(\exp h)\exp(\alpha h)} - \exp(\alpha\cdot)\right)^{(k)} \check{K}^{(j)}\right|, h > 0\right\}$$

is bounded by some $\mathscr{L}^1$-function. Let us prove this assertion. If $x > 0$, by (10) we have

$$\left|\left(\frac{c_0(x+h)}{L(\exp h)\exp(\alpha h)} - \exp(\alpha x)\right)^{(k)} \check{K}^{(j)}\right| \leqslant c_k |\check{K}^{(j)}(x)| \exp((\alpha+\delta)x), \quad h > 0.$$

Since (ii) holds for some $\delta > 0$, it holds for every δ', $0 < \delta' \leqslant \delta$. So we can take δ so small that $\delta \in (0, \alpha-\beta)$. If $x < 0$, both cases in (10) can appear and this implies that

$$\begin{aligned}\left|\left(\frac{c_0(x+h)}{L(\exp h)\exp(\alpha h)} - \exp(\alpha x)\right)^{(k)} \check{K}^{(j)}(x)\right| &\leqslant c_k(\exp(\alpha x + \delta|x|) + \exp(\beta x))\,|\check{K}^{(j)}(x)| \\ &\leqslant 2c_k |\check{K}^{(j)}(x)| \exp(\beta x).\end{aligned}$$

Thus we have proved the quoted assertion and the proof of (14) is completed.

Fourth step: the proof of (c). From (14) and assumption (iii), by Theorem 1 we have, for every $\phi \in \mathscr{D}_{\mathscr{L}^1}$,

$$((f/c_0) * \phi)(h) \longrightarrow a \int_{\mathbb{R}} \phi(x)\,dx, \textit{ as } h \longrightarrow \infty.$$

If $\psi \in C^\infty(\mathbb{R})$ satisfies the assumption given in (c), then $\check{\psi}\exp(\alpha\cdot) \in \mathscr{D}_{\mathscr{L}^1}$ and we have

$$((f/c_0) * (\psi\exp(-\alpha\cdot)))(h) \longrightarrow a \int_{\mathbb{R}} \psi(x)\exp(-\alpha x)\,dx, \text{ as } h \longrightarrow \infty.$$

As in the third step, we prove that $(f*\psi)(h)$, $h\in\mathbb{R}$, exists and that (c) holds. Since we have to prove that

$$\left\langle\frac{f(x+h)}{L(\exp h)\exp(\alpha h)},\check{\psi}(x)\right\rangle\longrightarrow a\int_{\mathbb{R}}\psi(x)\exp(-\alpha x)\,dx,\text{ as } h\longrightarrow\infty,$$

it is enough to prove that

$$\left\langle\frac{f(x+h)}{c_0(x+h)},\left(\frac{c_0(x+h)}{L(\exp h)\exp(\alpha h)}-\exp(\alpha x)\right)\check{\psi}(x)\right\rangle\longrightarrow 0,\text{ as } h\longrightarrow\infty.$$

But this has already been done (with K instead of ψ) in the third step and so the proof of the theorem is complete.

Proofs of Propositions 1 *and* 2. If in Theorem 2 we take $K\in\mathscr{D}$, then assumption (ii) is satisfied for every $\alpha,\beta,\delta\in\mathbb{R}$, and assumptions (i), (iii) and (iv) of Theorem 2 imply the assertion in Proposition 1.

Proposition 2 is not a direct consequence of Theorem 2, but the proof of it is a simpler version of the proof of Theorem 2.

The first step in the proof of Theorem 2 is not needed.

Assumption (ii) implies that $\check{k}\exp(\alpha\cdot)\in\mathscr{L}^1(\mathbb{R})$ and that for every $x\in\mathbb{R}$, $c(x+\cdot)\check{k}\in\mathscr{L}^1(\mathbb{R})$. Since

$$(f*k)(x)=\int_{\mathbb{R}}\frac{f(t+x)}{c(t+x)}c(t+x)\check{k}(t)\,dt,\quad x\in\mathbb{R},$$

(i) implies that $(f*k)(x)$ exists for every $x\in\mathbb{R}$.

We next need:

$$\left(\frac{f}{c}*k\exp(-\alpha\cdot)\right)(h)\longrightarrow a\int_{\mathbb{R}}k(t)\exp(-\alpha t)\,dt,\text{ as } h\longrightarrow\infty,\tag{17}$$

for which it is enough to prove that

$$\int_{\mathbb{R}}\frac{f(x+h)}{c(x+h)}\left(\frac{c(x+h)}{L(\exp h)\exp(\alpha h)}-\exp(\alpha x)\right)\check{k}(x)\,dx\longrightarrow 0,\text{ as } h\longrightarrow\infty$$

and this follows from (16) with $j=0$ and with k instead of K.

By applying the Wiener theorem to (17) we deduce that, for ψ satisfying the conditions in Proposition 2,

$$\left(\frac{f}{c}*(\psi\exp(-\alpha\cdot))\right)(h)\longrightarrow a\int_{\mathbb{R}}\psi(t)\exp(-\alpha t)\,dt,\text{ as } h\longrightarrow\infty.$$

Finally, for (2′) it is enough to prove that

$$\int_{\mathbb{R}}\frac{f(x+h)}{c(x+h)}\left(\frac{c(x+h)}{L(\exp h)\exp(\alpha h)}-e^{\alpha x}\right)\check{\psi}(x)\,dx\longrightarrow 0,\text{ as } h\longrightarrow\infty$$

and this again follows from (16) with $j=0$ and with ψ instead of K.

In particular, if $\beta+1>0$, then the function ψ defined by

$$\psi(t)=\begin{cases}0 & t<0,\\ \exp(-t) & t\geqslant 0,\end{cases}$$

satisfies the conditions in Proposition 2, (iv). Therefore, (2′) implies that the last assertion in Proposition 2 holds.

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