THEOREM 4. *Let a be an arbitrarily often differentiable function on the real line having countably many isolated zeros $s_1, s_2, \ldots$ of finite orders $m_1, m_2, \ldots$, respectively. Denote by A the multiplication operator mapping x into $Ax = ax$ and let E be any linear space satisfying $X = E \oplus R^*$. Then there is a unique continuous linear operator $A^{-1}:\bar{X} \rightarrow \bar{X}$ whose range is orthogonal to E and is such that $AA^{-1} = I$.*

The boundedness and local property of P_E can be proved by using the fact that every E satisfying $E \cap R^* = \{\theta\}$ is locally finite dimensional. Precisely, $X = E \oplus R^*$ if and only if for every root s_k there exist elements $\epsilon_\lambda \in E$ ($\lambda = 0, \ldots, m_k - 1$) with the property that $\epsilon_\lambda^{(\mu)}(s_k) = \delta_{\lambda\mu}$ for $\lambda, \mu = 0, \ldots, m_k - 1$, where $\delta_{\lambda\mu}$ is the Kronecker delta and at each s_n ($n \pm k$) the functions ϵ_λ vanish with order m_n. A natural division operator $(A^{-1})^*:R^* \rightarrow X$ is defined on R^*, which maps $r^* \in R^*$ into $r^*/a \in X$. Although the operator $(A^{-1})^*$ is not continuous, it can be extended in many ways to a linear operator $(A^{-1})^*:X \rightarrow X$. Namely, each A^{-1} has an adjoint $(A^{-1})^*:X \rightarrow X$ whose restriction to R^* is the natural division operator. The null-space of the adjoint operator associated with E is the linear space E.

The crucial point in the proof is the verification of the condition "$[u:au \in W]$ *belongs to* $\mathfrak{U}$ *for every* W *in* $\mathfrak{U}$." The only non-trivial tool needed in the proof of this proposition is the generalized mean-value theorem of the differential calculus. The same proposition leads to the solution of the division problem for special types of divisors in the case of several variables. This includes division by linear functions and by quadratic functions of the form $a(s_1, \ldots, s_n) = \sum a_k s_k^2$, where $a_k \geq 0$ ($k = 1, \ldots, n$). Earlier Schwartz proved that division of individual distributions by "regular" functions is possible.[1] The principle of localization and a change in variables show that division by regular functions can be treated globally:

THEOREM 5. *Let a be a regular function on a finite dimensional Euclidean space S and let $A:\bar{X} \rightarrow \bar{X}$ be the operator corresponding to multiplication by a. Then there exist continuous linear operators $A^{-1}:\bar{X} \rightarrow \bar{X}$ such that $AA^{-1} = I$.*

A satisfactory solution of the division problem in general would depend on the proof of the proposition "$[u:au \in W]$ *belongs to* $\mathfrak{U}$ *for every* W *in* $\mathfrak{U}$."

[1] L. Schwartz, *Théorie des distributions* ("Actualités scientifiques et industrielles," No. 1245 [Paris: Hermann, 1957]), 1, 126–28.

Note added in proof: I was recently informed that the problem of division of fixed distributions was solved by both L. Hörmander and S. Lojasiewicz working independently.

ON THE SPECTRAL THEORY OF SINGULAR INTEGRAL OPERATORS

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The purpose of this note is to present some results on the spectral theory of singular integral operators of the form

$$Lx(\lambda) = f(\lambda)x(\lambda) + \frac{1}{\pi i}\int_a^b \frac{\sqrt{k(\lambda)k(\mu)}}{\mu - \lambda}\, x(\mu)d\mu$$

acting on functions $x(\lambda)$ belonging to the class $L^2(a, b)$. (The integral which appears here is to be interpreted as a Cauchy principal value.) We shall assume that $f(\lambda)$ and $k(\lambda)$ are real, continuously differentiable functions on the interval $a \leq \lambda \leq b$, such that the functions $f'(\lambda) \pm k'(\lambda)$ have only isolated zeros and where $k(\lambda) > 0$ almost everywhere on the interval $a \leq \lambda \leq b$.

From the theory of singular integral equations[1] and some results on integrals of Cauchy type,[2] one can prove

THEOREM 1. *The limit points of the spectrum of the operator L lie in the closed interval* $[A, B]$, *where* $A = \min\{f(\lambda) - k(\lambda)\}$, $B = \max\{f(\lambda) + k(\lambda)\}$.

Next we introduce the functions

$$E(l, z) = \exp\left\{\frac{1}{2\pi i}\int_a^b \log\frac{f(u) - l - k(u)}{f(u) - l + k(u)}\,\frac{du}{u - z}\right\},$$

$$H(l, \lambda) = E(l, \lambda + i0) - E(l, \lambda - i0),$$

$$F(\xi, z) = E(\xi + i0, z) - E(\xi - i0, z),$$

and

$$A(\xi) = -\frac{1}{2\pi i}\lim_{\substack{\eta\to 0\\ \eta > 0}}\int_a^b \{H(\xi + i\eta, \lambda) - H(\xi - i\eta, \lambda)\}d\lambda.$$

Here ξ and λ are real. Then one can prove

LEMMA 1. $A(\xi) > 0$ *almost everywhere on the interval* $[A, B]$.

and

THEOREM 2. *The transformations*

$$\hat{R}h(\lambda) = -\frac{1}{2\pi i}\lim_{\substack{\eta\to 0\\ \eta > 0}}\int_a^b \frac{\overline{H(\xi + i\eta, \lambda) - H(\xi - i\eta, \lambda)}}{\sqrt{2k(\lambda)A(\xi)}}\,h(\lambda)d\lambda$$

and

$$\hat{S}g(\xi) = \frac{1}{2\pi i}\lim_{\substack{\eta\to 0\\ \eta > 0}}\int_A^B \frac{F(\xi, \lambda + i\eta) - F(\xi, \lambda - i\eta)}{\sqrt{2k(\lambda)A(\xi)}}\,g(\xi)d\xi$$

generate isometries R and S between the subspace $\bar{L}_\lambda$ *of* $L^2(a, b)$, *which is the closure of the linear manifold* L_λ *generated by the elements* $\left\{\sqrt{2k(\lambda)}, \frac{H(\omega, \lambda)}{\sqrt{2k(\lambda)}}\right\}$ where ω *ranges over all complex numbers in the exterior of the interval* $[A, B]$ *and the space* $L^2(A, B)$. *S is the inverse of R. The space* L_λ *is invariant under the operation L. Furthermore, if* $h(\lambda)$ *belongs to* $\bar{L}_\lambda$ *and if* $g(\xi) = Rh(\lambda)$, *then* $\xi g(\xi) = RLh(\lambda)$.[3]

From Theorems 1 and 2 we obtain

COROLLARY. *The set of limit points of the spectrum of the operator L is the entire interval* $[A, B]$ *and if* $k(\lambda) \pm 0$, $a \leq \lambda \leq b$, *then the spectrum of L consists of precisely all the points of the interval above.*

Now, one is in the position to prove

THEOREM 3. *Consider the function* $G(\lambda; \xi) = \{f(\lambda) - \xi - k(\lambda)\}/\{f(\lambda) - \xi + k(\lambda)\}$, $a \leq \lambda \leq b$. *Suppose that when* $f(a) - k(a) \leq \xi \leq f(a) + k(a)$, *the function* $G(\lambda; \xi)$ *changes sign at most once in the interval* $a \leq \lambda \leq b$, *and that for all other values of* ξ, *the function changes signs at most twice. Then the operator L has a simple continuous spectrum from* $A = \min\{f(\lambda) - k(\lambda)\}$ *to* $B = \max\{f(\lambda) + k(\lambda)\}$ *and the transformations R and S of Theorem 2 furnish a complete spectral representation for L.*

$F(\xi, z)$ *satisfies an identity of the form*

$$\frac{(z_2 - z_1)P(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} F(\xi, z_1)\overline{F(\xi, \bar{z}_2)} = \{E(\xi + i0, z_1)\overline{E(\xi - i0, \bar{z}_2)} - E(\xi - i0, z_1)\overline{E(\xi + i0, \bar{z}_2)}\}, \quad A < \xi < B,$$

where for each fixed ξ, $P(\xi, z_1, z_2)$ is a polynomial in z_1 and z_2 with real coefficients and $Q(\xi, z)$ is a polynomial in z with real coefficients. We now make the following hypothesis:

HYPOTHESIS A.

$$\frac{P(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} = \frac{1}{A(\xi)} + \sum_{j=1}^{N} \frac{C_j(\xi, z_1)C_j(\xi, z_2)}{Q(\xi, z_1)Q(\xi, z_2)},$$

where the $C_j(\xi, z)$ are polynomials in z, with real coefficients.

One can now prove

LEMMA 2. *Suppose that when* $f(a) - k(a) \leq \xi \leq f(a) + k(a)$, *the function* $G(\lambda;\ \xi)$ *of Theorem 3 changes signs at most three times in the interval* $a \leq \lambda \leq b$, *and that for all other values of* ξ, *the function changes signs at most four times. Then hypothesis A is satisfied and*

$$\frac{P(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} = \frac{1}{A(\xi)} + \frac{\sqrt{B(\xi)/A(\xi)}}{Q(\xi, z_1)} \frac{\sqrt{B(\xi)/A(\xi)}}{Q(\xi, z_2)},$$

where $B(\xi) \geq 0$.

THEOREM 4. *If the function* $G(\lambda;\ \xi)$ *satisfies the conditions stated in Lemma 2, then there exists an isometry* $\mathcal{R}$ *of* $L^2(a, b)$ *onto the direct sum* $\mathcal{H}$ *of the two Hilbert spaces* $L^2(A, B)$ *and* $L^2(\sum)$, *where* $\sum = \{\xi | B(\xi) \pm 0, A < \xi < B\}$. $\mathcal{R}$ "coincides" with the transformation R of Theorem 2 on the subspace $\bar{L}_\lambda$ of $L^2(a, b)$. Furthermore, $\mathcal{R}$ furnishes a spectral representation of the operator L. Thus the spectrum of L has multiplicity ≤ 2.

More general self-adjoint operators of the form

$$Mx(\lambda) = f(\lambda)x(\lambda) + \frac{1}{\pi i}\int_d^b \frac{g(\lambda, \mu)}{\mu - \lambda} x(\mu)d\mu$$

can be shown differ from a direct sum of operators of the form L and multiplication operators by completely continuous operators. Thus we have a method for obtaining the essential spectrum of operators of the form M.

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1 N. S. Muskhelishvili, *Singular Integral Equations* (Groningen: Noordhoff, 1953).

2 E. C. Titchmarch, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937).

3 W. Koppelman and J. D. Pincus, "Perturbation of Continuous Spectra and Singular Integral Operators," *Comm. on Pure and Appl. Math.* (to appear), contains a special case of this result.