6. Nonlinear problems

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Problem $$F(x,u)=0$$

Aproximations $\varphi \in S_h :\quad (F(\cdot,\varphi),\chi)=0$ for $\chi \in S_h$

Regularity assumptions (roughly)

i) There is a unique solution,

ii) F, F_u are Lischitz continuous.

Error analysis

Put $f(x):=F_u(x,u(x))$ and $e:=u-\varphi$, then

$$(fe,\chi)=(R,\chi) \quad \text{for} \quad \chi \in S_h$$

with a remainder term

$$R=R(e)=F(\cdot,u-\varphi)+fe$$

resp.

$$(f\varphi,\chi)=(fu-R(e),\chi) \quad \text{for} \quad \chi \in S_h \ .$$

Let P_h denote the L_2- projection relative to $(f\cdot,\cdot)$. Then

$$\varphi = P_h(u-\frac{1}{f}R(e)) \ ,$$

resp.

$$e=(I-P_h)(u)+P_h(\frac{1}{f}R(e))=:T(e) \ .$$

This means, that the difference $e:=u-\varphi$ is a fixpoint solution of the operator T.

Poperties of the operator T

Lemma 1: There is a $\kappa > 0$ such that for sufficiently small

$$\bar{\varepsilon} := \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty}$$

the operator T maps the ball

$$B_{\kappa\bar{\varepsilon}} := \left\{ e \middle| \|e\|_{L_\infty} \leq \kappa\bar{\varepsilon} \right\}$$

into itself.

Proof:

i) Because of P_h being bounded we have

$$\|(I - P_h)u\|_{L_\infty} \leq c_1 \bar{\varepsilon}$$

ii) For the same reason ($f^{-1} < \infty$)

$$\left\| P_h (\frac{1}{f} R(e)) \right\|_{L_\infty} \leq c_2 \|\mathrm{Re}\|_{L_\infty}$$

iii) It is

$$\|F(\cdot, u - e) + f \cdot e\|_{L_\infty} \leq c_3 \|e\|_{L_\infty}$$

with c_3 being the Lischitz constant of F_u .

From i)-iii) it follows

$$\|Te\|_{L_\infty} \leq c_1 \bar{\varepsilon} + c_3 c_2 \kappa^2 \bar{\varepsilon}^{-2} \leq \bar{\varepsilon}(c_1 + c_2 c_3 \kappa^2 \bar{\varepsilon}) \ .$$

Now fix $\kappa > c_1$ and choose $\bar{\varepsilon}_0$ according to

$$\kappa = c_1 + c_2 c_3 \kappa^2 \bar{\varepsilon}_0 \ .$$

Lemma 2: For $\bar{\varepsilon}$ small, the operator T is a contradiction in

$$B_{\kappa\bar{\varepsilon}} := \{ e \| \|e\|_{L_\infty} \le \kappa\bar{\varepsilon} \} \ .$$

Proof:

$$\|T(e_1) - T(e_2)\|_{L_\infty} \le \left\| P_h \left(\frac{1}{f} R(e_1) - R(e_2)\right) \right\|_{L_\infty} \le c_2 \|R(e_1) - R(e_2)\|_{L_\infty}$$

Now

$$R(e_1) - R(e_2) = F(\cdot, u - e_1) - F(\cdot, u - e_2) + f \cdot (e_1 - e_2)$$

$$= (F_u(\cdot, \vartheta) - F_u(\cdot, u))(e_1 - e_2)$$

with

$$(F_u(\cdot, \vartheta) := F_u(\cdot, u - \vartheta e_1 - (1 - \vartheta) e_2)$$

and

$$\|F_u(\cdot, \vartheta) - F_u(\cdot, u)\|_{L_\infty} \le \kappa\bar{\varepsilon} c_3 \ .$$

From this the assertion of the lemma follows for

$$\bar{\varepsilon} < \min\left\{ \bar{\varepsilon}_0 , \frac{1}{c_2 c_3 \kappa} \right\} \ .$$

Consequence: The operator T has a unique fixpoint in $B_{\kappa\bar{\varepsilon}} := \{ e \| \|e\|_{L_\infty} \le \kappa\bar{\varepsilon} \}$

Theorem 3: The FEM admits the error estimate

$$\|u - \varphi\|_{L_\infty} \le c \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty}$$