On Boltzmann and Landau equations

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We present here various compactness properties enjoyed by global solutions of the so-called Boltzmann and Landau equations. These properties, which are crucial for the existence of global solutions, are shown to depend heavily on the structure of the collision operators.

1. Introduction

We study here some properties of solutions of the following kinetic equations

$$\partial f/\partial t + v\cdot\nabla_x f = Q(f,f) \quad \text{for } t \geqslant 0,\ x\in\mathbb{R}^N,\ v\in\mathbb{R}^N, \tag{1}$$

where $N \geqslant 1$, f is a non-negative function and $Q(f,f)$ is a non-local, quadratic operator. Physically, such equations provide a mathematical model for the statistical evolution of a large number of particles interacting through 'collisions'. They are used for the description of a moderately rarefied gas or of plasmas. The unknown function f corresponds at each time t to the density of particles at the point x with velocity v. If the operator Q were 0, (1) would simply mean that the particles do not interact and f would be constant along particle paths ($\dot{x} = v$, $\dot{v} = 0$). This conservation no longer holds if collisions occur, in which case the rate of changes of f has to be specified. Such a description was introduced by Maxwell (1886, 1890) and Boltzmann (1872) and involves an integral operator described below. This model is derived under the assumption of stochastic independence of pairs of particles at (x,t) with different velocities (molecular chaos assumption). For further detail on the derivation of this model (Boltzmann collision operator), we refer the reader to Chapman & Cowling (1952), Grad (1958), Cercignani (1988), Truesdell & Muncaster (1960) and the references therein.

To explain the mathematical results we shall present here, we need to detail the structure of Boltzmann collision operator B. If φ is a smooth function (say $\varphi \in C_0^\infty(\mathbb{R}^N)$) of v then $Q(\varphi,\varphi)$ is a function of v given by

$$Q(\varphi,\varphi) = \int_{\mathbb{R}^N} \mathrm{d}v_* \int_{S^{N-1}} \mathrm{d}\omega \{\varphi(v')\varphi(v'_*) - \varphi(v)\varphi(v_*)\} B(v-v_*, \omega), \tag{2}$$

where $v' = v-(v-v_*,\omega)\,\omega$, $v'_* = v_* + (v-v_*,\omega)\,\omega$, and we denote by $a\cdot b$ or (a,b) the scalar product in $\mathbb{R}^N$. The collision kernel B depends on the nature of the interaction between particles and always satisfies at least

$$B \geqslant 0, \quad B(z,\omega) \text{ is a function of } |z|,\ |(z,\omega)| \text{ only}. \tag{3}$$

It is worth recalling the significance of the velocities $v, v_*, v', v'_* : v', v'_*$ are the velocities of two 'colliding' particles before a collision that will bring them to have velocities v, v_*. Elastic collisions must obey the conservation of momentum and

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kinetic energy ($v+v_* = v'+v'_*$, $|v|^2+|v_*|^2 = |v'|^2+|v'_*|^2$) and the formula written above for v' and v'_* are a description of all possible solutions of these two balance laws.

Of course, $Q(f,f)$ in (1) means $Q(f(t,x,\cdot), f(t,x,\cdot))$, provided such a quantity makes sense or in other words provided the integrals in (2) make sense first for a smooth φ and next for a solution of (1). The second part of this difficulty is of course related to *a priori* estimates and regularity informations on the solutions. But, even the first part is a serious mathematical issue since realistic collision kernels B can be rather singular as we explain now. Indeed, if the so-called hard-spheres model where $B(z,\omega) = |(z,\omega)|$ does not present real singularities, for inverse power intermolecular potentials, B takes the following form

$$B(z,\omega) = b(\theta)\,|z|^{-\gamma} \quad \text{with} \quad \gamma = 1-2(N-1)(s-1)^{-1}$$

where $s>1$ is the exponent of the potential, θ is the angle between $v-v_*$ and ω so that $\cos\theta = (v-v_*, \omega)\,|v-v_*|^{-1}$. In addition, b is smooth except at $\theta = \pm\frac{1}{2}\pi$ where it has a singularity of the form $|\cos\theta|^{-\alpha}$ with $\alpha = (s+1)(s-1)^{-1}$ if $N=3$. In other words, B presents singularities of an arbitrary high order when $(v-v_*, \omega) = 0$, condition that corresponds to the so-called *grazing collisions*. A classical approach consists in avoiding this difficulty, neglecting thus grazing collisions, and one simply truncates b assuming for instance

$$B \in L^1_{\text{loc}}(\mathbb{R}^N \times S^{N-1})$$

(see Grad 1958; Cercignani 1988; Truesdell & Muncaster 1960).

On the other hand, when almost all collisions are grazing, phenomenological arguments introduced by Landau (see Lifschitz & Pitaerskii 1981) and by Chapman & Cowling (1952) lead to another collision operator

$$Q(f,f) = \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^N} \mathrm{d}v_*\, a_{ij}(v-v_*)\left[f(v_*)\frac{\partial f(v)}{\partial v_j} - f(v)\frac{\partial f(v_*)}{\partial v_{*,j}}\right]\right\} \tag{5}$$

in which case (1) becomes the Landau equation (it also called the Fokker–Planck equation). The matrix $(a_{ij}(z))$ is symmetric, non-negative, even in z and is typically of the following form if $N=3$,

$$a_{ij}(z) = (a(z)/|z|)\,\{\delta_{ij} - z_i z_j/|z|^2\}, \tag{6}$$

where a is even, smooth (for instance) and positive on $\mathbb{R}^N$. In (5) and everywhere below, we use the standard convention of implicit summation over repeated indices.

Justifications of the collision operator given in (5) can be found in Desvillettes (1994) (through an asymptotic expansion of Boltzmann collision operators with small parameters) and in Degond & Lucquin-Desreux (1994) (via an expansion of a physically realistic Boltzmann collision operator around grazing collisions). These works strongly suggest that, in addition to the intrinsic interest in Landau equation, some insight on the Boltzmann equation when one does not make the angular cut-off might be gained by an analysis of the Landau model.

This is precisely our goal here and we shall prove that solutions of the Landau equation enjoy a rather striking compactness property and that this property does not hold for the Boltzmann equation with angular cut-off.

Let us mention at this stage that compactness properties of solutions of nonlinear partial differential equations are often a replacement for regularity results (that seem out of reach) and play a fundamental role in global existence results. Even if we are not concerned here with existence issues, we would like to mention that this type of compactness properties is one of the key ingredients in the results by DiPerna & Lions (1989*a*, 1991) on the global existence of weak solutions for the Boltzmann equation with general initial conditions and general collision kernels B with angular cut-off (see also DiPerna & Lions (1988*a*, 1989*b*) for results concerning related kinetic models). References to previous work on Boltzmann equations can be found in DiPerna & Lions (1989*a*).

In §2 below, we state our main compactness result for the Landau model: we prove that sequences of solutions with natural bounds are compact in $L^p(0,T;L^1(\mathbb{R}^{2n}))$ $(\forall 1 \leqslant p < \infty, \forall T \in (0,\infty))$. And we recall a weaker compactness result for solutions of the Boltzmann equation with angular cut-off, a result shown in DiPerna & Lions (1989*a*). Finally, we also show that the result for the Landau model does not hold for the Boltzmann equation with angular cut-off. The proofs are given in §3. We will also mention in §3 how the method of proof also yields some apparently new results on linear equations in cases which are related but more general than some typical hypoelliptic equations.

2. Compactness results

We shall consider a sequence of solutions $(f^n)_n$ of (1) corresponding to a sequence $(f_0^n)_n$ of initial conditions. We shall assume natural bounds which are straightforward consequences of the following formal identities that hold for solutions of (1) in the case of Boltzmann collision operators (2) or Landau collision operators (5). For any solution f of (1) (in these two cases), we have at least formally

$$\iint_{\mathbb{R}^{2N}} f\psi \,\mathrm{d}x\,\mathrm{d}v \quad \text{is independent of } t \quad \text{for } \psi \equiv 1,\ v_j (1 \leqslant j \leqslant N),\ |v|^2,\ |x-vt|^2. \tag{7}$$

In addition,

$$\iint_{\mathbb{R}^{2N}} f\log f \,\mathrm{d}x\,\mathrm{d}v \quad \text{is non-increasing with respect to } t. \tag{8}$$

In fact, a more precise formulation of (8) is the following formal identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^{2N}} f\log f\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{4}\iiiint \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega B(f'f'_* - ff_*)\log\frac{f'f'_*}{ff_*} = 0, \tag{9}$$

where $f_* = f(t,x,v_*)$, $f' = f(t,x,v')$, $f'_* = f(t,x,v'_*)$, in the case of the Boltzmann model. For the Landau model, (9) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^{2N}} f\log f\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\iiint \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}v_*\, a_{ij}(v-v_*)ff_*\left(\frac{\partial}{\partial v_i}(\log f) - \frac{\partial}{\partial v_{*,i}}(\log f_*)\right)$$
$$\times\left(\frac{\partial}{\partial v_j}(\log f) - \frac{\partial}{\partial v_{*,j}}(\log f_*)\right) = 0. \tag{10}$$

These conservations or identities lead to the following 'natural' bounds that we assume throughout this paper

$$\sup_{\substack{t\geqslant 0\\ n\geqslant 1}}\left\{\iint_{\mathbb{R}^{2N}} f^n[1+|x-vt|^2+|v|^2+|\log f^n|]\,\mathrm{d}x\,\mathrm{d}v\right\} < +\infty, \tag{11}$$

let us also recall that $f^n \geqslant 0$ on $[0, \infty) \times \mathbb{R}_x^N \times \mathbb{R}_v^N$.

To avoid unnecessary technicalities, we simply assume that f^n are smooth solutions of (1) say C^∞ with fast decay as (x, v) goes to infinity. In fact, the results below hold for convenient weak solutions, i.e. renormalized solutions – a notion introduced in DiPerna & Lions (1989*a*) – and for various approximations or regularizations of (1). Finally, they can be used to deduce global existence results of such weak solutions.

We begin with the Boltzmann model with angular cut-off.

Theorem 1. *We assume that B satisfies for all* $R \in (0, \infty)$

$$\left.\begin{array}{c} B \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times S^{N-1}), \quad \displaystyle\iint_{|z-\xi| \leqslant R} B \,\mathrm{d}z\,\mathrm{d}\omega \leqslant C_R(1+|z|^2) \\ \forall z \in \mathbb{R}^N \quad \textit{for some} \quad C_R \geqslant 0. \end{array}\right\} \tag{12}$$

Then, for each $\psi \in L^\infty(\mathbb{R}_x^N \times \mathbb{R}_v^N)$, $\int_{\mathbb{R}^N} f^n \psi \,\mathrm{d}v$ *is relatively compact in* $L^r(0, T; L^1(\mathbb{R}_x^N))$ *for all* $1 \leqslant r < \infty$, $T \in (0, \infty)$.

Remarks. (i) This result is shown in DiPerna & Lions (1989*a*) with a slightly more restrictive assumption on B and under another bound on f^n namely a bound on the dissipation of entropy that follows from (9)). This minor improvement is explained in DiPerna *et al.* (1991).

(ii) The compactness stated in Theorem 1 is sufficient to pass to the limit in the collision terms (and in the entropy inequality) as shown in DiPerna & Lions (1989*a*, 1991).

We next consider the Landau model and we recall that (a_{ij}) is symmetric, non-negative and even in z. In addition, we assume

$$\left.\begin{array}{l} \text{for all } R \in (0, \infty) \text{ there exists } \nu > 0 \text{ such that} \\ a_{ij}(z)\,\eta_i\,\eta_j \geqslant \nu|\eta|^2 \text{ if } \eta \cdot z = 0,\ |z| \leqslant R, \end{array}\right\} \tag{13}$$

$$a_{ij} \in L^1 + L^\infty; \quad \frac{\partial a_{ij}}{\partial z_j}, \frac{\partial^2}{\partial z_i\,\partial z_j} a_{ij} \in \mathcal{M} + L^\infty, \tag{14}$$

where $\mathcal{M}$ denotes the space of bounded measures on $\mathbb{R}^N$.

For instance, if $N \geqslant 3$, $a_{ij}(z) = (a(z)/|z|^\theta)\{\delta_{ij} - z_i z_j/|z|^2\}$, where $0 \leqslant \theta \leqslant N-2$, a is smooth (say in $W^{2,\infty}(\mathbb{R}^N)$) and positive on $\mathbb{R}^N$, (13) and (14) hold. Notice that when $\theta = N-2$, $a \equiv 1$,

$$\frac{\partial^2}{\partial z_i\,\partial z_j} a_{ij} = -c_N\,\delta_0.$$

Theorem 2. *Under the assumptions* (13) *and* (14), f^n *is relatively compact in* $L^r(O, T; L^1(\mathbb{R}^{2N}_{x,v}))$ *for all* $1 \leqslant r < \infty$, $T \in (0, \infty)$.

Remarks. (i) It is possible to improve a bit the assumption on $\partial^2 a_{ij}/\partial z_i\,\partial z_j$ allowing 'singular integrals' distributions but we will skip such a technical extension here.

(ii) If we do not make the assumption (13), then Theorem 1 still holds. Also, Theorem 2 holds if we add the Boltzmann collision and the Landau collision operators.

(iii) As we shall see in the proof, we also prove that $a_{ij} \underset{v}{*} f^n$ is relatively compact in L^1_{loc}.

Of course, it is then natural to ask whether Theorem 2 holds for Boltzmann models.

This remains an outstanding problem if we do not make the angular cut-off but if we do then we have

Theorem 3. *Under the assumption* (12), *the conclusion of Theorem 2 implies in fact that f^n is relatively compact in $C([0,T];L^1(\mathbb{R}^{2N}_{x,v}))$ and thus, in particular $f_0^n = f^n(0)$ is relatively compact in $L^1(\mathbb{R}^{2N}_{x,v})$.*

Theorems 2 and 3 are shown in §3.

3. Proofs of Theorems 2 and 3

We begin with the proof of Theorem 2. The first step consists in writing the Landau equation in renormalized form in the sense of DiPerna & Lions (1989*a*; see also DiPerna & Lions (1988*b*) and Lions & Murat (1994) for similar formulations in the case of equations involving second-order operators). Let $\beta \in C^2([0,+\infty),\mathbb{R})$. We clearly have for any (smooth) solution f of (1)

$$\frac{\partial}{\partial t}\beta(f) + \xi \cdot \nabla_x \beta(f) = \beta'(f)\, Q(f,f).$$

Next, if the collision operator Q is given by (5)

$$\beta'(f)\,Q(f,f) = \beta'(f)\frac{\partial}{\partial v_i}\left\{\bar{a}_{ij}\frac{\partial}{\partial v_j}f - \bar{b}_i f\right\},$$

where $$\bar{a}_{ij} = \int_{\mathbb{R}^N} a_{ij}(v-v_*) f_*\,\mathrm{d}v_*,\quad \bar{b}_i = \int_{\mathbb{R}^N}\left(\frac{\partial}{\partial z_j}a_{ij}\right)(v-v_*)f_*\,\mathrm{d}v.$$

Therefore, we have

$$\beta'(f)\,Q(f,f) = \frac{\partial}{\partial v_i}\left\{\bar{a}_{ij}\frac{\partial}{\partial v_j}(\beta(f)) - \bar{b}_i\,\beta(f)\right\} - \beta''(f)\,\bar{a}_{ij}\frac{\partial f}{\partial v_i}\frac{\partial f}{\partial v_j} + \bar{c}\{\beta(f)-\beta'(f)f\},$$

where $$\bar{c} = \int_{\mathbb{R}^N}\frac{\partial^2 a_{ij}}{\partial z_i\,\partial z_j}(v-v_*)f_*\,\mathrm{d}v.$$

In conclusion, we find for any solution of (1) and (5)

$$\frac{\partial}{\partial t}\beta(f) + \xi\cdot\nabla_x\beta(f) = \frac{\partial}{\partial v_i}\left\{\bar{a}_{ij}\frac{\partial}{\partial v_j}(\beta(f)) - \bar{b}_i\,\beta(f)\right\}$$
$$- \beta''(f)\,\bar{a}_{ij}\frac{\partial f}{\partial v_i}\frac{\partial f}{\partial v_j} + \bar{c}\{\beta(f)-\beta'(f)f\}. \quad (15)$$

In particular, this equation holds for each f^n where $\bar{a}_{ij}$, $\bar{b}_i$, $\bar{c}$ are replaced by

$$\left.\begin{aligned}
\bar{a}^n_{ij} &= \int_{\mathbb{R}^N} a_{ij}(v-v_*) f^n(t,x,v_*)\,\mathrm{d}v_*,\\
\bar{b}^n_i &= \int_{\mathbb{R}^N}\frac{\partial a_{ij}}{\partial z_j}(v-v_*)f^n(t,x,v_*)\,\mathrm{d}v_*,\\
\bar{c}^n &= \int_{\mathbb{R}^N}\frac{\partial^2 a_{ij}}{\partial z_i\,\partial z_j}(v-v_*)f^n(t,x,v_*)\,\mathrm{d}v_*.
\end{aligned}\right\} \quad (16)$$

To conclude this first step, we want to deduce a few simple bounds from (15). We choose $\beta(t) = t/(1+t)$ and we find

$$-\beta''(f^n)\,\bar{a}^n_{ij}\frac{\partial f^n}{\partial v_i}\frac{\partial f^n}{\partial v_j} = \bar{a}^n_{ij}\frac{\partial f^n}{\partial v_i}\frac{\partial f^n}{\partial v_i}\frac{1}{(1+f^n)^3}$$

$$= 4\bar{a}^n_{ij}\frac{\partial}{\partial v_i}(\gamma(f^n))\frac{\partial}{\partial v_j}(\gamma(f^n)),$$

where $\gamma(t) = (1-(1+t)^{-\frac{1}{2}})$.

Therefore, if we multiply (15) by $\varphi \in C_0^\infty(\mathbb{R}^N_v)$, $\varphi \geqslant 0$ on $\mathbb{R}^N_v$ and integrate over $[O,T]\times\mathbb{R}^N_x\times\mathbb{R}^N_v$ (for any fixed $T\in(0,\infty)$), we obtain

$$\int_0^T dt \iint_{\mathbb{R}^{2N}} \mathrm{d}x\,\mathrm{d}v\,\varphi\bar{a}^n_{ij}\frac{\partial}{\partial v_i}(\gamma(f^n))\frac{\partial}{\partial v_j}(\gamma(f^n))$$

$$\leqslant \iint_{\mathbb{R}^{2N}} \beta(f^n)(T)\,\varphi\,\mathrm{d}x\,\mathrm{d}v + \int_0^T \mathrm{d}t \iint_{\mathbb{R}^{2N}} \mathrm{d}x\,\mathrm{d}v\,\varphi\bar{c}^n\left\{\frac{f^n}{(1+f^n)^2} - \frac{f^n}{1+f^n}\right\}$$

$$+\int_0^T \mathrm{d}t \iint_{\mathbb{R}^{2N}} \mathrm{d}x\,\mathrm{d}v\,\frac{\partial\varphi}{\partial v_i}\left\{\bar{a}^n_{ij}\frac{\partial\beta(f^n)}{\partial v_j} - \bar{b}^n_i\,\beta(f^n)\right\}.$$

We next want to prove that the three terms in the right-hand side are bounded. This is clear for the first one in view of (11) since $\beta(f^n) \leqslant f^n$. The second term is also bounded since

$$\varphi\left|\frac{f^n}{(1+f^n)^2} - \frac{f^n}{1+f^n}\right| = \varphi\frac{(f^n)^2}{(1+f^n)^2}$$

is bounded in $L^\infty(0,\infty; L^\infty(\mathbb{R}^N_x; L^1\cap L^\infty(\mathbb{R}^N_v)))$ and in view of (14), $\bar{c}^n = c_1 \underset{v}{*} f^n + c_2 \underset{v}{*} f^n$, where $\partial^2 a_{ij}/\partial z_i\,\partial z_j = c_1 + c_2$ and $c_1 \in \mathscr{M}$, $c_2 \in L^\infty$, hence $c_1 \underset{v}{*} f^n$ is bounded in $L^\infty(0,\infty; L^1(\mathbb{R}^{2N}_{x,v}))$ while $c_2 \underset{v}{*} f^n$ is bounded in $L^\infty(0,\infty; L^1(\mathbb{R}^N_x; L^\infty(\mathbb{R}^N_v)))$ in view of (11). The same argument applies to the term

$$\int_0^T \mathrm{d}t \iint_{\mathbb{R}^{2N}} \mathrm{d}x\,\mathrm{d}v\,\frac{\partial\varphi}{\partial v_i}\,\bar{b}^n_i\,\beta(f^n)$$

using (14) again. The last we have to handle is

$$\int_0^T \mathrm{d}t \iint_{\mathbb{R}^{2N}} \mathrm{d}x\,\mathrm{d}v\,\frac{\partial\varphi}{\partial v_i}\,\bar{a}^n_{ij}\frac{\partial\beta(f^n)}{\partial v_j}$$

$$= -\int_0^T \mathrm{d}t \iint_{\mathbb{R}^{2N}} \mathrm{d}x\,\mathrm{d}v\,\frac{\partial^2\varphi}{\partial v_i\,\partial v_j}\,\bar{a}^n_{ij}\,\beta(f^n) - \int_0^T \mathrm{d}t \iint_{\mathbb{R}^{2N}} \mathrm{d}x\,\mathrm{d}v\,\frac{\partial\varphi}{\partial v_i}\,\bar{b}^n_i\,\beta(f^n).$$

And we show exactly as above, using (14), that each of those terms is bounded. In conclusion, we obtain for each $T, R \in (0,\infty)$

$$\sup_n \int_0^T \mathrm{d}t \int_{\mathbb{R}^N} \mathrm{d}x \int_{B_R} \mathrm{d}v\,\bar{a}^n_{ij}\frac{\partial}{\partial v_i}(\gamma(f^n))\frac{\partial}{\partial v_j}(\gamma(f^n)) < \infty, \tag{17}$$

where $B_R = \{z \in \mathbb{R}^n / |z| \leqslant R\}$.

The second step consists in showing that averages in v of f^n and $\beta(f^n)$ are compact in $L^1((0,T)\times B_R)$. We again write (15) with the choice $\beta = t/(1+t)$ we find for all n

$$\frac{\partial}{\partial t}\beta(f^n)+\xi\cdot\nabla_x\beta(f^n)=\frac{\partial}{\partial v_i}\left(\bar{a}^n_{ij}\frac{\partial}{\partial v_j}(\beta(f^n))-\bar{b}^n_i\,\beta(f^n)\right)$$

$$-\beta''(f^n)\,\bar{a}^n_{ij}\frac{\partial f^n}{\partial v_i}\frac{\partial f^n}{\partial v_j}+\bar{c}^n(\beta(f^n)-\beta'(f^n)f^n). \qquad (18)$$

We have shown above that $\bar{b}^n_i\,\beta(f^n)$, $\bar{c}^n\{\beta(f^n)-\beta'(f^n)f^n\}$ are bounded in $L^\infty(0,\infty;L^1(\mathbb{R}^N_x\times B_R))$ and that

$$\beta''(f^n)\,\bar{a}^n_{ij}\frac{\partial f^n}{\partial v_i}\frac{\partial f^n}{\partial v_j}$$

is bounded in $L^1(0,T;L^1(\mathbb{R}^N_x\times B_R))$ for all $R,T\in(0,\infty)$. We then want to show that

$$\sum_i\left|\bar{a}^n_{ij}\frac{\partial}{\partial v_j}(\beta(f^n))\right|$$

is bounded in $L^1(0,T;L^1(\mathbb{R}^N_x\times B_R)$ for all $R,T\in(0,\infty)$. Indeed, we have by Cauchy–Schwarz inequality

$$\sum_i\left|\bar{a}^n_{ij}\frac{\partial}{\partial v_j}(\beta(f^n))\right|\leqslant\Big(\sum_{i,j}|\bar{a}^n_{ij}|\Big)^{\frac12}\left(\bar{a}^n_{ij}\frac{\partial}{\partial v_i}(\beta(f^n))\frac{\partial}{\partial v_j}(\beta(f^n))\right)^{\frac12}$$

$$\leqslant\Big(\sum_{i,j}|\bar{a}^n_{ij}|\Big)^{\frac12}\left(\bar{a}^n_{ij}(1+f^n)^{-3}\frac{\partial f^n}{\partial v_i}\frac{\partial f^n}{\partial v_j}\right)^{\frac12}$$

$$=\Big(\sum_{i,j}|\bar{a}^n_{ij}|\Big)^{\frac12}\left(\bar{a}^n_{ij}\frac{\partial}{\partial v_i}(\gamma(f^n))\frac{\partial}{\partial v_j}(\gamma(f^n))\right)^{\frac12}.$$

In view of (14), $\sum_{i,j}|\bar{a}^n_{ij}|$ is bounded in $L^\infty(0,\infty;L^1(\mathbb{R}^N_x\times B_R))$ for all $R\in(0,\infty)$ and our claim is proved using (17).

We deduce from all these bounds

$$\frac{\partial}{\partial t}\beta(f^n)+\xi\cdot\nabla_x\beta(f^n)=\frac{\partial}{\partial v_i}(g^n_i)+g^n, \qquad (19)$$

where $g^n_i(1\leqslant i\leqslant N)$, g^n are bounded in $L^1(0,T;L^1(\mathbb{R}^N_x\times B_R))$ for all $R,T\in(0,\infty)$. In addition, $\beta(f^n)$ is obviously bounded in $C([0,\infty);L^1\cap L^\infty(\mathbb{R}^{2N}))$. These bounds are enough to ensure that for each $\varphi\in C_0^\infty(\mathbb{R}^N_v)$

$$\int_{\mathbb{R}^N}\beta(f^n)\,\varphi\,\mathrm{d}v\quad\text{is relatively compact in } L^1((0,T)\times B_R) \qquad (20)$$

for all $R,T\in(0,\infty)$. This is a consequence of the general velocity averaging results shown in DiPerna *et al.* (1991), extending the previous results due to Golse *et al.* (1985, 1988), DiPerna & Lions (1986*b*). In fact, one can even show a Sobolev type regularity for such averages. In view of (11) and the boundedness of β, one sees that (20) holds in fact for each $\varphi\in L^1+L^\infty(\mathbb{R}^N)$. And as in DiPerna & Lions (1989*a*), we can deduce from (20)

$$\beta(f^n)\underset{v}{*}\varphi\quad\text{is relatively compact in } L^1((0,T)\times B_R\times B_R) \qquad (21)$$

for all $R, T \in (0, \infty)$, $\varphi \in L^1 + L^\infty(\mathbb{R}^N)$. In particular, if we introduce for $\delta > 0$, $\rho_\delta(z) = (1/\delta^N)\,\rho(z/\delta)$, where $\rho \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{Supp}\rho \subset B_1$, $\rho \geqslant 0$ in $\mathbb{R}^N$, $\int_{\mathbb{R}^N} \rho\,\mathrm{d}z = 1$, we see that

$$(F_\delta^n)_n \text{ is relatively compact in } L^1((0,T)\times B_R \times B_R) \tag{22}$$

for all $R, T \in (0, \infty)$, $\delta > 0$, where $F^N = \beta(f^n)$, $F_\delta^n = \beta(f^n) \underset{v}{*} \rho_\delta$.

Let us also remark that since

$$\beta'(t)^2 = 1/(1+t)^4 \leqslant \gamma'(t)^2 = 1/(1+t)^3$$

on $[0, \infty)$, (17) implies

$$\sup_n \int_0^T \mathrm{d}t \int_{\mathbb{R}^N} \mathrm{d}x \int_{B_R} \mathrm{d}v\, \bar{a}_{ij}^n \frac{\partial F^n}{\partial v_i} \frac{\partial F^n}{\partial v_j} < \infty. \tag{23}$$

We then want to show that (20) and (21) hold with $\beta(f^n)$ replaced by f^n. To this end, we argue as in DiPerna & Lions (1989a) and observe that everything we did with β is still true with $\beta_\nu(t) = t/(1+\nu t)$ for any $\nu > 0$. In particular (20), (21) hold with $\beta(f^n)$ replaced by $\beta_\nu(f^n)$ (for all $R, T \in (0, \infty)$ and for all $\varphi \in L^1 \cap L^\infty(\mathbb{R}^N)$). But then we notice that for each $K > 1$ there exists $C_K > 0$ such that

$$|\beta_\nu(f^n) - f^n| \leqslant C_K\, \nu f^n 1_{f^n \leqslant K} + \frac{1}{\log K} f^n \log f^n 1_{f^n > K}$$

$$\leqslant C_K\, \nu f^n + \frac{1}{\log K} f^n |\log f^n|.$$

This allows to deduce, using (11), the following facts

$$\left.\begin{array}{c}\displaystyle\int_{\mathbb{R}^N} f^n \varphi\,\mathrm{d}v \quad \text{is relatively compact in } L^1((0,T)\times B_R)\\ \text{for all } \varphi \in L^\infty(\mathbb{R}^N),\end{array}\right\} \tag{24}$$

$$\left.\begin{array}{c} f^n \underset{v}{*} \varphi \text{ is relatively compact in } L^1((0,T)\times B_R + B_R)\\ \text{for all } \varphi \in L^1 + L^\infty(\mathbb{R}^N).\end{array}\right\} \tag{25}$$

Let us finally point out that, using again (11), the L^1 compactness stated in (20) and (24) holds in fact on $(0,T) \times \mathbb{R}_x^N$ for all $T \in (0, \infty)$.

We are now ready to show, in a third step, the relative compactness in $L^1((0,T) \times B_R \times B_R)$ of F^n for all $R, T \in (0, \infty)$. We thus fix $T \in (0, \infty)$, $R_0 \in (0, \infty)$ and we deduce from (11), (20), (22), (24), (25), extracting subsequences if necessary, that

$$f^n \underset{n}{\rightharpoonup} f \text{ weakly in } L^1((0,T)\times \mathbb{R}^{2N}_{x,v}), \quad f \geqslant 0 \text{ a.e.} \tag{26}$$

$$a_{ij} \underset{v}{*} f^n = a_{ij}^n \underset{n}{\to} a_{ij} \underset{v}{*} f \text{ a.e. and in } L^1((0,T)\times B_{R_1} \times B_{R_1}) \tag{27}$$

$$\rho^n \underset{n}{\to} \rho \text{ a.e. and in } L^1((0,T)\times B_{R_1}) \tag{28}$$

$$F_\delta^n \underset{n}{\to} F_\delta \text{ a.e. and in } L^p((0,T)\times B_{R_1})\,(\forall p < \infty) \tag{29}$$

for all $\delta > 0$, where $R_1 = R_0 + 1$, where

$$\rho^n = \int_{\mathbb{R}^N} f^n\,\mathrm{d}v, \quad \rho = \int_{\mathbb{R}^N} f\,\mathrm{d}v.$$

We will denote by $\bar{a}_{ij} = a_{ij} \underset{v}{*} f$.

Next, we want to analyse the positivity of the matrix $\bar{a}_{ij}$. Let $\mu \in (0,1)$, for all $\eta \in \mathbb{R}^N$, $|\eta| = 1$, we observe that (13) yields some $\nu = \nu(\mu) > 0$ such that

$$\bar{a}_{ij}(t,x,v)\,\eta_i\,\eta_j \geqslant \nu(\mu) \int 1_{V(\mu,\eta)}(v_*) f(t,x,v_*)\,\mathrm{d}v_* = \nu(\mu)\,\rho_\mu(t,x,v,\eta), \tag{30}$$

where $V(\mu,\eta) = \{|v_*| < (1-\mu)^{-1},\ |(v-v_*,\eta)| < \mu|v-v_*|\}$. Then, we observe that as $\mu \uparrow 1$, $V(\mu,\eta) \uparrow \mathbb{R}^N - (v+\mathbb{R}_\eta)$. Therefore, as $\mu \uparrow 1$, $\rho_\mu(t,x,v,\eta) \uparrow \rho(t,x)$. This implies, by Dini's lemma and Lebesgue theorem, that $\rho_\mu \to \rho$ as $\mu \to 1$ uniformly in $v \in B_{R_1}$; $\eta \in S^{N-1}$, a.e. $x \in B_{R_1}$, $t \in (0,T)$ and in $L^1((0,T)\times B_{R_1}; C(B_{R_1}\times S^{N-1}))$. In particular, if we denote by $K_\alpha = \{(t,x) \in (0,T)\times B_{R_1} / \rho(t,x) > \alpha\}$ for $\alpha > 0$, we deduce from the Egorov theorem that, for each $\epsilon > 0$, there exists $E_1 \subset K_\alpha$ with meas$(E_1) < \frac{1}{2}\epsilon$ such that on $(E_1^c \cap K_\alpha)\times B_{R_1}\times S^{N-1}$, $\rho_\mu \geqslant \frac{1}{2}\alpha$ if μ is close to 1, i.e. $\mu \in [\mu_0(\alpha,\epsilon),1)$. We then choose μ in that interval and remark that (30) obviously holds with $\bar{a}_{ij}$ replaced by $\bar{a}^n_{ij}$, f by f^n and ρ_μ by ρ^n_μ. Next, $\rho^n_\mu \to_n \rho_\mu$ uniformly in

$$v \in B_{R_1},\ \eta \in S^{N-1},\ \text{a.e. } x \in B_{R_1},\ t \in (0,T)$$

and in
$$L^1((0,T)\times B_{R_1}; C(B_{R_1}\times S^{N-1})).$$

We may then apply again the Egorov theorem to deduce that there exists $E_2 \subset K_\alpha$ such that meas$(E_2) < \frac{1}{2}\epsilon$ and on $(E^c)\times B_{R_1}\times S^{N-1}$, $\rho^n_\mu \geqslant \frac{1}{4}\alpha$ for n large enough ($n \geqslant n_0(\alpha,\epsilon)$) where $E = E_1 \cup E_2$ so that meas$(E) < \epsilon$. We have thus shown for all $\alpha > 0$ and $\epsilon > 0$, the existence of a measurable set $E \subset K_\alpha$ such that meas$(E) < \epsilon$ and for all $(t,x) \in E^c \cap K_\alpha$, for all $v \in B_{R_1}$, $\eta \in S^{N-1}$

$$\bar{a}^n_{ij}(t,x,v)\,\eta_i\,\eta_j \geqslant \nu > 0 \tag{31}$$

for some $\nu = \nu(\alpha,\epsilon)$, for $n \geqslant n_0(\alpha,\epsilon)$.

Then, (23) implies for $n \geqslant n_0(\alpha,\epsilon)$

$$\int_{E^c\cap K_\alpha} \mathrm{d}t\,\mathrm{d}x \int_{B_{R_1}} \mathrm{d}v |\nabla_v F^n|^2 \leqslant C(\alpha,\epsilon). \tag{32}$$

This bound implies in turn that we have for $n \geqslant n_0(\alpha,\epsilon)$

$$\int_{E^c\cap K_\alpha} \mathrm{d}t\,\mathrm{d}x \int_{B_{R_0}} \mathrm{d}v |F^n - F^n_\delta| \leqslant C(\alpha,\epsilon)\,\delta^2. \tag{33}$$

We may now show that F^n is a Cauchy sequence in $L^1((O,T)\times B_{R_0}\times B_{R_0})$. Indeed, we have

$$\begin{aligned}\int_{(0,T)\times B_{R_0}} \mathrm{d}t\,\mathrm{d}x \int_{B_{R_0}} \mathrm{d}v|F^n - F^m| &\leqslant \int_{K^c_\alpha\cap((O,T)\times B_{R_1})} \mathrm{d}t\,\mathrm{d}x \int_{B_{R_0}} \mathrm{d}v(F^n+F^m)\\ &+ \int_E \mathrm{d}t\,\mathrm{d}x \int_{B_{R_0}} \mathrm{d}v(F^n+F^m) + \int_{E^c\cap K_\alpha} \mathrm{d}t\,\mathrm{d}x \int_{B_{R_0}} \mathrm{d}v|F^n - F^n_\delta| + |F^m - F^m_\delta|\\ &+ \int_{(0,T)\times B_{R_0}\times B_{R_0}} \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v|F^n_\delta - F^m_\delta|.\end{aligned}$$

The first integral is bounded by

$$\int_{K_\alpha^c \cap ((O,T)\times B_{R_1})} \mathrm{d}t\,\mathrm{d}x \int_{B_{R_0}} \mathrm{d}v (f^n + f^m),$$

which converges to

$$2\int_{K_\alpha^c \cap ((O,T)\times B_{R_1})} \rho\,\mathrm{d}t\,\mathrm{d}x$$

as n and m go to $+\infty$, and this last integral is bounded by $C\alpha$ where C denotes various positive constants independent of α and ϵ.

The second integral is bounded by $2\,\mathrm{meas}\,(B_{R_0})\,\mathrm{meas}\,(E) \leqslant C\epsilon$. The third integral is bounded for $n, m \geqslant n_0(\alpha, \epsilon)$ by $C'(\alpha, \epsilon)\,\delta$ because of (33). And the fourth integral goes to 0 and n, m got to $+\infty$ because of (29). Collecting all those estimates, we find

$$\limsup_{n,m\to+\infty} \int_{(0,T)\times B_{R_0}} \mathrm{d}t\,\mathrm{d}x \int_{B_{R_0}} \mathrm{d}v |F^n - F^m| \leqslant C\alpha + C\epsilon + C'(\alpha,\epsilon)\,\delta.$$

And we conclude letting first δ go to 0 and then α, ϵ go to 0.

We may now conclude the proof of Theorem 2. Indeed, extracting subsequences if necessary, we may assume that F^n converges a.e. on $(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$. And since β is $1-1$ we immediately see that f^n converges a.e. on $(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$. The conclusion of Theorem 2 follows then immediately in view of the bounds (11). □

Remark. With a little more effort, one can modify the above proof to allow the following condition on a_{ij} replacing (13)

$$\text{a.e. } z\in\mathbb{R}^N, \quad a_{ij}(z)\,\eta_i\,\eta_j > 0 \text{ if } \eta\in S^{N-1}, \quad \eta\cdot z = 0. \tag{13$'$}$$

Before proving Theorem 3, we wish to state without proof two consequences of the method of proof used above. The first one concerns another collision model (sometimes called Fokker–Planck model) presented for instance in Cercignani (1988):

$$Q(f,f) = \nu\{(e - |j|^2\rho^{-1})\,\Delta_v f + N\,\mathrm{div}_v\{(\rho v - j) f\}\}, \tag{34}$$

where $\eta > 0$, $\rho = \int_{\mathbb{R}^N} f\,\mathrm{d}v$, $j = \int_{\mathbb{R}^N} f v\,\mathrm{d}v$, $e = \int_{\mathbb{R}^N} f|v|^2\,\mathrm{d}v$.

Then, the bounds (11) are still available because (7), (8) are still valid and the analogue of (9), (10) is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^{2N}} f\log f\,\mathrm{d}x\,\mathrm{d}v + \eta\int_{\mathbb{R}^N}\mathrm{d}x\left\{(e-|j|^2\rho^{-1})\int_{\mathbb{R}^N}\frac{|\nabla_v f|^2}{f}\,\mathrm{d}v - N^2\rho^2\right\} = 0. \tag{35}$$

and
$$(e-|j|^2\rho^{-1})\int_{\mathbb{R}^N}\frac{|\nabla_v f|^2}{f}\,\mathrm{d}v - N^2\rho^2 \geqslant 0 \text{ a.e.}$$

in view of the following

Lemma 4. *Let $g\in H^1(\mathbb{R}^N)$ be such that $\int_{\mathbb{R}^N}|g|^2|v|^2\,\mathrm{d}v < \infty$. Then,*

$$\left(\int_{\mathbb{R}^N}|g|^2|v-u|^2\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^N}|\nabla g|^2\,\mathrm{d}v\right) \geqslant \tfrac{1}{4}N^2\left(\int_{\mathbb{R}^N}|g|^2\,\mathrm{d}v\right)^2, \tag{36}$$

where
$$u = \left(\int_{\mathbb{R}^N} |g|^2\, v\, \mathrm{d}v\right)\left(\int_{\mathbb{R}^N} |g|^2\, \mathrm{d}v\right)^{-1}.$$

In addition, the equality holds if and only if $g = \rho(2\pi T)^{-N/4}\,\mathrm{e}^{-|v-u|^2/4T}$ *for some* $T > 0$, ρ, $u \in \mathbb{R}^N$.

Remarks. (i) This lemma replaces the famous H-theorem for the Boltzmann model.
(ii) It is obviously one form of the Heisenberg uncertainty principle.

Proof. By a simple translation and scaling argument, we may assume without loss of generality that
$$\int_{\mathbb{R}^N} |g|^2\, v\, \mathrm{d}v = 0, \qquad \int_{\mathbb{R}^N} |g|^2\, |v|^2\, \mathrm{d}v = 1, \qquad \int_{\mathbb{R}^N} |\nabla g|^2\, \mathrm{d}v = 1.$$
Then, we may maximize $\int_{\mathbb{R}^N} |g|^2\, \mathrm{d}v$ over all functions g satisfying
$$\int_{\mathbb{R}^N} |g|^2\, v\, \mathrm{d}v = 0, \qquad \int_{\mathbb{R}^N} |g|^2\, |v|^2\, \mathrm{d}v \leqslant 1, \qquad \int_{\mathbb{R}^N} |\nabla g|^2\, \mathrm{d}v \leqslant 1.$$
The existence of a maximizer g_0 follows from easy functional analysis considerations. In addition, one can show that
$$\int_{\mathbb{R}^N} |g_0|^2\, |v|^2\, \mathrm{d}v = 1, \qquad \int_{\mathbb{R}^N} |\nabla g_0|^2\, \mathrm{d}v = 1$$
(again by scaling arguments for instance) and that, by the strong maximum principle, $\pm g_0$ is the ground state of an operator of the form $-\Delta + \lambda|v|^2$ for some $\lambda > 0$. Therefore, $g_0 = \rho(2\pi T)^{-N/4}\,\mathrm{e}^{-|v|^2/4T}$ for some $T > 0$, $\rho \in \mathbb{R}$ such that $N\rho^2 T = 1$ and $N\rho^2/4T = 1$. Hence $N^2\rho^4 = 4$ and $\int_{\mathbb{R}^N} |g_0|^2\, \mathrm{d}v = \rho^2 = 2/N$. □

Then, adapting the proof of Theorem 2, we find the

Theorem 5. *The conclusion of Theorem* 2 *holds for the collision model given by* (34).

Another applications of the method of proof of Theorem 2 concerns linear equations having certain hypoelliptic features (but which are not in general hypoelliptic). Let $(f^n)_n$ be a sequence of smooth solutions of
$$\frac{\partial f^n}{\partial t} + v\cdot\nabla_x f^n - a_{ij}(t,x)\,\frac{\partial^2}{\partial v_i\,\partial v_j} f^n = 0, \tag{37}$$
where $a_{ij} \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N_x)$. We assume that
$$\left.\begin{array}{c} f^n \text{ is bounded in } L^p_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{2N}_{x,v}) \text{ if } 1 < p < \infty \\ f^n \text{ is bounded in } L^p_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{2N}_{x,v}) \text{ and} \\ \text{uniformly locally integrable if } p = 1 \end{array}\right\} \tag{38}$$
and
$$\left.\begin{array}{c} \text{a.e. } (t,x) \in (0,\infty)\times\mathbb{R}^N_x, \\ a_{ij}(t,x)\,\eta_i\,\eta_j > 0 \quad \text{for all} \quad \eta \in S^{N-1}. \end{array}\right\} \tag{39}$$
Then, the method of proof of Theorem 2 yields the following result

Theorem 6. *The sequence* $(f^n)_n$ *is relatively compact in* $L^q_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{2N}_{x,v})$ *for* $1 \leqslant q < p$ *if* $p > 1$ *and for* $q = 1$ *if* $p = 1$.

Remarks. (i) If a_{ij} is smooth and if $p>1$, this result can be deduced from hypoelliptic theory (see Hörmander 1985). Indeed, f^n is then uniformly locally smooth on the open set

$$\omega = \{(t,x,v) \in (0,\infty)\times \mathbb{R}^{2N}_{x,v} / a_{ij}(t,x)\,\eta_i\,\eta_j > 0 \text{ for all } \eta \in S^{N-1}\}.$$

In particular, f^n is relatively compact in $L^p_{\mathrm{loc}}(\omega)$ and Theorem 6 follows by Hölder inequalities since (39) implies that meas $(\omega^c)=0$.

(ii) Notice also that f^n is not in general smooth nor compact in L^p_{loc}. Indeed, consider $a_{ij}(t,x) = |x|^2\,\delta_{ij}$. By a simple scaling argument, it is easy to construct a sequence of solutions such that $|f^n|^p$ converges weakly in the sense of measures to a Dirac mass at $x=0$, $v=0$.

(iii) This result also holds for stationary equations and in fact for more general operators. However, we will not pursue this direction (by the lack of applications). Also, one could include other terms in the equation (37) (right-hand sides bounded in L^1_{loc}, first-order terms, sequences of a_{ij}).

(iv) Even if $p=2$, the remark (ii) above shows that this type of compactness phenomena cannot be handled by the H-measures of Tartar (1990) and Gérard (1991). □

We now conclude this paper with the proof of Theorem 3. We begin with the case when $B \in L^1(\mathbb{R}^N \times S^{N-1})$. Indeed, by the arguments introduced in DiPerna & Lions (1989*a*), one sees that $Q^-(f^n,f^n)(1+f^n)^{-1}$, $Q^+(f^n,f^n)(1+f^n)^{-1}$ are bounded respectively in $L^\infty(0,\infty;L^1(\mathbb{R}^{2N}_{x,v}))$, $L^1(0,T;L^1(\mathbb{R}^{2N}_{x,v}))$ for all $T\in(0,\infty)$ where

$$Q^-(\varphi,\varphi) = \int_{\mathbb{R}^N} \mathrm{d}v_* \int_{S^{N-1}} \mathrm{d}\omega\, B\varphi\varphi_*,$$

$$Q^+(\varphi,\varphi) = \int_{\mathbb{R}^N} \mathrm{d}v_* \int_{S^{N-1}} \mathrm{d}\omega\, B\varphi'\varphi'_*.$$

In addition, we have for each $K>1$,

$$Q^+(f^n,f^n) \leqslant KQ^-(f^n,f^n) + \frac{1}{\log K} D^n, \tag{40}$$

where D^n is bounded in $L^1(0,\infty;L^1(\mathbb{R}^{2N}_{x,v}))$. The estimate (40) is shown in DiPerna & Lions (1989*a*). Therefore, if we set $F^n_\delta = \delta^{-1}\log(1+\delta f^n)$, observing that we have

$$\frac{\partial f^n_\delta}{\partial t} + v\cdot\nabla_x f^n_\delta = \frac{1}{1+\delta f^n} Q(f^n,f^n),$$

we deduce from the bounds recalled above and (40) for all $t,s\geqslant 0$

$$\|(f^n_\delta)^\#(t) - (f^n_\delta)^\#(s)\|_{L^1(\mathbb{R}^{2N}_{x,v})} \leqslant \omega(|t-s|), \tag{41}$$

where ω is a continuous, non-negative, non-decreasing function on $[0,\infty)$ such that $\omega(0)$, and ω depends on δ but is independent on n. Here and below, we denote, as in DiPerna & Lions (1989*a*), $g^\#(x,v,t) = g(x+vt,v,t)$ for any function g on $[0,\infty)\times\mathbb{R}^{2N}_{x,v}$. Using (11) in the second step of the proof of Theorem 2, we deduce from (41) letting δ go to 0 that (41) also holds with f^n_δ replaced by f. Next, if f^n is relatively compact in $L^1((0,T)\times\mathbb{R}^{2N}_{x,v})$ for some $T>0$, $(f^n)^\#$ is also relatively compact in $L^1((0,T)\times\mathbb{R}^{2N}_{x,v})$. And this combined with the fact that (41) holds for $(f^n)^\#$ yields the relative compactness in $C([0,T];L^1(\mathbb{R}^{2N}_{x,v}))$ of f^n. In particular, $(f^n)^{\#}|_{t=0} = f^n|_{t=0} = f^n_0$ is relatively compact in $L^1(\mathbb{R}^{2N}_{x,v})$.

We now conclude the proof of Theorem 3 by adapting the above argument. We consider, instead of $f^n_\delta, f^n_{\delta,R} = \delta^{-1}\log(1+\delta f^n)\,1_{B_R}(v)$ for $R \in (0,\infty)$ and we have

$$\left(\frac{\partial}{\partial t} + \sigma\cdot\nabla_x\right) f^n_{\delta,R} = \frac{1}{1+\delta f^n} Q(f^n, f^n)\,1_{B_R}(v). \tag{42}$$

then $Q^-(f^n,f^n)(1+\delta f^n)^{-1}1_{B_R}(v)$ is bounded in $L^\infty(0,\infty;L^1(\mathbb{R}^{2N}_{x,v}))$. This bound combined with (40) implies (41) with f^n_δ replaced by $f^n_{\delta,R}$. And letting δ go to 0_+, R go to $+\infty$ and using (11) (as we sketched above), we recover the fact that (41) hold for f^n and we conclude as above. □

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