Generalized Wavelet Theory and Non-Linear, Non-Periodic Boundary Value Problems

Dr. Klaus Braun
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Abstract

"*When Fourier meets Navier*" in the physical $H_{-1/2}$ – Hilbert space there is a global unique solution of the corresponding weak variation equation of the non-linear, non-stationary Navier-Stokes equations ([BrK1]). The corresponding numerical approximation method is the Ritz-Galerkin method which is usually equipped with finite elements approximation spaces. In case of negative scale physical spaces this is called boundary element method related to the underlying singular equation representation. The finite (boundary) element approximation properties face some challenges in case of non-linearity and/or non-periodic boundary conditions. The wavelet extension method is used to represent functions that have discontinuities and sharp peaks, and for accurately deconstructing and reconstructing finite, non-periodic and/or non-stationary signals [MeM].

The wavelet theory is established in the Fourier L_2 – Hilbert space framework. As a consequence in order to apply the Calderón repreducing formula this requires to the so-called admissibility condition defining a wavelet $\psi \in L_2$ analyzing a (signal) function $f \in L_2$. The challenge for the wavelet extension (approximation) method is about the combination of finite elements ("to approximate on regular function domains") applied to within the physical space and wavelets (treating the "nasty" non-linear terms (shocks) or the non-periodic (sharp edges) boundary conditions applied to within the wavelet space.

Following the slogan "*when Fourier (generalized waves) meets Calderón (generalized wavelets)*" we provide a Galerkin-expansion-wavelet method which operates on same physical and wavelet space which is $H_{-1/2}$. The isometry of the generalized wavelet transform is given by

$$H_\beta(R^2, \frac{dadb}{a^2}) \cong H_\beta .$$

The space $H_{-1/2}$ – is motivated by the original Calderón formula:

let ψ real, radial with vanishing average, i.e. $\hat{\psi}(0) = 0$ and ($a > 0$)

$$\psi_a(x) := \frac{1}{a^n}\psi(\frac{x}{a}) \quad , \quad \int_0^\infty \left|\hat{\psi}(a\omega)\right|^2 \frac{da}{a} \equiv 1 .$$

Then for $f \in L_2$ it holds the reproducing formula

$$f = \int_0^\infty \psi_a * \psi_a * f \frac{da}{a} .$$

This reflects a not balanced "relationship" between the Hilbert spaces $H_0 \leftrightarrow H_{-1}$.

Fourier (physical $H_{-1/2}$ – space) **meets Calderón** (wavelet $H_{-1/2}$ – space)

Most common numerical methods used for numerical solution of partial differential equations are spectral, finite element (FEM) or wavelet methods. While spectral methods have good accuracy its spatial localization is poor (Gibbs phenomenon) the FEM for PDE with smooth boundary conditions have good accuracy and good spatial localization [BrK]. In case of singularity at the boundary good accuracy of FEM can be achieved, as well. However this requires additional problem depending modification of the approximation spaces (e.g. [BlH] [NiJ]).

In [ChF] a Nitsche-based domain composition method is defined and analyzed for the solution of hypersingular integral equations governing the Laplacian in R^3 exterior to an open surface, subject to a Neumann boundary problem. The method can be applied to linear elasticity and acoustics problems, but in case the operator governing the Lamé equation causes major difficulties.

A wavelet is a function used to divide a given function or continuous signal into different scale components. A wavelet is a L_2 – integrable function ψ fulfilling the admissibility condition

$$c_\psi := 2\pi \int_R \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \, .$$

For the parameters for $a \in R - \{0\}$, $b \in R$ it enables the definition of the wavelet transformation by

$$L_\psi[f](a,b) := \frac{1}{\sqrt{|a|}} \int_R f(t) \psi\left(\frac{t-b}{a}\right) dt \quad , \quad f \in L_2(R) \quad .$$

The parameters a,b are called dilation (re-scale, zooming) resp. translation (shift) parameter. For $a > 0$ the parameter can be interpreted as a reciprocal of frequency. The admissibility condition ensures the validity of the Calderón reconstruction formula

$$f(x) = \frac{1}{c_\psi} \int_R \int_R L_\psi(b,a) \overline{\psi\left(\frac{x-b}{a}\right)} \frac{dadb}{a^2}$$

whereby it holds

$$\left\| L_\psi[f] \right\|^2_{L_2(R^2, \frac{dadb}{a^2})} == \|f\|^2_{L_2} \, .$$

Wavelet transformation analysis is applied especially to nonlinear PDEs having solutions containing local phenomena (e.g. formation of shocks like hurricanes) and interactions between several scales (e.g. atmospheric turbulence where there is motion on continuous range of length scales). Such solutions can be well represented in wavelet bases because of properties like compact support (locality in space) and vanishing moment (locality in time).

Most of the wavelet algorithms can handle periodic boundary conditions. However, different possibilities of dealing general boundary conditions have been studied and the advantage of wavelet transforms over traditional Fourier transforms for representing functions that have discontinuities and sharp peaks, and for accurately deconstructing and reconstructing finite, non-periodic and/or non-stationary signals are still pending ([MeM]).

The Hilbert transformed wavelet approach could allow the direct numerical simulation of scattering and radiation phenomena while avoiding the limitations of boundary element methods (non-uniqueness) and the constraints of artificial, non-reflective boundary conditions.

The various properties of the Hilbert transform and its related Hilbert transformed wavelets suggest that they might be useful for solving exterior boundary value problems with prescribed behavior at the point at ∞. For instance, acoustic radiation from a compact object is described by a solution of the wave equation that satisfies the Sommerfeld radiation condition at ∞ .

The scaling and wavelets functions of compact support can be defined on the real line, R, or on the circle (is period) ([WeJ]). The Hilbert transform of a compactly-supported wavelet is also a wavelet ([WeJ]). That is, the Hilbert transformed wavelets are orthogonal to their translates and form a basis for $L_2(R)$. The Hilbert transform scaling and wavelet functions do not have compact support. Their support is all of R. The periodic Hilbert transform is defined by a $\cot(x)$ kernel. The periodic Hilbert transform of a periodic, compact support scaling or wavelet function is not a periodic solution of a scaling relation.

In Galerkin method the degrees of freedom are the expansion coefficients of a set of basis functions and these expansion coefficients are not in physical state in case of a wavelet Galerkin method, means in wavelet space. Moreover, in wavelet Galerkin methods the treatment of nonlinearities is complicated when can be handled with couple of techniques ([MeM]):

- using the connection coefficients
- using the quadrature formula
- using the “pseudo-approach” (first map wavelet space to physical space, compute nonlinear term in physical space and then back to wavelet space); this approach is not very practical because it requires transformation between the physical and the wavelet space.

We propose generalized wavelets to be used by the “pseudo-approach” for the treatment of nonlinearities where the wavelet space is identical with the appropriately defined physical space as proposed in [BrK1]. At the same time the approach overcomes the limitations of boundary element methods (non-uniqueness) and the constraints of artificial, non-reflective boundary conditions of scattering and radiation phenomena.

The approach can be extended to continuous radial wavelet transform on R^n with scales a and orientations $R = SO(n)$ being related to a Riesz basis enabling dilations, rotations and translation of a single function ψ ([RaH]). The corresponding admissibility condition is given by

$$c_\psi := \iint_{R^n} \frac{|\hat{\psi}(\omega)|^2}{|\omega|^n} d\omega < \infty$$

with

$$\hat{\psi}(\omega) := \frac{1}{(2\pi)^{n/2}} \iint_{R^n} \psi(x) e^{-ix\cdot\omega} d\omega \ .$$

For a radial function $f(x) = F(|x|) = F(r)$ it holds

$$\hat{f}(\omega) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty F(r) \cdot \hat{v}(r\omega) \cdot r^{n-1} dr$$

with the Fourier transform of the uniform distribution of unit mass over the unit sphere with center at the origin given by

$$\hat{v}(\omega = 2^{(n-2)/2} \Gamma(\frac{n+1}{2}) \cdot |\omega|^{(2-n)/2} J_{(n-2)/2}(|\omega|) \cdot$$

With respect to the Hilbert-Courant conjecture in the context of the wave radiation problem (regarding the existence of families of distortion-free, progressive waves and the Huygens principle, [BrK2]) and the straightforward generalization of the below for a continuous radial wavelet theory on R^n ([RaH]v1.5.2) we note that

- $d\hat{v}(0 = 0$ iff the space-time dimension is $n = 4$, ([WaG]2.13, 15.5: $J_0(2\sqrt{0}) = 0$)
- the spatial lattice is equidistant only in the special cases of space-time dimensions $n = 2,4$, ([RaH] chapter 3)
- the analog properties of classical Riesz transforms on spheres ([ArN]).

In this paper we consider the Poisson boundary value (model) problem with non-periodic boundary condition for space(-time) dimension $n=2$ and generalized periodic function of the Hilbert spaces $H_\beta = H_\beta(\Gamma)$. The related FEM/BEM optimal Galerkin approximation analysis also in case of locally reduced regularity assumptions to the solution is given in [BrK].

Let $H = L_2^*(\Gamma)$ with $\Gamma = S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β Fourier coefficients and norms are defined by

$$u_\nu := \frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx \qquad \|u\|_\beta^2 := \sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 \quad .$$

Then the Fourier coefficients of the convolution operator

$$(Au)(x) := -\oint \log 2\sin\frac{x-y}{2}u(y)dy =: \oint k(x-y)u(y)dy$$

are given by

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|}u_\nu \quad .$$

The operator A enables characterization of the Hilbert spaces and $H_{-1/2}$ in the form

$$H_{-1/2} = \left\{\psi \middle| \|\psi\|_{-1/2}^2 = (A\psi,\psi)_0 < \infty\right\} \quad .$$

We note that $k_0 = 0$. In this framework the admissibility condition for standard wavelets ψ is equivalent to $\psi \in H_{-1/2}$ and the admissibility condition constant is given by the norm $\|\psi\|_{-1/2}^2$.

The (simple layer potential) integral equation $A\psi = g$ solves the boundary value problem

$$\begin{aligned} \Delta\psi = 0 \quad &\text{in} \quad D := \left\{(x,y)\in R^2 \middle| x^2+y^2<1\right\} \\ \psi = g \quad &\text{on} \quad \Gamma := S^1(R^2) = \partial D \ . \end{aligned}$$

The related Ritz method using spline spaces is called boundary finite element method (BEM). In the case above this defines a (Ritz-) Galerkin approximation $\psi_h := R_h\psi \in S_h \subset H_{-1/2}$

$$(A\psi,\chi) = (A\psi_h,\chi) \text{ for all } \chi \in S_h \subset H_{-1/2}.$$

with optimal approximation behavior with respect to the related regularity of the solution ψ and the triangle parameter h ([BrK]). Let $\phi_h = A\psi_h$ the corresponding approximation of the Ritz approximation ψ_h of $A\psi = g$. Then the error $\phi - \phi_h$ can be represented in the form $(\phi-\phi_h) = (\ln|\xi - \circ|, \psi - \psi_h)$ whereby

$$\phi_h(\xi) = \oint \ln|\xi - \eta|\psi_h(y)dy$$

and $\xi = \xi_1 + i\xi_2$, $\eta = \eta_1 + i\eta_2 = e^{iy}$. For $\xi = \xi_1 + i\xi_2$ with $|\xi| < 1$ fixed the function $\mu(y) := \ln|\xi - e^{iy}|$ is analytical with respect to y. From the Schwarz inequality it therefore follows

$$|\phi(\xi) - \phi_h(\xi)| \le \|\mu\|_\beta \|\psi - \psi_h\|_{-\beta} \le ch^{2t+1}\|\psi\|_t \|\mu\|_{t+1} \quad \text{in case } S_h = S_h^{k.t} \ .$$

The limitation of the "pseudo approach" ([MeM]) in the sense that there is no mapping required from the wavelet space to the physical space and then back to compute nonlinear terms in physical space is enable by the following isometry properties of generalized wavelets defined as $\psi \in H_{-1/2}$.

Theorem: For every $\psi \in H_{-1/2}$ Calderón's reproducing identity holds true, i.e.

$$\left\| L_\psi [f] \right\|^2_{H_\beta(R^2, \frac{dadb}{a^2})} = \| f \|^2_{H_\beta} \quad \text{for } f \in H_\beta$$

Proof:

$$\left\| L_\psi [f] \right\|^2_{H_\beta(R^2, \frac{dadb}{a^2})} = \int_R \int_R \left| L_\psi [f]^\wedge (a, \omega) \right|^2 |\omega|^{2\beta+1} \frac{dad\omega}{a^2 \cdot |\omega|}$$

$$= \frac{2\pi}{c_\psi} \int_R \int_R \cdot \left| \hat{\psi}(a\omega) \right| \cdot \left| \hat{f}(\omega) \right|^2 |\omega|^{2\beta+1} \frac{dad\omega}{|a| \cdot |\omega|}$$

$$= \frac{2\pi}{c_\psi} \int_R \int_R \frac{\left| \hat{\psi}(\tau) \right|}{|\tau|} \cdot \left| \hat{f}(\omega) \right|^2 d\tau |\omega|^{2\beta+1} \frac{d\omega}{|\omega|} = \| f \|^2_{H_\beta} .$$

Remark: due to corresponding properties of the Hilbert transform (i.e. $\| H\eta \| = \| \eta \|$, $(\eta, H\eta) = 0$, $(H\eta)^\wedge(0) = 0$, $H\eta = -A\eta'$) the relationship between the generalized wavelets ($\psi \in H_{-1/2}$) and the standard wavelet ($\psi \in H_0$, $c_\psi < \infty$, $\hat{\psi}(0) = 0$) is given by

$$\psi = \xi + \lambda = H\eta + \lambda \quad \text{whereby } \xi, H\eta \in H_0, \; \lambda \in H_0^{\perp \|\cdot\|_{-1/2}}, \; \hat{\xi}(0) = 0 .$$

Remark: the extension of the above to space dimensions $n > 2$ is enabled by the Prandtl and the multi-dimensional analog to the Hilbert transform, the Riesz operators. With respect to the Prandtl operator we recall from [LiI]

> Theorem: The Prandtl operator $\prod : H_{1/2} \to H_{-1/2}$ is bounded and coercive, the range $R(\prod) = H_1(R^3 - S)$ and the exterior Neumann problem admit one and only on generalized solution.

The theorem prompts to introduce in case of a physical $H_{-1/2}-$ space the corresponding energy inner product in the form $(\Pi u, \Pi v)_{-1/2}$.

In the above Hilbert space framework the Prandtl operator corresponds to the normal derivative of the double layer potential operator $T : H_{1/2}(\Gamma) \to H_{-1/2}(\Gamma)$ ([KrR] theorem 8.21):

$$Tv(x) := \frac{1}{\pi} \frac{\partial}{\partial n(x)} \oint_\Gamma v(y) \frac{\partial}{\partial n(y)} \ln \frac{1}{|x - y|} ds(y) , \quad x \in \Gamma$$

which is selfadjoint with respect to the dual systems $(H_{1/2}(\Gamma), H_{-1/2}(\Gamma))_{L_2(\Gamma)}$ and $(H_{-1/2}(\Gamma), H_{1/2}(\Gamma))_{L_2(\Gamma)}$, that is,

$$(Tv, w)_{L_2(\Gamma)} = (v, Tw)_{L_2(\Gamma)} , \text{ for all } v, w \in H_{1/2}(\Gamma) .$$

The Gaussian function and its Fourier transform have same decay. In case of the wavelet transform the situation is different depending from the wavelet definition and the parameter a e.g. leading to the Mexican hat wavelet function

$$\mu(x) = -\frac{d^2}{dx^2} e^{-x^2/2} = (1-x^2) \cdot e^{-x^2/2} .$$

In case of the proposed generalized wavelet model concept we note that for

$$\varphi_a(x) = \frac{1}{\sqrt{a}} e^{-\pi x^2/a} =: \frac{1}{\sqrt{a}} f(\frac{x}{\sqrt{a}})$$

and its related Hilbert transform $\varphi_{H;a}(x) := H[\varphi_a](x)$ it holds

$$\hat{\varphi}_a(x) = \frac{a^{n/2}}{\sqrt{a}} e^{-\pi \cdot a \cdot x^2} , \quad \int_{-\infty}^{\infty} \varphi_a(x) \equiv 1 \quad , \quad \varphi_{H;a}(0) = 0 \quad , \quad \|\varphi_{H;a}\| = \|\varphi_a\| .$$

The relationship $H[f](x) = A[f'](x)$ then defines a generalized wavelet alternatively to the Mexican hat with an appropriate decay in sync with the decay of the related Fourier transform of the to-be transformed function.

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