Some Commutator Properties

The below indicates the usefulness of the theory of complementary variational principles ([ArA]), e.g. the method of Trefftz, the hyper circle method of Prager/Synge, the method of orthogonal projection ([VeW] 4).

The Hilbert operator

$$(Hu)(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{u(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy$$

fulfills the following properties

Lemma:

i) The constant Fourier term vanishes, i.e. $(Hu)_0 = 0$

ii) $$H(xu(x)) = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy$$

iii) For odd functions it hold

$$H(xu(x)) = x(Hu)(x)$$

iv) If $u, Hu \in L_2$ then u and Hu are orthogonal, i.e.

$$\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = 0$$

v) $\|H\| = 1$, $H^* = -H$, $H^2 = -I$, $H^{-1} = H^3$,

vi) $H(f * g) = f * Hg = Hf * g$, $f * g = -Hf * Hg$

vii) If $(\varphi_n)_{n \in N}$ is an orthogonal system, so it is for the system $(H(\varphi_n))_{n \in N}$,

i.e. $(H\varphi_n, H\varphi_n) = -(\varphi_n, H^2 \varphi_n) = (\varphi_n, \varphi_n)$.

viii) $\|Hu\|^2 = \|u\|^2$, i.e. if $u \in L_2$, then $Hu \in L_2$.

Proof:

i) , v)-viii) see [PeB], 2.9

ii) Consider the Hilbert transform of $xu(x)$

$$H(xu(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yu(y)}{x-y}dy \; .$$

The insertion of a new variable $z = x - y$ yields

$$H(xu(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x-z)u(x-z)}{z}dz = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(xu(x-z)}{z}dz - \frac{1}{\pi}\int_{-\infty}^{\infty}u(x-z)dz = xH(u(x)) - \frac{1}{\pi}\int_{-\infty}^{\infty}u(y)dy$$

iii) It follows from i) and ii)

iv) $$\int_{-\infty}^{\infty}u(y)(Hu)(y)dy = \frac{i}{2\pi}\int_{-\infty}^{\infty}sign((\omega)\left|\hat{u}(\omega)\right|^{2}d\omega$$

whereby $\left|\hat{u}(\omega)\right|^{2}$ is even. This gives the result •

Let Q, P, T denote the location operator, the momentum operator and the normal derivative operator of the double layer potential. Then it holds for the related commutators for

$$u \in \dot{L}_2 := \left\{ u \in L_2 \left| \int_{-\infty}^{\infty}u(y)dy = 0 \right. \right\}$$

Corollary:

i) $[Q,H]u = (QH - HQ)u = 0$

ii) $[Q,P]u = (QP - PQ)u = u$

iii) $[Q,T]u = (QT - TQ)u = Hu$

and therefore

iv) $([Q,P]u,[Q,T]u) = 0$

APPENDIX

Calderon-Zygmund and Riesz Operators

The Calderon-Zygmund operator with symbol $|v|$ ([EsG] (3.17), (3.35)) is defined by

$$(\Lambda u)(x) = (\sum_{k=1}^{n} R_k D_k u)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{n} p.v. \int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta \Lambda^{-1}) u(x)$$

whereby R_k denotes the Riesz operators ([AbH] p. 19, 106, [PeB] example 9.9)

$$R_k u = -i \pi^{-(n+1)/2} \Gamma(\frac{n+1}{2}) p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy \; .$$

For $n \geq 2$ it holds ([EsG] (3.15))

$$\Lambda^{-1} u = \frac{1}{2} \pi^{-(n+1)/2} \Gamma(\frac{n-1}{2}) \int_{-\infty}^{\infty} \frac{u(y)dy}{|x-y|^{n-1}} \quad .$$

The Riesz operators fulfill certain properties with respect to commutation with translations homothesis and rotation ([PeB], [StE]). Let $SO(n)$ denote the rotation group. If $j \neq j$ then $R_j R_k$ is a singular convolution operator. On the other hand it holds $R_j^2 = -(1/n)I + A_j$ where A_j is a convolution operator. The following identities are valid

$$\|R_j\| = 1 \quad , \quad R_j^* = -R_j \quad , \quad \sum R_j^2 = -I \quad , \quad \sum \|R_j u\|^2 = \|u\|^2 \quad , u \in L_2 \; .$$

Let

$$m := m(x) := (m_1(x), ... m_n(x))$$

be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then

$$m(\rho(x)) = \rho(m(x)) \, ,$$

whereby

$$m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$$

and

$$m(\rho(x)) = c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(x\rho^{-1}(y)) + \log\left|\frac{1}{x\rho^{-1}(y)}\right|) \frac{y}{|y|} d\sigma(y)$$

$$= c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(xy) + \log\left|\frac{1}{xy}\right|) \frac{y}{|y|} d\sigma(y) \; .$$

One-dimensional Calderon-Zygmund and Riesz Operators

In case of n=1 the Calderon-Zygmund and the Riesz operators are given by **(**[LiI] (1.2.31)-(1.2.33), [LiI1]):

$$Su(x) := \oint \frac{1}{4\sin^2 \frac{x-y}{2}} u(y)dy$$

$$Hu(x) := \oint \frac{1}{2}\cot\frac{x-y}{2} u(y)dy \ .$$

For

$$Au(x) := -\oint \log 2\sin\frac{x-y}{2} u(y)dy$$

we note the relationship

$$A[u_x](x) = -H[u](x)$$

resp.

$$(HA)[u_x](x) = u(x) \ .$$

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