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ON M. G. KREIN'S PAPERS IN THE THEORY OF SPACES WITH AN INDEFINITE METRIC

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This is a survey of the papers by M. G. Krein (and his disciples) devoted to the theory of operators in spaces with an indefinite metric and its applications.

In order to reflect M. G. Krein's contribution to the theory of spaces with an indefinite metric in the limited volume of this article, we focus our attention on a series of papers by M. Krein (including papers written with coauthors). Unfortunately, we are unable to consider here the problems of priority and to trace the development of his ideas in numerous researches of other authors [in particular, in the works of his disciples {for this see, e.g., [1–3])}.

1. A linear space $\mathfrak{H}$ equipped with a sesquilinear form $[x, y]$, $x, y \in \mathfrak{H}$, is called a space with an indefinite metric $[\cdot, \cdot]$. The vectors of the space $\mathfrak{H}$ and its lineals are characterized by a sign determined by an indefinite scalar product. For example, a vector $x \in \mathfrak{H}$ is called positive (negative or neutral) if $[x, x] > 0$ (< 0 or $= 0$, respectively). The notions of nonnegative lineal, nonpositive lineal, neutral lineal, indefinite lineal, etc., have a natural meaning. Denote by $\mathfrak{M}^+$ a set of maximal nonnegative lineals. With rare exceptions, we will omit the definitions having analogs in the theory of Hilbert spaces or are clear from the presentation. In particular, this is true for the concept of $[\cdot, \cdot]$-orthogonality of vectors and lineals.

Let

$$\mathfrak{H} = \mathfrak{H}^+ [\dot{+}] \mathfrak{H}^- \tag{1}$$

be a decomposition of the space $\mathfrak{H}$ into a $[\cdot, \cdot]$-orthogonal sum of a positive subspace $\mathfrak{H}^+$ and a negative subspace $\mathfrak{H}^-$. A space $\mathfrak{H}$ admitting decomposition (1) into spaces $\mathfrak{H}^\pm$ that are complete with respect to the norms $|[x, x]|^{1/2}$, $x \in \mathfrak{H}^\pm$, is called a Krein space; if, in addition, $\kappa := \min\{\dim \mathfrak{H}^+, \dim \mathfrak{H}^-\} < \infty$, then it is called a Pontryagin space Π_κ (for definiteness, we assume that $\kappa = \dim \mathfrak{H}^-$). The Krein space is a Hilbert space with a scalar product

$$(x, y) = [x_+, y_+] - [x_-, y_-]$$

for

$$x = x_+ + x_-, \quad y = y_+ + y_-, \quad x_\pm, y_\pm \in \mathfrak{H}^\pm.$$

This implies that $[x, y] = (Jx, y)$, where J is the difference of mutually complementary orthoprojectors P^+ and P^- onto $\mathfrak{H}^+$ and $\mathfrak{H}^-$, respectively, i.e., $J = P^+ - P^-$. In what follows, instead of $[\cdot, \cdot]$-orthogonality, we consider J-orthogonality (or π-orthogonality in the special case of the Pontryagin space). This remark also relates to the other concepts. Note that decomposition (1) is called the canonical decomposition of $\mathfrak{H}$.

Let $\mathfrak{H}$ be the Krein J-space, let (1) be its canonical decomposition, and let $\mathfrak{K}$ be a set of contractions acting

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from $\mathfrak{H}^+$ into $\mathfrak{H}^-$. There is a one-to-one correspondence between the sets $\mathfrak{M}^+$ and $\mathfrak{K}$ established by the fact that every subspace $\mathfrak{L} \in \mathfrak{M}^+$ is a graph of a certain operator $K \in \mathfrak{K}$, i.e.,

$$\mathfrak{L} = \{x_+ + Kx_+ \mid x_+ \in \mathfrak{H}^+\}$$

and, vice versa, the graph of an arbitrary operator in $\mathfrak{K}$ is an element of $\mathfrak{M}^+$; K is called an angular operator of the subspace $\mathfrak{L}$.

Finally, note that the definitions of almost all considered classes of operators acting in the spaces with indefinite metrics are introduced by analogy with the corresponding "definite" definitions. The so-called plus-operators prove to be an exception to this rule. An operator T with a domain $\mathfrak{D}_T$ is called a plus-operator if it maps the nonnegative vectors in $\mathfrak{D}_T$ into the set $\mathfrak{F}^+$ of nonnegative vectors (it is supposed that $\mathfrak{D}_T \cap \mathfrak{F}^+ \neq \emptyset$).

2. The communication [4] was apparently M. Krein's first work devoted to the theory of operators in spaces with indefinite metrics (for a detailed presentation of these results, see [5]). In fact, these articles dealt with J-nonnegative completely continuous integral operators. It was proved that the spectrum of a J-nonnegative completely continuous integral operator A is real and the kernels $\operatorname{Ker}(A - \lambda I)$ and $\operatorname{Ker}(A - \mu I)$ are J-orthogonal for $\lambda \neq \mu$. Actually, the J-spectral decomposition of the operator A was also constructed there. More precisely, it was shown that the operator A is representable as a sum $A = A_0 + A_1$ of two J-nonnegative operators A_0 and A_1 satisfying the following conditions:

(a) $A_0^2 = A_0 A_1 = A_1 A_0 = 0$;

(b) A_1 is representable as an integral in terms of a "J-spectral function".

(The corresponding definition and the theorem of existence can be found in Section 6.)

Furthermore, it was proved that a J-positive completely continuous operator possesses a complete system of eigenvectors and, in addition, its negative spectrum consists of finitely many κ eigenvalues if and only if the J-space is the Pontryagin space Π_κ.

The results cited above are a reformulation of M. Krein's corresponding results for loaded integral equations whose distribution functions are not monotone. Although an indefinite scalar product was actually introduced in the indicated papers and its indefiniteness was stressed, M. Krein, in his subsequent papers, always said that Pontryagin's famous work [6] opened a new direction in functional analysis and operator theory and was pioneering.

3. The first papers by M. Krein and his students [7–10] (see also [11]) in the abstract theory of indefinite spaces were devoted to the Pontryagin spaces. The papers [9, 10] play a distinguished role in this series. They were written with I. S. Iokhvidov who was Krein's first student involved in the investigations of the problems connected with indefiniteness. For about 20 years, these papers were, in fact, the only systematic exposition of the theory of the Pontryagin spaces Π_κ, and a generation of mathematicians regarded them as an introduction to this theory. Unlike Pontryagin's paper [6] where fine analytic methods were used, papers [7–11] were based on geometric methods. This made it possible to construct the axiomatics of the spaces Π_κ, to get much simpler proofs of the principal results in [6], to develop these results significantly, and to give rise to numerous new trends of investigation. In particular, these works laid the foundation of the theory of extensions of π-isometric and π-symmetric operators and enabled the authors to describe various types of extensions depending on the signature of the defect subspaces.

The indicated papers were mainly devoted to the study of plus-operators, in particular, of π-noncontracting, π-isometric, and π-unitary operators. Operators of this sort appear in applications as naturally as π-Hermitian. The special attention paid to these operators is explained, on the one hand, by the fact that geometric methods are much simpler for them and, on the other hand, it was shown by Iokhvidov that the Cayley–Neumann transformation enables one to establish the properties of π-Hermitian operators as the consequences of the corresponding assertions for π-isometric operators.

Papers [7 – 11] also contain the investigation of the root lineals and elementary divisors of π-isometric operators, the deduction of the general form of π-unitary and π-semiunitary operators, and the classification of the invariant subspaces of π-unitary and π-self-adjoint operators. Among the possible applications indicated in these papers, we especially mention the investigation of indefinite Toeplitz forms, the problem of extension of the helical arcs in the Lobachevski space, etc. Here, we dwell in more detail upon two theorems in the diverse collection of profound results obtained in [7 – 11]. They are selected for their significance (see, e.g., their applications to the linear-fractional transformations and quadratic bundles discussed in Sections 4 and 6) and short proofs.

Theorem 1. *Let* $\Pi_\kappa = \Pi^+ [+] \Pi^-$ *be a canonical decomposition of the Pontryagin space. In order that an operator* $U: \Pi_\kappa \to \Pi_\kappa$ *given by a matrix* $\| U_{ij} \|_{i,j=1}^2$ *with respect to decomposition (1) be* π*-unitary, it is necessary and sufficient that the following relations be true:*

$$U_{11} = (I - \Gamma^*\Gamma)^{-1/2} U_+, \quad U_{12} = \Gamma^*(I - \Gamma\Gamma^*)^{-1/2} U_-,$$

$$U_{21} = \Gamma(I - \Gamma^*\Gamma)^{-1/2} U_+, \quad U_{22} = (I - \Gamma\Gamma^*)^{-1/2} U_-,$$

where U_+ *and* U_- *are unitary operators acting in* Π^+ *and* Π^-, *respectively, and* $\Gamma: \Pi^+ \to \Pi^-$ *is a uniform contraction, i.e.,* $\|\Gamma\| < 1$.

Proof. *Sufficiency* is verified directly.

To prove *necessity*, one must take the angular operator of the subspace $U\Pi^+$ as Γ and $(I - \Gamma^*\Gamma)^{1/2} U_{11}$ and $(I - \Gamma\Gamma^*)^{1/2} U_{22}$ as U_+ and U_-, respectively.

Later, Theorem 1 was developed further; in particular, paper [12] contains the proof of the fact that similar relations are also valid for J-unitary operators in the Krein space. However, this follows directly from the sketch of the proof given above.

The problem of existence for the maximal nonnegative invariant subspaces of the operators under consideration appears to be one of the key problems in the theory of operators in the spaces with indefinite metrics. Below, we formulate one of the basic theorems of this type and prove it by two different methods. The first method was applied by M. G. Krein in the case of a π-noncontracting operator in [8]; the second method was used in the case of a J-noncontracting operator [13].

Theorem 2. *Let* $V = \| V_{ij} \|_{i,j=1}^2$ *be the matrix representation of a J-noncontracting operator* V *with respect to decomposition (1), let* $V\mathfrak{H}^+ \in \mathfrak{M}^+$, *and let* V_{12} *be a completely continuous operator* $V_{12} \in \gamma_\infty$. *Then the operator* V *has the maximal nonnegative invariant subspace.*

Proof. (a) *Approximation method.* Consider operators $V_\varepsilon = VI_\varepsilon$, $\varepsilon > 0$, where

$$I_\varepsilon = \sqrt{1+\varepsilon}\, P^+ + \sqrt{1-\varepsilon}\, P^-.$$

Since $[V_\varepsilon x, V_\varepsilon x] \geq [I_\varepsilon x, I_\varepsilon x] = [x, x] + \varepsilon(x, x)$, the unit circle does not contain the elements of the spectrum of the operator V_ε. By using the Riesz projectors, we choose an invariant subspace $\mathfrak{L}_\varepsilon$ of the operator V_ε corresponding to its spectrum lying outside of the unit circle. The subspace $\mathfrak{L}_\varepsilon$ is maximal nonnegative. Let K_ε be the angular operator of $\mathfrak{L}_\varepsilon$. The invariance of $\mathfrak{L}_\varepsilon$ with respect to V_ε is equivalent to the equality

$$\sqrt{1+\varepsilon}\,K_\varepsilon V_{11} + \sqrt{1-\varepsilon}\,K_\varepsilon V_{12}K_\varepsilon - \sqrt{1+\varepsilon}\,V_{21} - \sqrt{1-\varepsilon}\,V_{22}K_\varepsilon = 0. \tag{2}$$

In view of $K_\varepsilon \in \mathfrak{K}$, we may assume, without loss of generality, that K_ε converges as $\varepsilon \to 0$ to an operator K_0 in the weak operator topology. The complete continuity of V_{12} implies that equality (2) turns into the equality

$$K_0 V_{11} + K_0 V_{12} K_0 - V_{21} - V_{22} K_0 = 0$$

as $\varepsilon \to 0$. This is equivalent to the invariance of the subspace $\mathfrak{L}_0 \in \mathfrak{M}^+$ with the angular operator K_0 under V.

(b) *Fixed-point method.* If $\mathfrak{L} \in \mathfrak{M}^+$ and K is its angular operator, then $V\mathfrak{L} \in \mathfrak{M}^+$ and its angular operator has the form

$$F(K) = (V_{21} + V_{22}K)(V_{11} + V_{12}K)^{-1}. \tag{3}$$

Therefore, $V\mathfrak{L} = \mathfrak{L}$ if and only if K is a fixed point of the linear-fractional transformation F. It follows from the complete continuity of V_{12} that the function F is continuous in the weak operator topology. Since $F\colon \mathfrak{K} \to \mathfrak{K}$ (see Section 4) and $\mathfrak{K}$ is a convex bicompact set in the weak operator topology, the function F possesses a fixed point.

Note that each of these methods gives certain additional information on the invariant subspaces. In the first one, it was proved that the spectrum $\sigma(V|\mathfrak{L}_0)$ of the operator $V|\mathfrak{L}_0$ does not contain points of the open unit disk. The second one allowed us to establish that every invariant nonnegative subspace can be extended to the maximal nonnegative invariant subspace.

If the operator V is J-unitary under the conditions of Theorem 2, then its nonunitary spectrum $\sigma_{\mathrm{nun}}(V)$ consists of normal eigenvalues. Let

$$\sigma_{\mathrm{nun}}(V) = \Omega_1 \cup \Omega_2, \quad \Omega_2 = \{\bar{\lambda}^{-1} \mid \lambda \in \Omega_1\} \quad \text{and} \quad \Omega_1 \cap \Omega_2 = \emptyset.$$

Then there exists a subspace $\mathfrak{L} \in \mathfrak{M}^+$ invariant under V and such that $\sigma_{\mathrm{nun}}(V|\mathfrak{L}) = \Omega_1$ [13].

4. Among Krein's researches into the theory of indefinite spaces, we should especially mention a series of papers [12, 14–16] written together with Shmul'yan and containing a deep and comprehensive study of the plus-operators. Below, we present some results obtained in these papers.

Let V be a plus-operator. Define the functions

$$\mu_+(V) = \inf\{[Vx, Vx] \mid [x, x] = 1\},$$

$$\mu_-(V) = \sup\{-[Vx, Vx] \mid [x, x] = -1\}.$$

Clearly, $\mu_+(V) \ge \mu_-(V)$ and $[Vx, Vx] \ge \mu[x, x]$ if and only if $\mu \in [\mu_-(V), \mu_+(V)]$. Therefore, if $\mu_+(V) = 0$, then the range of the plus-operator V is a nonnegative lineal. A plus-operator V is called strict whenever $\mu_+(V) > 0$. This operator is collinear to a J-noncontracting operator; indeed, $\mu_+(V)^{-1/2}V$ is a J-noncontracting operator.

An operator V is called J-binoncontracting if both V and its J-adjoint operator V^c are J-noncontracting.

Let V be a strict plus-operator in the Krein space and $\mathfrak{L} \in \mathfrak{M}^+$. Let $\operatorname{def} V := \dim(\mathfrak{H}^+ / P^+ V\mathfrak{L})$. This value is independent of the choice of $\mathfrak{L}$ and $\operatorname{def} V = 0$ if and only if V is collinear to a J-binoncontracting operator.

We say that an operator V is uniformly J-expanding if $[Vx, Vx] \ge [x, x] + \delta\|x\|^2$ for some $\delta > 0$.

A plus-operator V is called stable if all the operators in a certain neighborhood of it are plus-operators.

It was proved that the following conditions are equivalent:

(a) V is a stable plus-operator;

(b) V is a strict plus-operator and $\mu_+(V) > \mu_-(V)$;

(c) V is collinear to a uniformly J-expanding operator.

Let A be a bounded operator. A J-self-adjoint operator R is called its J-module provided that

$$\sigma(R) \subset [0, \infty), \qquad R^2 = A^c A, \qquad \text{and} \qquad \operatorname{Ker} R = \operatorname{Ker} A^c A.$$

It was proved that if V is a strict plus-operator and $\sigma(V^c V) \subset [0, \infty)$, then V possesses a J-module R. The condition $\sigma(V^c V) \subset [0, \infty)$ holds, for example, whenever V^c is also a plus-operator. Under the indicated conditions there exists a J-isometric operator W such that $R = WV$. This, in particular, implies that every J-binoncontracting operator V admits a J-polar decomposition $V = UR$, where U is a partially J-isometric operator and R is the J-module of the operator.

Papers [12] and [16] contain an extensive study of transformation (3), which is now called the Krein–Shmul'yan linear-fractional transformation. We now cite several results obtained in these papers. Let $\mathfrak{K}^0$ be the interior of the set $\mathfrak{K}$ of contractions acting from $\mathfrak{H}^+$ into $\mathfrak{H}^-$. In order that transformation (3) map $\mathfrak{K}$ into $\mathfrak{K}$ and $\mathfrak{K}^0$ into $\mathfrak{K}^0$, it is necessary and sufficient that the operator $V = \left\| V_{ij} \right\|_{i,j=1}^2$ be collinear to a J-binoncontracting operator. This mapping is bijective on $\mathfrak{K}$ if and only if V is collinear to a J-unitary operator.

This statement and Theorem 1 (in the strengthened form) immediately give a parametric description of the bijective linear-fractional transformations of the unit ball. Further, in order that transformation (3) map $\mathfrak{K}$ into $r\mathfrak{K}$ for some $r \in (0, 1)$, it is necessary and sufficient that V be collinear to a uniformly J-biexpanding operator.

The corresponding statements concerning linear-fractional transformations of the operator "upper half plane" into itself can be expressed in the same terms. These transformations are used in describing the sets of solutions of certain extrapolational problems for operator functions (cf. Section 10).

5. The publication of M. Krein's lectures [17] on the theory of indefinite spaces, where the contribution of many mathematicians was generalized for the first time, was an event of great mathematical significance. These lectures reflected not only theory but also its various applications, in particular, to the problems of stability of solutions of differential equations in Hilbert spaces. The monograph by M. Krein and Daletskii [18] was also largely devoted to applications of this sort. Here, we only recall the criterion of the exponential dichotomy of equations $dx/dt = A(t)x$ with periodic Hamiltonians A in a Hilbert space $\mathfrak{H}$. This criterion was expressed in terms of the behavior of the operator-function $A(t)$ when the space $\mathfrak{H}$ is equipped with an indefinite metric.

6. In 1961–1962, M. Krein and Langer intensely studied the operators in the Krein spaces and their applications to the quadratic operator sheafs (see [19 – 21, 13] and lectures [17]). Here, we outline some of these results.

After an integral representation of a Toeplitz sequence with κ $(< \infty)$ negative squares was constructed in [10] (as is known, for $\kappa = 0$, this is equivalent to constructing a resolution of the identity for a unitary operator in a Hilbert space) and the general spectral theory of operators with real spectrum and tempered resolvents was built (see, e.g., [22]), it became possible to prove the existence of the spectral function of a π-self-adjoint operator in the space Π_κ. This was done by M. Krein. Later, a more complete theory was developed by him together with Langer.

Recall that a bounded J-self-adjoint operator A in the Krein space $\mathfrak{H}$ is called definitizable if there exists a polynomial p such that $[p(A)x, x] \geq 0$, $x \in \mathfrak{H}$ (the importance of this concept was first mentioned in [10]). Let us give two examples of definitizable operators:

(i) an integral operator with positive definite kernel and indefinite weight (in this case, $p(\lambda) \equiv \lambda$; these operators were considered in Section 2);

(ii) any bounded π-self-adjoint operator in Π_κ.

The last fact is an analytical consequence of the Pontryagin geometric theorem on the existence of a κ-dimensional nonpositive invariant subspace $\mathfrak{L}$ of a π-self-adjoint operator A in Π_κ (see Sections 2 and 3). Indeed, if q is the minimal polynomial of the restriction $A \,|\, \mathfrak{L}$, then it is easy to see that

$$[\bar{q}(A)q(A)x, x] = [q(A)x, q(A)x] \geq 0$$

for any $x \in \Pi_\kappa$ (here, $\bar{q}$ denotes the polynomial $\overline{q(\bar{\lambda})}$).

Let us formulate the main result concerning the J-spectral function of a bounded operator with real spectrum acting in the Krein space $\mathfrak{H}$ and possessing a definitizing polynomial p of the lowest degree (cf. Section 2). Denote by $\mathbb{R}_p$ the ring of subsets of the real axis generated by all the intervals whose endpoints are not the roots of the polynomial p.

Theorem 3. *Assume that an operator A possesses the properties indicated above. Then every $\Delta \in \mathbb{R}_p$ can be associated with a bounded J-self-adjoint projector $E(\Delta)$ in $\mathfrak{H}$ such that*

(i) $E(\Delta)E(\Delta') = E(\Delta \cap \Delta')$;

(ii) $E(\Delta \cup \Delta') = E(\Delta) + E(\Delta')$ *if* $\Delta \cap \Delta' = \emptyset$;

(iii) $E(\Delta)\mathfrak{H}$ *is the positive (negative) subspace for* $p(\Delta) > 0$ $(p(\Delta) < 0)$;

(iv) $E(\mathbb{R}) = I$;

(v) $AE(\Delta) = E(\Delta)A$;

(vi) $\sigma(A \,|\, E(\Delta)\mathfrak{H}) = \operatorname{clos} \Delta$.

For the space Π_κ, this theorem was announced in [19]. In the general form, it was proved by Langer in his thesis (1965). For a special case of J-nonnegative operators, this theorem was proved in a different way by M. Krein and Shmul'yan [15] who used this result in solving the problem of polar representations of plus-operators (see Section 4).

Note that this situation differs from the definite case by the fact that the J-spectral function $E(\Delta)$ possesses finitely many critical points. (A point $\lambda_0 \in \mathbb{R}$ is called critical if $\|E(\Delta)\| \to \infty$ when one of the ends of the interval Δ approaches λ_0.)

By using the J-spectral function of an operator A, one can construct its J-spectral decomposition (cf. Section 2).

7. Having read Duffins' paper [23], M. Krein reformulated his theorem on the existence of two bases of eigenvectors for a second-order overdamped matrix sheaf as an assertion on the existence of matrix solutions Z_+ and Z_- of the equation $Z^2 + BZ + C = 0$. Later, by using his own generalization of the Pontryagin theorem, Langer extended this result to the sheaf $\lambda^2 + \lambda B + C$, where B is a self-adjoint operator and $C \geq 0$ is a completely continuous operator in a Hilbert space. Further investigations in this direction, carried out by M. Krein and Langer (see [20, 21]), were stimulated by the problems of the theory of nonself-adjoint operators [24].

Note that paper [21] (and even its title) demonstrates a distinctive feature of many of M. Krein's investigations, namely, their close relation with the problems of mechanics. Here, we present one of the main results of this paper.

Let $B = B^*$, let $C > 0$ be a completely continuous operator in a Hilbert space $\mathfrak{H}$, and let $L(\lambda) := \lambda^2 I + \lambda B + C$. One can easily see that the nonreal spectrum $\sigma_0(L)$ of the sheaf L is discrete.

Theorem 4. *Every representation*

$$\sigma_0(L) = \Lambda \cup \bar{\Lambda} \quad (\bar{\Lambda} := \{\lambda \mid \bar{\lambda} \in \Lambda\},\ \Lambda \cap \bar{\Lambda} = \emptyset)$$

can be associated with a completely continuous operator Z *in* $\mathfrak{H}$ *with the following properties:*

(i) $Z^2 + BZ + C = 0$;

(ii) $Z^* Z \le C$;

(iii) the nonreal spectrum of the operator Z *coincides with* Λ; *for* $\lambda \in \Lambda$, *the Jordan chains of the sheaf* L *and the operator* Z *are identical.*

8. In the mid-1960s, M. Krein had an idea to write with Langer a monograph devoted to the investigation of operators in the J-spaces. He thought that it would be desirable first to develop the applications of the already existing general theory. This originated a series of investigations, the results of which are discussed in this and subsequent sections.

A series of papers published in 1968 – 1971 (see, e.g., [25 – 27]) laid the foundations of the theory which is now called the Adamyan – Arov – Krein theory. For years, this theory served as a basis of numerous researchers into different fields of pure and applied mathematics. Recall that some results of this theory deal with the description of all possible solutions of extrapolational problems in the case where certain analytic functions have poles. Roughly speaking, the last statement is equivalent to the fact that certain Hermitian forms possess finitely many negative squares (this is discussed in Section 9 in more detail). These considerations enabled M. Krein to conclude that, by analogy with the corresponding classical problems, the solutions of the indicated more general problems can be obtained from the theory of π-self-adjoint extensions and generalized resolvents of a π-Hermitian operator A. For a densely defined operator A with equal defect numbers and for a π-isometry with a nondegenerate domain of definition, the corresponding theory was developed by M. Krein and Langer [28–30]. Note that a part of these results related to the description of generalized resolvents threw a new light on M. Krein's corresponding earlier results for symmetric operators in a Hilbert space, raising them to a new level of generality. Furthermore, some assertions, e.g., the characteristic properties of the Q-function (see Section 9), were first obtained just for the case of the space Π_κ.

We now formulate the main result of [28].

Let A be a densely defined closed π-Hermitian operator in Π_κ, let $\sigma_p(A)$ be its point spectrum, and let $\mathbb{C}^+$ and $\mathbb{C}^-$ be the open upper and lower half planes, respectively,

$$\sigma_p^+(A) := \sigma_p(A) \cap \mathbb{C}^+, \qquad \sigma_p^-(A) := \sigma_p(A) \cap \mathbb{C}^-, \qquad \mathfrak{N}_{\bar{z}} := \Pi_\kappa [-] (A - zI)\mathfrak{D}_A.$$

Then $\sigma_p^+(A)$ ($\sigma_p^-(A)$) consists of the eigenvalues of A whose sum of algebraic multiplicities is at most κ and $\dim \mathfrak{N}_{\bar{z}}$ remains constant for $z \in \mathbb{C}^+ \setminus \sigma_p^+(A)$ ($z \in \mathbb{C}^- \setminus \sigma_p^-(A)$). This dimensionality $n^+(A)$ ($n^-(A)$) is called the upper (lower) defect number of the operator A.

Further, we consider the case where $n^+(A) = n^-(A) = n \le \infty$. Let $\mathfrak{G}$ be a Hilbert space with dimensionality n, let $\mathring{A}$ be a π-self-adjoint extension of A that does not lead out of Π_κ, let $z_0 \in \mathbb{C}^+ \cap \rho(\mathring{A})$, let Γ_{z_0} be a linear continuous operator that maps $\mathfrak{G}$ onto $\mathfrak{N}_{z_0}$ bijectively, and let

$$\Gamma_z := (\mathring{A} - z_0 I)(\mathring{A} - zI)^{-1}\Gamma_{z_0} \quad (z \in \rho(\mathring{A})).$$

Consider a set (class) $\tilde{N}_0(\mathfrak{G})$ of holomorphic functions $\mathcal{T}(z)$ defined symmetrically with respect to the real axis and such that, for $z \in \mathbb{C}^+$, their values are densely defined maximal dissipative operators in $\mathfrak{G}$ (including improper ones). Omitting the details, we only note that for $n = 1$ the class $\tilde{N}_0(\mathfrak{G})$ $(= \tilde{N}_0(\mathbb{C}))$ is defined as a collection of all scalar functions locally holomorphic outside the real axis and such that

$$\mathcal{T}(z) = \overline{\mathcal{T}(\bar{z})}, \quad \operatorname{Im}\mathcal{T}(z)/\operatorname{Im} z \geq 0 \quad \text{for} \quad \operatorname{Im} z \neq 0;$$

moreover, it also contains the constant ∞.

If $\tilde{A}$ is a π-self-adjoint extension of the operator A to the space $\tilde{\Pi}_\kappa \supset \Pi_\kappa$ (κ has the same value for both these spaces) and P is the π-orthogonal projector of $\tilde{\Pi}_\kappa$ onto Π_κ, then, by analogy with the definite case, the operator function $R_z := P(\tilde{A} - zI)^{-1} \mid \Pi_\kappa$ is called a generalized resolvent of the operator A.

Theorem 5. *There exists a bijective correspondence between the set of generalized resolvents R_z of the operator A and the set $\tilde{N}_0(\mathfrak{G})$ of operator functions $\mathcal{T}(z)$. This correspondence is given by the equality*

$$R_z = (\mathring{A} - zI)^{-1} - \Gamma_z(\mathcal{T}(z) + Q(z))^{-1}\Gamma_{\bar{z}}^c, \qquad z \in \rho(A) \cap \rho(\mathring{A}).$$

Here,

$$Q(z) = C - iy_0\Gamma_{z_0}^c\Gamma_{z_0} + (z - \bar{z}_0)\Gamma_{z_0}^c\Gamma_z, \quad y_0 = \operatorname{Im} z_0,$$

is a so-called Q-function of the operator A determined to within a bounded self-adjoint operator C in $\mathfrak{G}$. Furthermore, the extension $\tilde{A}$ can be constructed in the original space if and only if $\mathcal{T}$ is a constant self-adjoint operator in $\mathfrak{G}$ (generally speaking, improper).

9. The investigation of the generalized resolvents of Hermitian and isometric operators in Π_κ clarified the significance of certain classes of complex-valued and operator-valued functions defined in a half plane or in a unit circle. We give the relevant definitions for scalar functions.

First of all, we recall the following: We say that a complex-valued kernel $K(s, t) = \overline{K(t, s)}$ defined for s, t from a nonempty set $D \subset \mathbb{C}$ has κ (≥ 0) negative squares if, for any $n \in \mathbb{N}$ and $s_1, \dots, s_n \in D$, the number of negative eigenvalues of the matrix $\| K(s_i, s_j)\|_{i,j=1}^n$ does not exceed κ, and, at least for one choice of $n, s_1, \dots, s_n$, it is equal to κ.

A function Q meromorphic in the upper half plane $\mathbb{C}^+$ belongs to the generalized Nevanlinna class N_κ if its kernel

$$N_Q(z, \zeta) = (z - \bar{\zeta})^{-1}(Q(z) - \overline{Q(\zeta)})$$

has κ negative squares for z and ζ belonging to the domain D_Q where Q is holomorphic.

A function F meromorphic in the open unit circle $\mathbb{D}$ belongs to the generalized Carathéodory class C_κ if its kernel

$$C_F(z, \zeta) = (1 - z\bar{\zeta})^{-1}(F(z) + \overline{F(\zeta)}), \quad z, \zeta \in D_F,$$

has κ negative squares.

A function θ meromorphic in $\mathbb{D}$ belongs to the generalized Schur class S_κ if the kernel

$$S_\theta(z,\zeta) = (1-\bar{\zeta}z)^{-1}(1-\overline{\theta(\zeta)}\theta(z)), \quad z,\zeta \in D_\theta,$$

has κ negative squares.

The corresponding operator classes $N_\kappa(\mathfrak{G})$, $C_\kappa(\mathfrak{G})$, and $S_\kappa(\mathfrak{G})$, where $\mathfrak{G}$ is a Hilbert space, are defined similarly.

The main result of [30] can be formulated as follows: An operator function $Q(z) = Q^*(\bar{z})$ is a Q-function of a simple π-Hermitian operator A with deficiency index (n, n) $(n \le \infty)$ in Π_κ if and only if

(i) $Q \in N_\kappa(\mathfrak{G})$, where $\dim \mathfrak{G} = n$;

(ii) $w-\lim\limits_{y\uparrow\infty} y^{-1}Q(iy) = 0$;

(iii) $\lim\limits_{y\uparrow\infty} y(\operatorname{Im} Q(iy)\xi,\xi) = \infty$ for all $\xi \in \mathfrak{G}$, $\xi \ne 0$;

(iv) the operator $\operatorname{Im} Q(z)$ is uniformly positive at least for one z.

Recall that a π-Hermitian operator in Π_κ is called simple if all its eigenvalues are real and

$$\bigvee_{z\ne\bar{z}} \mathfrak{N}_z = \Pi_\kappa.$$

Every Q-function of an operator is a Q-function of its simple part and vice versa. A simple operator is defined by its Q-function to within π-unitary equivalence.

M. Krein and Langer (see, e.g., [31]) constructed integral representations for the functions belonging to the classes N_κ and C_κ with a measure that may have singularities at finitely many points. It was also shown that $\theta \in S_\kappa$ if and only if $\theta(z) = B(z)^{-1}\theta_0(z)$, $z \in D_\theta$, where $\theta_0 \in S_0$ and B is the Blaschke product of the κth order. The indicated paper also contains the proof of the assertion that every function $\theta(z)$ from the class S_κ holomorphic for $z = 0$ is a characteristic function of a certain π-unitary knot U, namely,

$$\theta(z) = U_{22} - zU_{21}(I - zU_{11})^{-1}U_{12}, \tag{4}$$

where the matrix $U = \|U_{ij}\|_{i,j=1}^2$ defines a π-unitary operator in the direct sum $\Pi_\kappa[+]\,\mathbb{C}$. By the way, M. Krein was probably the first who understood that all types of characteristic functions of linear operators can be obtained from relation (4), where the operator U possesses special properties.

As in the definite case $(\kappa = 0)$, various problems of extrapolation and representation are connected with the classes N_κ, C_κ, and S_κ. Some of these problems are mentioned in the introduction to [31]. Here, we formulate two problems of this sort.

I. *Indefinite problem of moments.* Given a sequence of complex numbers $(s_j)_{j=0}^\infty$, establish the conditions under which there exists a function $Q \in N_\kappa$ with the following asymptotic expansion:

$$Q(z) \sim -\frac{1}{z}\left(s_0 + \frac{s_1}{z} + \frac{s_2}{z^2} + \ldots\right), \quad z = iy, \quad y \uparrow \infty.$$

If Q is not unique, describe all these functions.

It is known that, for $\kappa = 0$, this problem is equivalent to the classical Hamburger moment problem. If we restrict the class N_κ by introducing an additional requirement that $Q_0 \in N_0$, where $Q_0(z) = zQ(z)$, then we arrive at an indefinite analog of the Stieltjes moment problem.

II. *The problem of extension of a Hermitian indefinite function with* κ *negative squares.* Denote by $\mathfrak{F}_{\kappa;a}$, $0 < a \le \infty$, a set of continuous functions f defined on $(-2a, 2a)$ and such that $f(t) = \overline{f(-t)}$ $(|t| < 2a)$ and the kernel $f(t-s)$ $(|s|, |t| < a)$ has κ negative squares. The equality

$$Q(z) = i\int_0^\infty e^{izt}\overline{f(t)}\,dt, \qquad \operatorname{Im} z > \gamma_f,$$

establishes a bijective correspondence between $\mathfrak{F}_{\kappa;\infty}$ and a set selected from N_κ by certain conditions imposed on the behavior at infinity.

Is it possible to construct an extension $\tilde{f} \in \mathfrak{F}_{\kappa;\infty}$ of a given function $f \in \mathfrak{F}_{\kappa;a}$, $a < \infty$? If this extension $\tilde{f}$ exists and is not unique, it is necessary describe all $\tilde{f}$.

Note that the problem of extension of a function $f \in \mathfrak{F}_{0;a}$, $a < \infty$, i.e., the problem of extension of a Hermitian positive continuous function to the entire axis, permanently drew M. Krein's attention, who suggested several solutions of it. This problem is quite important for probability theory, where $f \in \mathfrak{F}_{0;\infty}$ plays the roles of a characteristic function of a random distribution and of a correlation function of a stationary random process. At the same time, it is more surprising that this problem also plays a key role in M. Krein's investigations of inverse spectral problems for second-order differential operators. He explained this deep relationship by intuitive considerations provoked by the mechanics of an oscillating string (for details, see the paper by I. S. Kats in the next issue of the journal).

10. The above-mentioned (and some other) extension problems are closely related to the theory of entire π-Hermitian operators (as in the case of $\kappa = 0$). For Hilbert spaces, this deep theory was constructed at the end of the 1940s. M. Krein always regarded these results as one of the most significant of his personal achievements. This is why he dedicated the paper [33] to his teacher N. G. Chebotarev as a sign of his deepest gratitude which was stressed quite often. The generalization of M. Krein's results carried out together with Langer was no longer difficult. By using the theory of resolvent matrices, the following result was established in [32] (at that time, it was also new for the case of a Hilbert space):

Let A be a simple π-Hermitian operator with the deficiency index $(1, 1)$ in the space Π_κ; we say that $u \in \Pi_\kappa$ is a module of A if $u \notin \mathfrak{R}(A - zI)$ at least for one point $z \in \mathbb{C}^+$ and one point $z \in \mathbb{C}^-$ and, consequently, for all $z \in \mathbb{C}^+ \cup \mathbb{C}^-$ except, possibly, a set of isolated points. A module u is called an entire if

$$\mathfrak{R}(A - zI) \dotplus \{\lambda u \,|\, \lambda \in \mathbb{C}\} = \Pi_\kappa$$

for all $z \in \mathbb{C}$. An operator A is called entire if it has an entire module. Such operators can be realized as operators of multiplication by z in the space of entire functions $f(z)$ of exponential type $\le a$. The smallest a (≥ 0) of this sort is called the type of operator A. Suppose that the space Π_κ is equipped with an involution and, hence, we can speak about real elements and operators.

Theorem 6. *Let A be a simple real entire π-Hermitian operator of type a (≥ 0) with deficiency index $(1, 1)$ acting in the space in P_κ and let u be its real module (determined, for a given A, uniquely to within a real factor). Then the operator A possesses a u-resolvent matrix $\omega(z) = \|\omega_{ij}(z)\|_{i,j=1}^2$ whose elements are real entire functions of exponential type a. Under the normalization conditions $\omega(0) = I_2$ and $\det\omega(z) \equiv 1$, $z \in \mathbb{C}$, the* *matrix* ω *is determined uniquely.*

Here, a u-resolvent matrix is defined as a matrix function $\omega(z)$ such that the relation

$$\left[\left(\tilde{A} - zI\right)^{-1}u, u\right] = \frac{\omega_{11}(z)\mathcal{T}(z) + \omega_{12}(z)}{\omega_{21}(z)\mathcal{T}(z) + \omega_{22}(z)}$$

defines a bijective correspondence between all minimal π-self-adjoint extensions $\tilde{A}$ of the operator A (that may lead to the space $\tilde{\Pi}_\kappa \supset \Pi_\kappa$ with the same κ) and all $\mathcal{T} \in \tilde{N}_0(\mathbb{C})$.

In the case where problems I and II posed above admit more than one solution, the complete description of their solutions can be obtained by using Theorem 6. This approach was first outlined in the introduction to [31]. For problem I, it was realized immediately [34]. At the same time, its realization for problem II was delayed for many reasons, although the main result (the description of the set of solutions in the case of nonuniqueness) was obtained relatively soon and announced in [35]. One of the reasons for this delay can be described as follows: In [34], the indefinite Stieltjes moment problem was connected with a generalized Stieltjes string with positive and negative masses and dipoles. Although there is no doubt that the indicated result should admit a generalization to even functions from the class $\mathfrak{F}_{\kappa;\infty}$, this assertion remains unproven up until now. At the same time, for a special case of the functions $f \in \mathfrak{F}_{\kappa;\infty}$ with accelerant (i.e.,

$$f(t) = f(0) - \alpha|t| - \int_0^t (t-s)H(s)\,ds, \qquad \alpha > 0, \qquad H \in L^1_{\text{loc}}(\mathbb{R})),$$

the corresponding results were obtained rather soon. The amount of accumulated material was so large that it was decided to speed up the publication without waiting until the investigations of the general case could be completed (see [36]). It should be noted that, unlike the general case, the resolvent matrix of the function f with accelerant is constructed by an efficient analytical procedure. In the beginning of the 1980s, when M. Krein and Langer were prepared to publish the complete proofs of the results related to problem II, M. Krein had an idea to anticipate the presentation by a general overview of his old results on extensions of positive definite functions (from the class $\mathfrak{F}_{0;a}$) and their relation to the inverse problems for differential operators, helical arcs in a Hilbert space, etc. In preparing this paper, the authors wrote a new article, where the concept of indefinite metric was not mentioned at all but new results were obtained, e.g., in the theory of extrapolation of stationary random processes. But both this article and the complete solution of problem II have not been published yet. Unfortunately, this is a fate of many of M. Krein's important results that either remain unpublished until now or have been published only partially.

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