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ADDENDUM TO "A HILBERT SPACE OF DIRICHLET SERIES AND SYSTEMS OF DILATED FUNCTIONS IN $L^2(0,1)$"

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In the present note we comment on some relevant references of which we were unaware when we published our work in the *Duke Mathematical Journal* in 1997 [HLS]. We also take the opportunity to report about the present state of a conjecture in this work. We would like to thank M. Balazard, J. Neuwirth, and E. Saksman for valuable pieces of information.

The main problem treated in our work was to find necessary and sufficient conditions for completeness and basis properties of the system $\varphi(x), \varphi(2x), \varphi(3x), \ldots$ of functions in $L^2(0,\pi)$, where

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sin(nx), \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

Thus series of the type

$$c_1\varphi(x) + c_2\varphi(2x) + c_3\varphi(3x) + \cdots, \quad \sum_{n=1}^{\infty} |c_n|^2 < \infty, \tag{1}$$

are considered. The auxiliary Dirichlet series

$$\mathcal{S}\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad s = \sigma + it, \tag{2}$$

plays a central role. We assume that $a_1 \neq 0$. As in [HLS], we denote by $\mathcal{H}$ the Hilbert space of square-summable Dirichlet series.

In 1944 A. Wintner [W2] introduced series (2) and derived a necessary condition for the L^2-convergence of *all* series (1) with square-summable coefficients: *the auxiliary Dirichlet series (2) has an extension as an analytic and bounded function to the half-plane* $\sigma > 0$. A. Beurling in [B] used the same Dirichlet series in a seminar in 1945. In hindsight one may say that Wintner's condition means that $\sum_{n=1}^{\infty} a_n n^{-s}$ is a *multiplier*. This characterization of the multipliers is the content of our Theorem 3.1.

Wintner also gave some sufficient conditions for the system $\{\varphi(nx)\}_n$ to be complete in $L^2(0,\pi)$. In particular, he studied the example

$$\varphi(x) = \sum_{n=1}^{\infty} n^{-\lambda} \sin(nx)$$

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and proved that the corresponding system of functions is complete if and only if $\Re e(\lambda) > 1/2$. This is now a special case of our Theorem 5.8. The completeness has been studied in [B], [Bo], and [NGN].

The Riesz bases were given the following characterization in our work.

THEOREM 1. *We have a Riesz basis in* $L^2(0,\pi)$ *if and only if* $\mathcal{S}\varphi(s)$ *is analytic and bounded away from* 0 *and* ∞ *in the whole right half-plane* $\Re e(s) > 0$.

In other words,

$$\delta \le \left| \sum\nolimits_{n=1}^{\infty} a_n n^{-\sigma - it} \right| \le M \quad \text{when } \sigma > 0, \tag{3}$$

for some positive constants δ and M. Here we want to acknowledge that R. Gosselin and J. Neuwirth earlier had given a similar criterion, though under an additional assumption that, strictly speaking, makes their condition sufficient but not necessary. In [GN], they formulated their results in terms of the boundary function

$$\mathcal{S}\varphi(it) = \sum_{n=1}^{\infty} a_n n^{-it}, \quad -\infty < t < \infty.$$

THEOREM 2 (Gosselin-Neuwirth). *Suppose that* $\mathcal{S}\varphi(it)$ *is almost periodic in the sense of Bohr. Then we have a Riesz basis if and only if* $1/\mathcal{S}\varphi \in \mathcal{H}$ *and*

$$\inf_{-\infty < t < \infty} |\mathcal{S}\varphi(it)| > 0.$$

The auxiliary Dirichlet series $\mathcal{S}\varphi$ now defines a function that is uniformly continuous in the closed half-plane $\Re e(s) \ge 0$ and analytic in the open half-plane. We refer the reader to [Bes] for the relevant theory of Besicovitch almost-periodic functions. Under the assumptions of Theorem 2, it is easy to see that we have (3). However, the almost periodicity of $\mathcal{S}\varphi(it)$ is far from needed to guarantee (3). An example is induced by the Dirichlet series

$$\mathcal{S}\varphi(s) = \frac{1}{4}\left\{3 + e^{-(1+2^{-s})/(1-2^{-s})}\right\} = \frac{3e+1}{4e} - \frac{1}{2e}2^{-s} + \frac{1}{16e}8^{-s} + \cdots,$$

satisfying the inequalities

$$\frac{1}{2} < |\mathcal{S}\varphi(s)| < 1 \quad \text{when } \Re e(s) > 0.$$

The inequalities follow from the estimate

$$0 < \left| e^{-(1+z)(1-z)} \right| < 1, \quad |z| < 1,$$

which also shows that we get square-summable coefficients. The corresponding system is a Riesz basis, according to our theorem. However, the function $\mathcal{S}\varphi(it)$, $-\infty <$

$t < \infty$, is not uniformly continuous, because of the behaviour at the points $t = 2n\pi/\log 2, n = 0, \pm 1, \pm 2, \ldots$, and *a fortiori* not almost periodic in the sense of Bohr. This Riesz basis is outside the scope of Gosselin's and Neuwirth's theorem.

Let us turn to the conjecture of almost everywhere convergence of the Fourier series associated with a Dirichlet series. A "character" χ is defined as a totally multiplicative function of absolute value 1: $\chi(mn) = \chi(m)\chi(n)$ when $m, n = 1, 2, 3, \ldots$ and $|\chi(n)| = 1$. We conjectured that for an arbitrary square-summable sequence $\{a_n\}_n$ of complex numbers, the series

$$\sum_{n=1}^{\infty} a_n \chi(n)$$

should converge for almost every character χ. This is analogous to L. Carleson's convergence theorem for Fourier series [C]. Recently, E. Saksman and H. Hedenmalm [HS] proved that this is so by adapting C. Fefferman's technique in [F], which transfers the problem of convergence of multiple Fourier series back to Carleson's one-dimensional Fourier series setting. It is a simple application of this result that

$$\chi(1) + \chi(2) + \cdots + \chi(N) = \mathcal{O}\big(\sqrt{N} \log^{1/2} N (\log\log N)^{1/2+\varepsilon}\big)$$

holds as $N \to +\infty$ for *almost every* character χ. This settles the conjecture on character sums made in [HLS].

Finally, we point out that two intriguing series of the form (2) were mentioned by B. Riemann in the sixth section of his famous *Habilitationsschrift*, published in 1854. See [R], [CW], [W1], and [H]. The first study of "general" series of this type seems to be due to F. Jerosch and H. Weyl in 1909 [JW]. Some more recent work about pointwise behaviour is [G] and [Ber].

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