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Spectral and Scattering Theory for the Klein-Gordon Equation

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Abstract. Eigenfunction expansions associated with the Klein-Gordon equation, are derived in the static external field case. By employing these, we develop spectral and scattering theory. The results are almost as strong as those obtained in the Schrödinger case.

1. Introduction

We shall in this paper consider spectral and scattering theory for the Klein-Gordon (K-G) equation

$$(\Box + 2iq_0(x)\,\partial_t - q_0(x)^2 + q_s(x) + m^2)\,u(x,t) = 0\,, \tag{1.1}$$

where $x \in R^3, t \in R$ and $\Box = \partial_t^2 - \Delta$. The functions $q_0(x)$ and $q_s(x)$ are static external potentials coupled like a zero'th component of a four-vector and a scalar respectively.

Equations similar to (1.1) have been studied previously. Thoe [1] developed spectral and scattering theory in the case $m = 0$, $q_0(x) = 0$ and $q_s(x) > 0$ by employing a method due to Lax and Phillips [2] (only suited for the $m = 0$ case). Strauss [3] showed the existence and boundedness of the scattering operator and its inverse when $m > 0$ and $q_0(x) = 0$. Veselic [4] considered some spectral properties of Eq. (1.1) in the case $m > 0$, $q_s(x) = 0$ under very restrictive conditions on $q_0(x)$ (excluding for example the square well case).

We shall consider Eq. (1.1) when $m > 0$ and derive eigenfunction expansions. These will enable us to develop the spectral theory of Eq. (1.1) in detail and develop scattering theory to the same level as is possible in the Schrödinger case [5–8].

The main motivation for this investigation (from a physical point of view) is the fact that once we have developed spectral and scattering theory for Eq. (1.1) when considered as a classical field equation, the associated quantum field theoretic problem can be completely solved (and will be considered elsewhere).

In Section 2 we start by specifying the class of potentials we shall consider. Furthermore, we write the K-G equation in the form $i\partial_t \Psi = A\Psi$

and construct a Hilbert space $\mathscr{H}$ associated with it (by employing the field-energy as norm). The operator A is then shown to be self-adjoint and its essential spectrum $\sigma_e(A)$ is determined (Section 3). Eigenfunction expansions associated with A are derived and the absolutely continuous spectrum $\sigma_{a.c.}(A)$ is determined (Section 4).

Finally, the existence and completeness of the wave-operators $W_\pm$ are established and the explicit form of the S-matrix is given (Section 5).

2. The K-G Equation Written as $i\partial_t \Psi = A\Psi$ and the Hilbert Space $\mathscr{H}$ Associated with it

We shall consider Eq. (1.1) under the following conditions on the potentials $q_0(x)$ and $q_s(x)$;

i) $q_0(x)$ *and* $q_s(x)$ *are real-valued and locally Hölder-continuous except at a finite number of singularities;*

ii) $q_0(x)^2$ *and* $q_s(x)$ *are square integrable;*

iii) $q_0(x)$ *and* $q_s(x)$ *behave as* $\mathcal{O}(|x|^{-3-\varepsilon})$, $\varepsilon > 0$ *for* $|x| \to \infty$;

iv) $\int dx(-q_0(x)^2 + q_s(x))\,|f(x)|^2 \geqq -\alpha \int dx(|\nabla f(x)|^2 + m^2 |f(x)|^2)$, *with* $0 < \alpha < 1$ *and* $f(x) \in C_0^\infty(R^3)$.

Provided Eq. (1.1) together with $\{u(x,0), i\partial_t u(x,0)\} = \{f_1(x), f_2(x)\}$ defines a well-posed initial-value problem, it is well known that the field energy (energy integral)

$$E(u, i\partial_t u) = \int dx(|\nabla u|^2 + (m^2 - q_0^2 + q_s)\,|u|^2 + |\partial_t u|^2)\,, \tag{2.1}$$

is independent of t (if it is finite). Note that condition iv) on $q_0(x)$ and $q_s(x)$ ensures the positivity of $E(u, i\partial_t u)$.

Let us introduce the pair

$$\Psi(x,t) = \{u(x,t), i\partial_t u(x,t)\}\,, \tag{2.2}$$

and the time evolution operator $U(t)$; $\Psi(\cdot, t) = U(t)\,\Psi(\cdot, 0)$.

The operator $U(t)$ is easily seen to be unitary in the Hilbert space $\mathscr{H}$ consisting of initial data $f = \{f_1, f_2\}$ such that $\|f\|^2 = E(f_1, f_2) < \infty$, provided the initial-value problem is well-posed.

Definition 2.1. Let $\mathscr{H}$ denote the Hilbert space obtained by completing $\mathscr{D} = C_0^\infty(R^3) \times C_0^\infty(R^3)$ *in the norm given by*

$$\|f\|^2 = \int dx(|\nabla f_1|^2 + (m^2 - q_0^2 + q_s)\,|f_1|^2 + |f_2|^2)\,, \tag{2.3}$$

and with the scalar product defined in the obvious way. Let furthermore $\mathscr{H}_0$ denote $\mathscr{H}$ when $q_0 = q_s = 0$.

Remark 2.2. The norms in $\mathscr{H}_0$ and $\mathscr{H}$ are equivalent (this follows from condition ii) and iv) on the potentials), i.e. there exist constants c_1 and c_2 such that

$$0 < c_1 \|f\|_0 \leqq \|f\| \leqq c_2 \|f\|_0 , \tag{2.4}$$

where $\| \ \|_0$ denotes the norm in $\mathscr{H}_0$.

The K-G equation (1.1) can be written

$$i\partial_t \Psi = A\Psi , \tag{2.5}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ L & 2q_0 \end{pmatrix}, \quad L = -\Delta + m^2 + q, \quad q = -q_0^2 + q_s , \tag{2.6}$$

and with Ψ given by Eq. (2.2). One can easily check that A is symmetric on $\mathscr{D}$ in $\mathscr{H}$.

In the next section we shall prove that A is essentially self-adjoint on $\mathscr{D}$ in $\mathscr{H}$ and furthermore determine the essential spectrum of its closure (also denoted by A) in $\mathscr{H}$.

3. Self-adjointness of A in $\mathscr{H}$ and the Essential Spectrum $\sigma_e(A)$

In this section we show that A is essentially self-adjoint on $\mathscr{D}$ in $\mathscr{H}$ by using a well-known theorem of Kato and Rellich (generalized to the case of two Hilbert spaces by Thoe [1]). We furthermore determine the essential spectrum of $A(\sigma_e(A))$ in $\mathscr{H}$, by a compactness argument.

The operator A can be split into two parts

$$A = A_0 + V , \tag{3.1}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ L_0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ q & 2q_0 \end{pmatrix}, \quad L_0 = -\Delta + m^2 . \tag{3.2}$$

Let H^n denote the Sobolev space consisting of all functions, which together with their derivatives up to order n, are square integrable.

We note that $\mathscr{H} = \mathscr{H}_0 = H^1 \times H^0 = H^1 \times L^2(R^3)$. Let us start with the following well-known results (when $m > 0$).

Remark 3.1. A_0 is essentially self-adjoint on $\mathscr{D}$ in $\mathscr{H}_0$ and its closure (also denoted by A_0) has the domain $D(A_0) = H^2 \times H^1$.

Remark 3.2. The spectrum of A_0 in $\mathscr{H}_0$ satisfies $\sigma(A_0) = \sigma_e(A_0) = \sigma_{a.c.}(A_0) = (-\infty, -m] \cup [m, \infty)$. Here "e" stands for essential and "a.c." for absolutely continuous.

Remark 3.3. The unitary one-parameter group e^{-iA_0t} maps $\mathscr{D}$ into $\mathscr{D}$.

Remarks 3.1 and 3.2 are easily verified by Fourier transformation and 3.3 follows from the finite propagation velocity of solutions to Eq. (1.1) in the free case.

Theorem 3.4. *The operator A is essentially self-adjoint on $\mathscr{D}$ in $\mathscr{H}$ and its closure (also denoted by A) has the domain $D(A)=D(A_0)=H^2\times H^1$. The essential spectrum of A and A_0 coincide, $\sigma_e(A)=\sigma_e(A_0)$.*

Proof. We have seen that $A=A_0+V$ is symmetric on $\mathscr{D}$ in $\mathscr{H}$ and that A_0 is essentially self-adjoint on $\mathscr{D}$ in $\mathscr{H}_0$. It then follows from a theorem due to Thoe (see Appendix 1) that the condition

$$\|Vf\|_0 \leqq \beta\|A_0 f\|_0+\gamma\|f\|_0, \quad \beta<1, \quad f\in\mathscr{D}, \tag{3.3}$$

ensures the essential self-adjointness of A on $\mathscr{D}$ in $\mathscr{H}$. We will in fact show that β can be chosen arbitrarily small. It is sufficient for (3.3) to hold that

$$\|Vf\|_0^2 \leqq \beta'\|A_0 f\|_0^2+\gamma'\|f\|_0^2, \tag{3.4}$$

where β' can be chosen arbitrarily small. The estimate (3.4) follows from the following estimates[1] ($q(x)\in L^2(R^3)$, $q_0(x)\in L^4(R^3)$)

$$\|qf_1\|_2^2 \leqq \varepsilon_1\|L_0 f_1\|_2^2+\delta_1(\varepsilon_1)\,\|f_1\|_2^2, \tag{3.5}$$

and

$$\|q_0 f_2\|_2^2 \leqq \varepsilon_2(L_0 f_2; f_2)_2+\delta_2(\varepsilon_2)\,\|f_2\|_2^2, \tag{3.6}$$

where $\varepsilon_1,\varepsilon_2>0$ are arbitrary (see [9], p. 302 and p. 321). This finishes the proof of the essential self-adjointness of A on $\mathscr{D}$ in $\mathscr{H}$.

In order to ensure that $\sigma_e(A)=\sigma_e(A_0)$ it is sufficient to show that V is a compact operator from $D(A_0)$ to $\mathscr{H}_0$ (see [10], p. 18) or, equivalently that $q(q_0)$ is a compact operator from $H^2(H^1)$ to H^0. This, however follows from conditions ii) and iii) on $q_0(x)$ and $q_s(x)$ (see [10], p. 104).

4. Eigenfunction Expansions Associated with A and the Absolutely Continuous Spectrum $\sigma_{a.c.}(A)$

In this section we construct eigenfunctions and generalized eigenfunctions of A. These constitute a particular set of weak solutions to the "eigenvalue equation"

$$A\Phi=\omega\Phi, \quad \omega\in\sigma(A). \tag{4.1}$$

Equation (4.1) can be written

$$(\omega^2-2\omega q_0-L)\,u=0, \tag{4.2}$$

with $\Phi=\{u,\omega u\}$, and is of Schrödinger type. This means that we can make use of the well-known properties of solutions to the time-independent Schrödinger equation (see [5–8]).

Let us define

$$L(\omega)=L+2\omega q_0. \tag{4.3}$$

[1] The lower index "2" denotes $L^2(R^3)$.

Remark 4.1. The Schrödinger operator $L(\omega)$, $\omega \in R$ is self-adjoint on $D(L(\omega)) = H^2$ in $L^2(R^3)$ with $\sigma_e(L(\omega)) = \sigma_{a.c.}(L(\omega)) = [m^2, \infty)$, $L(\omega)$ has a finite number of eigenvalues (see [10], p. 218) and $L(0)$ is positive (follows from condition iv) on the potentials).

Let u_n denote a square integrable solution of Eq. (4.2) with $\omega = \omega_n \in R$. Remark 4.1 implies that $u_n \in H^2$ and that $|\omega_n| < m$.

Remark 4.2. The eigenfunctions Φ_n of A have the form $\Phi_n = \{u_n, \omega_n u_n\}$, i.e. $A\Phi_n = \omega_n \Phi_n$ with $\Phi_n \in D(A)$. We assume Φ_n to be normalized such that $(\Phi_n, \Phi_m) = \delta_{nm}$.

Remark 4.2 and the fact that $\sigma_e(A) = (-\infty, -m] \cup [m, \infty)$, implies that the projection operator P_d given by

$$P_d = E(m) - E(-m) = \int_{-m}^{m} dE(\lambda) \tag{4.4}$$

$\left(\text{with } A = \int_{-\infty}^{\infty} \lambda\, dE(\lambda)\right)$ has the following representation

$$P_d f = \sum_n \Phi_n (f, \Phi_n) \equiv \sum_n \Phi_n \hat{f}_n, \qquad f \in \mathscr{H}, \tag{4.5}$$

i.e. every element in $P_d \mathscr{H}$ can be expanded in terms of $\{\Phi_n\}$.

Let us introduce

$$P^+ = \int_m^{\infty} dE(\lambda), \qquad P^- = \int_{-\infty}^{-m} dE(\lambda), \qquad P = P^+ + P^-, \tag{4.6}$$

and

$$\mathscr{H}^\pm = P^\pm \mathscr{H}, \qquad A^\pm = P^\pm A. \tag{4.7}$$

Our goal is to obtain a formula similar to (4.5) for $P^\pm f$.

It is well-known that Eq. (4.2), under our conditions on the potentials, has solutions which satisfy the following integral equation (Lippmann-Schwinger equation) (see [5])

$$u^\pm(x,k) = e^{ikx} - \frac{1}{4\pi} \int dy \frac{e^{\pm i|k||x-y|}}{|x-y|} (\pm 2\omega(k) q_0(y) + q(y))\, u^\pm(y,k), \tag{4.8}$$

where $\omega(k) = \sqrt{k^2 + m^2}$.

Remark 4.3. A solution $u^\pm(x,k)$ of Eq. (4.8) has the following properties; a) it is unique, b) it is bounded on $R^3 \times K$, c) $u^\pm(x,k) - e^{ikx}$ is uniformly continuous on $R^3 \times K$. Here K is a compact set in $R^3 - \{0\}$. (For the proof see [5].)

Remark 4.4. The operator A has as a generalized eigenfunction $\Phi^\pm(x,k) = c_k \{u^\pm(x,k), \pm \omega(k) u^\pm(x,k)\}$. This is easily seen to be a weak solution of Eq. (4.1) with $\omega = \pm \omega(k)$. We choose $c_k = \dfrac{1}{\sqrt{2}(2\pi)^{3/2}\omega(k)}$.

Let $(\, , \,)_N$ denote the scalar product in $\mathscr{H}$ when the integration is only carried out over a sphere of radius N in R^3.

Theorem 4.5. *The mapping $F^{\pm}; \mathscr{H} \to L^2(R^3)$ defined by*

$$F^{\pm} f(k) = \lim_{N \to \infty} (f, \Phi^{\pm}(\cdot, k))_N = \hat{f}^{\pm}(k), \tag{4.9}$$

is isometric with initial domain $\mathscr{H}^{\pm}$ and the whole of $L^2(R^3)$ as range.

The adjoint $F^{\pm *}$ is given by

$$F^{\pm *} g(x) = \lim_{n \to \infty} \int_{K_n} dk \, \Phi^{\pm}(x, k) \, g(k), \tag{4.10}$$

where $K_n \subset R^3$ is an increasing set of compacts such that $U K_n = R^3$.

The operator $F^{\pm}$ diagonalizes $A^{\pm}$;

$$A^{\pm} = \pm F^{\pm *} M_{\omega(\cdot)} F^{\pm}, \tag{4.11}$$

where $M_{\omega(k)}$ stands for the multiplication operator $\omega(k)$.

The singular spectrum of A fulfills $\sigma_s(A) \subset (-m, m)$, i.e. $\sigma_{a.c.}(A) = \sigma_e(A)$.

Remark 4.6. The analog to formula (4.5) reads $P^{\pm} f = F^{\pm *} F^{\pm} f$.

Proof of Theorem 4.5. We shall start by showing that $F^{\pm}$ is isometric with initial domain $\mathscr{H}^{\pm}$. Our proof is based on the following formula

$$\begin{aligned} I(\lambda, \lambda', f) &= ((E(\lambda) - E(\lambda')) f, f) \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\lambda'}^{\lambda} d\mu ((R(\mu + i\varepsilon) - R(\mu - i\varepsilon)) f, f), \end{aligned} \tag{4.12}$$

where $R(z) = (A - z)^{-1}$, $\sigma_s(A) \cap (\lambda', \lambda) = \phi$, and $f \in \mathscr{D}$.

The right hand side of Eq. (4.12) will now be evaluated in a series of lemmas and takes the form

$$I(\lambda, \lambda', f) = \int_{\lambda' < \omega(k) < \lambda} dk |F^{\pm} f(k)|^2, \tag{4.13}$$

where the $\pm$ sign is chosen when $\begin{matrix} m \\ -\infty \end{matrix} < \lambda' < \lambda < \begin{matrix} \infty \\ -m \end{matrix}$. The integral $I(\lambda, \lambda', f)$ turns out to be continuous as a function of λ and λ' in the interval given above, which implies that $\sigma_s(A) \subset (-m, m)$ and furthermore that $F^{\pm}$ maps $\mathscr{H}^{\pm}$ isometrically into $L^2(R^3)$.

Lemma 4.6. *Let $f \in \mathscr{D}$, then*

$$R(z) f = \frac{1}{z} \{1, z\} G(z) (L f_1 + z f_2) - \frac{1}{z} \{f_1, 0\}, \quad \operatorname{Im} z \neq 0, \tag{4.14}$$

where

$$G(z) = (L + 2z q_0 - z^2)^{-1}, \tag{4.15}$$

is of the Carleman type in $L^2(R^3)$ for $\operatorname{Im} z \neq 0$.

Proof. The operator $G(z)$ is easily seen to exist and to be bounded in $L^2(R^3)$ for z sufficiently small (due to the positivity of L and the L-boundedness of q_0). Formula (4.14) can than be verified for small z by straightforward algebra and the analyticity of $R(z)f$ in $\mathscr{H}$ can be employed to extend the validity of Eq. (4.14) to all z such that $\operatorname{Im} z \neq 0$.

The first resolvant equation for $G(z)$ has the form

$$G(z) = G_0(z) - G(z)(2zq_0 + q)\,G_0(z), \qquad \operatorname{Im} z \neq 0, \tag{4.16}$$

where $G_0(z) = (L_0 - z^2)^{-1}$. The operator $G_0(z)$ is of the Carleman type and $(2zq_0 + q)\,G_0(z)$ is compact in $L^2(R^3)$ for $\operatorname{Im} z \neq 0$; thus the boundedness of $G(z)$ in $L^2(R^3)$ (which follows from Eq. (4.14)) and Eq. (4.16) implies that $G(z)$ is of the Carleman type for $\operatorname{Im} z \neq 0$ (for more details see [11]). This finishes the proof of the lemma.

Remark 4.7. Let $g \in H^2 \times L^2(R^3)$ and $h \in \mathscr{H}$, then

$$(g, h) = (Lg_1, h_1)_2 + (g_2, h_2)_2. \tag{4.17}$$

Lemma 4.8. *The kernel* $G(x, y; z)$ *of* $G(z)$ *satisfies*

$$|G(x, y; \mu - i0)| \leqq C_1\left(\frac{1}{|x-y|} + C_2\right), \qquad \mu \in \sigma_e(A) - \{\pm m\}. \tag{4.18}$$

For the proof see Ikebe [5].

The second resolvant equation for $G(z)$ has the form

$$G(z) - G(z') = (z - z')\,G(z)\,(z + z' - 2q_0)\,G(z'), \tag{4.19}$$

and by combining Eqs. (4.15), (4.17), (4.18) and (4.19) one gets

$$I(\lambda, \lambda', f) = \lim_{\varepsilon \downarrow 0} \frac{2\varepsilon}{\pi} \int_{\lambda'}^{\lambda} d\mu\, \|G(\mu - i\varepsilon) f_\mu\|_2^2, \tag{4.20}$$

where $f_\mu = Lf_1 + \mu f_2$. The estimate (4.18) is used to show that the last term in Eq. (4.19) does not survive in the limit $\varepsilon \downarrow 0$ (for details see [11]).

Lemma 4.9. *A solution Eq.* (4.8) *is related to the kernel of* $G(z)$ *by the following formula*

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} (|k|^2 + (\pm\omega(k) + i\varepsilon)^2 + m^2) \int dy\, G(x, y; \pm\omega(k) + i\varepsilon)\, e^{iky} \\ &\qquad = u^{\pm}(x, k) \quad \text{on} \quad R^3 \times K. \end{aligned} \tag{4.21}$$

Lemma 4.9 is proved in complete analogy with Lemma 9.2 in Ikebe [5].

Eq. (4.13) can now be verified by inserting Eq. (4.21) into Eq. (4.20), using the Parseval equality and finally employing the following formal identity

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \frac{\varepsilon}{(\omega(k)^2 - \mu^2 + \varepsilon^2)^2 + 4\mu^2\varepsilon^2} = \frac{1}{2\mu}\,\delta(\omega(\mathrm{k})^2 - \mu^2), \tag{4.22}$$

(Remark 4.3 and the fact that f_μ has compact support allows us to freely interchange the order of intergrations, and also to interchange the limit with the integrations.)

It then remains to prove that $F^\pm$ maps onto $L^2(R^3)$ and that Eqs. (4.10) and (4.11) hold. Eq. (4.10) is easily verified by using the isometric character of $F^\pm$ and the definition of the adjoint $(F^\pm f, g)_2 = (f, F^{\pm *} g)$, $f \in \mathscr{H}$, $g \in L^2(R^3)$. Eq. (4.11) is proved as follows: Let $f^\pm = P^\pm f$, with $f \in D(A)$; the spectral representation for A together with Eqs. (4.12) and (4.13) then gives

$$\begin{aligned}(Af^\pm, f^\pm) &= \int \lambda\, d(E(\lambda) f^\pm, f^\pm) = \int dk \pm \omega(k) |F^\pm f(k)|^2 \\ &= \pm (M_{\omega(\cdot)} F^\pm f, F^\pm f)_2 = \pm (F^{\pm *} M_{\omega(\cdot)} F^\pm f, f).\end{aligned} \tag{4.23}$$

The ontoness of $F^\pm$ is verified by showing that $F^{\pm *}$ has a trivial null-space. We follow a similar proof, given in the Schrödinger case, by Alsholm and Schmidt [8].

Lemma 4.10. *Let $g_1, g_2 \in L^2(R^3)$ and let χ_i be characteristic functions corresponding to sets $\{k; \omega(k) \in I_i\}$, where I_1 and I_2 are disjoint intervals on R^+; then*

$$(F^{\pm *} \chi_1 g_1, F^{\pm *} \chi_2 g_2) = 0. \tag{4.24}$$

Proof. See Appendix 2.

Let us assume that $F^{+ *} g = 0$. Lemma 4.10 and Eq. (4.10) then give

$$\int_{\lambda' < \omega(k) < \lambda} dk\, u^+(x, k)\, g(k) = 0, \qquad m < \lambda' < \lambda, \tag{4.25}$$

and

$$\int_{|\Omega| = 1} d\Omega\, u^+(x, |k|\, \Omega)\, g(|k|\, \Omega) = 0 \quad \text{for a.e.} \quad |k| > 0. \tag{4.25}$$

The Lippmann-Schwinger equation (4.8) finally gives

$$\int_{|\Omega| = 1} d\Omega\, e^{i|k|\Omega x} g(|k|\, \Omega) = 0, \quad \text{for a.e.} \quad |k| > 0. \tag{4.26}$$

But this shows that $g = 0$, due to the Fourier inversion theorem. The same conclusion holds of course for $F^{- *}$. The proof of Theorem 4.5 is thus completed.

5. Existence and Completeness of the Wave-Operators $W_\pm$ and an Explicit Representation of the S-Matrix

In scattering theory one is concerned with the asymptotic behaviour of solutions to some evolution equation for large positive and negative times.

We shall in this section study the asymptotic behaviour of solutions to the Klein-Gordon equation (1.1) or, equivalently Eq. (2.4) $i\partial_t \Psi = A\Psi$, with $\Psi = \{u, i\partial_t u\}$. We shall in fact show the existence of two wave-operators $W_\pm$ which have the property that

$$e^{-iAt} W_\pm f \to e^{-iA_0 t} f, \quad t \to \pm\infty, \; f \in \mathscr{H}_0 . \tag{5.1}$$

Our results are collected in the following theorem:

Theorem 5.1. a) *The wave-operators* $W_\pm$ *defined by*

$$W_\pm = \underset{t \to \pm\infty}{s\text{-lim}}\, e^{iAt} e^{-iA_0 t} , \tag{5.2}$$

exist as isometric mappings of $\mathscr{H}_0$ *into* $P\mathscr{H}$ *and they intertwine* A *and* A_0, $AW_\pm \supset W_\pm A_0$.

b) *One has*

$$P^\pm W_- = F^{\pm *} F_0^\pm , \qquad W_+ f = \overline{W_- \bar{f}}, \tag{5.3}$$

i.e. $W_\pm$ *maps onto* $P\mathscr{H}$.

c) *The scattering operator* S *defined by*

$$S = W_+^* W_- P_0^+ + W_-^* W_+ P_0^- , \tag{5.4}$$

is unitary in $\mathscr{H}_0$ *and commutes with* A_0.

The following representation for $\hat{S}^\pm = F_0^\pm S F_0^{\pm *}$ *holds*

$$\hat{S}^\pm g(|k|\,\Omega) = g(|k|\,\Omega) - \pi i |k| \int d\Omega'\, T^\pm(|k|, \Omega, \Omega')\, g(|k|\,\Omega') , \tag{5.5}$$

where

$$T^\pm(|k|, \Omega, \Omega') = \frac{1}{(2\pi)^3} \int dx\, e^{-i|k|\Omega' x} (\pm 2\omega(k)\, q_0(x) + q(x))\, u_\pm(x, |k|\,\Omega), \tag{5.6}$$

with $u_+(x,k) = u^+(x,k)$ *and* $u_-(x,k) = \overline{u^-(x,-k)}$.

Remark 5.2. $T^\pm(|k|, \Omega, \Omega')$ is the phase factor appearing in the asymptotic expansion of $u_\pm(x,k)$ for large $|x|$,

$$u_\pm(x,k) = e^{ikx} + 2\pi^2 T^\pm\left(|k|, \frac{x}{|x|}, \frac{k}{|k|}\right) \frac{e^{i|k|\,|x|}}{|x|} + \mathcal{O}\left(\frac{1}{x}\right). \tag{5.7}$$

The proof of Theorem 5.1 will be divided into three parts. First we show the existence of $W_\pm$ as defined in Eq. (5.2), then Eq. (5.3) is established, and finally Eq. (5.6) is derived.

Proof of Theorem 5.1. a) By differentiating and integrating $W(t) = e^{iAt} e^{-iA_0 t}$ we get

$$W(t) f = f + i \int_0^t dt\, e^{iAt} V e^{-iA_0 t} f, \qquad f \in \mathscr{D} \subset D(V), \tag{5.8}$$

and thus

$$\| W(t) f \| \leqq \| f \| + \int_0^t dt\, \| V e^{-iA_0 t} f \| , \tag{5.9}$$

which shows that it is sufficient for the existence of $W_\pm f$ that $\| Ve^{-iA_0t} f \|$ is integrable on $(0, \pm\infty)$.

We have

$$\| V e^{-iA_0 t} f \|^2 = \| q f_1(t) \|_2^2 + \| 2 q_0 f_2(t) \|_2^2 , \tag{5.10}$$

where $f(t) = \{f_1(t), f_2(t)\} = e^{-iA_0 t} f$.

Let Θ_N denote the characteristic function for a sphere of radius N around the origin in R^3.

Lemma 5.2. *The following estimate holds*

$$\| \Theta_N f_i(t) \|_\infty \leqq \text{const} |t|^{-3/2} , \quad |t| \geqq 4N , \quad i = 1, 2 , \tag{5.11}$$

where the constant is independent of N.

For the proof see [11], Lemma 8.1, or [12] where a similar result is proved in the Dirac case.

We are now in the position to estimate the right-hand side of Eq. (5.10). The first term is estimated as follows $(t = 4N)$;

$$\begin{aligned} \| q f_1(t) \|_2 &\leqq \| (1 - \Theta_N) q f_1(t) \|_2 + \| \Theta_N q f_1(t) \|_2 \\ &\leqq \| (1 - \Theta_N) q \|_\infty \, \| f_1(t) \|_2 + \| q \|_2 \, \| \Theta_N f_1(t) \|_\infty \\ &= \mathcal{O}(|t|^{-3/2}) , \end{aligned} \tag{5.12}$$

where we have used the fact that $q(x) = \mathcal{O}(|x|^{-3-\varepsilon})$, $|x| \to \infty$, and Lemma 5.2. We get a similar result for $\| q_0 f_1(t) \|_2$.

This shows that $\| V e^{iA_0 t} f \|$ is integrable and thus $W_\pm$ exists on $\mathcal{D}$. One can then easily verify that $\| W_\pm f \| = \| f \|_0$ by observing that $\| f \|^2 = \| f \|_0^2 + (q f_1, f_1)_2$ and using an estimate similar to (5.12).

The domain of $W_\pm$ can now be extended by continuity to the whole of $\mathcal{H}_0$ and we furthermore have $P W_\pm = W_\pm$ and $A W_\pm \supset W_\pm A_0$ (see Kato [13], p. 346). This completes the proof of a) in Theorem 5.1.

b) In this part of the proof we use a different but equivalent definition of $W_\pm$.

Lemma 5.3. *One has*

$$W_\pm = W_\pm(J) = s - \lim_{t \to \pm\infty} e^{iAt} J e^{-iA_0 t} , \tag{5.13}$$

where J is defined by $(J f, g) = (f, g)_0$, $f, g \in \mathcal{H}$.

Proof. See Appendix 3.

We now show that $P^\pm W_- = F^{\pm *} F_0^\pm$. Put $W_J(t) = e^{iAt} J e^{-iA_0 t}$, $B(t) = F_0^{+*} F^+ W_J(t)$, and choose f and g such that $f \in \mathcal{D}$ and $\hat{g}_0^+ \in C_0^\infty(R^3 - \{0\})$ [one can easily verify that this means that $g \in D(A)$].

Let us consider $(B(t)f, g)_0$. By differentiating and integrating it we get

$$(B(t)f, g)_0 = (F_0^{+*}F^+ Jf, g)_0 + i\int_0^t dt (F_0^{+*}F^+ e^{iAt}V' e^{-iA_0t}f, g)_0 , \quad (5.14)$$

where $V' = AJ - JA_0$. The integrand can be rearranged as follows (we put $F = F^+$ and $F_0 = F_0^+$),

$$\begin{aligned}(F_0^* F e^{iAt} V' e^{-iA_0t} f, g)_0 &= (V' e^{-iA_0t} f, e^{-iAt} F^* F_0 g)\\ &= (e^{-iA_0t} f, V e^{-iAt} F^* F_0 g)_0 = (e^{-iA_0t} f, V F^* e^{-i\omega(\cdot)t} F_0 g)_0\\ &= (e^{-iA_0t} f, V \int dk\, \Phi^+(\cdot, k)\, e^{-i\omega(k)t} \hat{g}_0^+(k))_0\\ &= \int dk (f, e^{i(A_0 - \omega(k))t} V \Phi^+(\cdot, k))_0 \overline{\hat{g}_0^+(k)},\end{aligned} \quad (5.15)$$

where we have used the definition of J and interchanged the order of integrations, which is allowed due to the absolute convergence of the integrals $(H^+(\cdot, k) = V\Phi^+(\cdot, k) \in \mathscr{H}_0)$.

Equations (5.2) and (5.13) ensures the existence of the limit as $t \to -\infty$ of Eq. (5.14); thus we are allowed to take the Abelian limit and get

$$\begin{aligned}(F_0^{+*}F^+ W_- f, g)_0 = &\int dk (f, \Phi^+(\cdot, k))_0 \overline{\hat{g}_0^+(k)}\\ &+ \lim_{\varepsilon \downarrow 0} \int dk (f, R_0(\omega(k) + i\varepsilon) H^+(\cdot, k))_0 \overline{\hat{g}_0^+(k)}.\end{aligned} \quad (5.16)$$

The Lippmann-Schwinger Equation (4.8) can be written

$$\Phi^\pm(\cdot, k) = \Phi_0^\pm(\cdot, k) - \left(R_0(\pm\omega(k) + i0)\, V\right) \Phi^\pm(\cdot, k), \quad (5.17)$$

where $\Phi_0^\pm(\cdot, k) \in L^\infty(R^3) \times L^\infty(R^3) \equiv (L^\infty)^2$ and $\left(R_0(\pm\omega(k) + i0)\, V\right)$ is a compact integral operator on $(L^\infty)^2$ (compare [5], p. 14).

By inserting Eq. (5.17) into Eq. (5.16) we finally get

$$(F_0^{+*}F^+ W_- f, g)_0 = \int dk (f, \Phi_0^+(\cdot, k))_0 \overline{\hat{g}_0^+(k)} = (P_0^+ f, g)_0 , \quad (5.18)$$

which implies that

$$F_0^{+*}F^+ W_- = P_0^+ , \quad (5.19)$$

and thus

$$P^+ W_- = F^{+*} F_0^+ . \quad (5.20)$$

In a similar way one shows that $P^- W_- = F^{-*} F_0^-$ and it is furthermore simple to verify that $W_+ f = \overline{W_- \bar{f}}$. This completes the proof of part b) of Theorem 5.1.

c) Let $f \in D(A_0) \subset D(V)$; we then get

$$W_+ f - W_- f = i \int_{-\infty}^{\infty} dt\, e^{iAt} V e^{-iA_0t} f , \quad (5.21)$$

and by applying W_+^* we get

$$W_+^* W_- f = f - i \int_{-\infty}^{\infty} dt\, e^{iA_0t} W_+ V e^{-iA_0t} f . \quad (5.22)$$

Let $f, g \in \mathscr{H}_0^+$ be such that $\hat{f}_0^+, \hat{g}_0^+ \in C_0^\infty(R^3 - \{0\})$, $(f, g \in D(A_0))$. By inserting Eq. (5.22) into Eq. (5.4) we obtain

$$\begin{aligned}(Sf, g)_0 &= (f, g)_0 - i \int_{-\infty}^{\infty} dt (e^{iA_0 t} W_+ V e^{-iA_0 t} f, g)_0 \\ &= (f, g)_0 - i \int_{-\infty}^{\infty} dt (V e^{-iA_0 t} f, W_- e^{-iA_0 t} g)\,.\end{aligned} \tag{5.23}$$

In complete analogy with the Schrödinger case (see Ikebe [7]) we get

$$\begin{aligned}(Sf, g)_0 = (f, g)_0 &- 2\pi i \int dk dk' \delta(\omega(k') - \omega(k)) \\ &\cdot (V\Phi_0^+(\cdot, k), \Phi^+(\cdot, -k'))\, \hat{f}_0^+(k)\, \overline{\hat{g}_0^+(k')}\,,\end{aligned} \tag{5.24}$$

when $f, g \in \mathscr{H}_0^+$. In the case when $f, g \in \mathscr{H}_0^-$ we obtain similarily

$$\begin{aligned}(Sf, g)_0 = (f, g)_0 &- 2\pi i \int dk dk' \delta(\omega(k') - \omega(k)) \\ &\cdot (V\Phi_0^-(\cdot, k), \Phi^-(\cdot, k'))\, \hat{f}_0^-(k)\, \overline{\hat{g}_0^-(k')}\,.\end{aligned} \tag{5.25}$$

Equation (5.5) follows directly from Eqs. (5.24) and (5.25), which completes the proof of Theorem 5.1.

6. Summary and Conclusions

We have in this paper developed the spectral and scattering theory for the Klein-Gordon equation with two static external potentials (coupled like a zeroth component of a four-vector and a scalar, respectively)

$$(\Box + 2iq_0(x)\,\partial_t - q_0(x)^2 + q_s(x) + m^2)\, u(x, t) = 0\,.$$

Our main results, which are contained in Theorems 4.5 and 5.1 are similar to those obtained in the Schrödinger case (see [8]). The main difference is that we have to impose a limit of the strength of q_0 and on the negative part of q_s (see condition iv), p. 244).

Theorem 4.5 can be employed to explicitly diagonalize the quantum-field-theoretic Hamiltonian for a charged scalar field interacting with two kinds of external static fields $q_0(x)$ and $q_s(x)$. Furthermore, Theorem 5.1 can be used to prove that the S-matrix, defined in the LSZ-sense, coincides with the classical S-matrix [Eq. (5.5)] and that asymptotic completeness holds (under our conditions on the potentials). All this will be considered elsewhere.

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Appendix 1

Theorem (Thoe [1], p. 373): *Let $\mathscr{V}$ be a complex linear vector space which forms a Hilbert space with respect to each of two scalar products $(\,,)$ and $(\,,)_0$. Suppose T_0 and V are closed linear operators on the Hilbert space $\mathscr{H}_0 = \{\mathscr{V}, (\,,)_0\}$, possessing the following properties:*

i) *T_0 is self-adjoint;*

ii) *$D(V) \supset D(T_0)$;*

iii) *$\|Vf\|_0 \leqq a\|T_0 f\|_0 + b\|f\|_0$, for $f \in D(T_0)$ and $0 < a < 1$;*

iv) *The operator $T = T_0 + V$ with domain $D(T) = D(T_0)$ is symmetric in the Hilbert space $\mathscr{H} = \{\mathscr{V}, (\,,)\}$.*

Then T is self-adjoint in $\mathscr{H}$.

Appendix 2

Proof of Lemma 4.10. Let us start with the following lemma;

Lemma A 2. *Let h_1 and h_2 be two functions in $C_0^\infty(R^3)$, such that the compact sets S_1 and S_2, defined by*

$$S_i = \{t \in R; \exists\, k \in \operatorname{supp} h_i \quad \text{with} \quad \omega(k) = t\},$$

are disjoint and contained in $R^+ - \{0\}$. Then

$$(F^{\pm *} h_1, F^{\pm *} h_2) = 0. \qquad \text{(A 2.1)}$$

We observe that Lemma 4.10 follows from Lemma A 2, since $\chi_1 f_1$ and $\chi_2 f_2$ can be approximated in $L^2(R^3)$ by sequences of C_0^∞ functions having their support in $\{k; \omega(k) \in I_1\}$ and $\{k; \omega(k) \in I_2\}$, respectively.

Proof of Lemma A2. We start by showing that if $g \in C_0^\infty(R^3)$ has support in $R^3 - \{0\}$, then $F^{\pm *} g \in D(A^n)$ for each integer n, and

$$p(A)\, F^{\pm *} g = F^{\pm *}(p(\pm\omega(\cdot))\, g), \qquad \text{(A 2.2)}$$

for each real polynomial p. To this end we consider

$$F^{\pm *} g = \int dk\, \Phi^{\pm}(\cdot, k)\, g(k).$$

It is easy [using Fubini's Theorem and our knowledge of $\Phi^{\pm}(x, k)$] to see that

$$A F^{\pm *} g = \int dk\, \Phi^{\pm}(\cdot, k)\, (\pm\omega(k))\, g(k) = F^{\pm *}(\pm\omega(\cdot)\, g), \qquad \text{(A 2.3)}$$

weakly. Explicitly this means that

$$(Af, F^{\pm *} g) = (f, F^{\pm *}(\pm\omega(\cdot)\, g)), \qquad \text{(A 2.4)}$$

for any $f \in \mathscr{D}$ and thus for any $f \in D(A)$. Since A is self-adjoint, it follows that $F^{\pm *} g$ lies in $D(A)$ and (A 2.3) holds. By iterating this fact we get (A 2.2).

Equation (A 2.1) is now easily established. In fact, the self-adjointness of $p(A)$ gives

$$(p(A)\,F^{\pm *}h_1, F^{\pm *}h_2) = (F^{\pm *}h_1, p(A)\,F^{\pm *}h_2)\,, \qquad \text{(A 2.5)}$$

and thus, due to (A 3.4) we get

$$(F^{\pm *}(p(\pm\omega(\cdot))\,h_1), F^{\pm *}h_2) = (F^{\pm *}h_1, F^{\pm *}(p(\pm\omega(\cdot))\,h_2))\,. \qquad \text{(A 2.6)}$$

We can now choose a function $\psi(t)$ in $C_0^\infty(R)$ such that $\psi(t)=1$ for $t\in S_1$ and $\psi(t)=0$ for $t\in S_2$. We can approximate ψ uniformly on $S_1\cup S_2$ by a sequence $\{p_n\}$ of polynomials and obtain (A 2.1) in the limit.

Appendix 3

Proof of Lemma 5.3. Let us start by observing that $Jf=\{L^{-1}L_0 f_1, f_2\}$ when $f\in\mathscr{D}$. Let $f\in\mathscr{D}$; we then get

$$\begin{aligned}\|e^{iAt}(J-1)\,e^{-iA_0t}f\| &= \|(J-1)\,e^{-iA_0t}f\| \qquad \text{(A 3.1)}\\ &= \|C^{-1}(C_0-C)\,e^{-iA_0t}f\| = \|C^{-1}Q\,e^{-iA_0t}f\|\,,\end{aligned}$$

where $C=\begin{pmatrix}L & 0\\ 0 & 1\end{pmatrix}$, $C_0=\begin{pmatrix}L_0 & 0\\ 0 & 1\end{pmatrix}$, and $Q=\begin{pmatrix}-q & 0\\ 0 & 0\end{pmatrix}$. Lemma 4.7 and (A 3.1) finally gives

$$\begin{aligned}\|C^{-1}Qe^{-iA_0t}f\|^2 &= (Qe^{-iA_0t}f, C^{-1}Qe^{-iA_0t}f)_{L^2\oplus L^2}\\ &\leqq \|C^{-1}\|_{L^2\oplus L^2}\,\|Qe^{-iA_0t}f\|^2_{L^2\oplus L^2}\\ &= \text{const}\,\|qf_1(t)\|_2^2\to 0,\ |t|\to\infty\,,\end{aligned}$$

where, in the last step, we have used Eq. (5.12).

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