Hilbert scales, Approximation Theory, Non Linear Problem
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The Eigenvalue problem for compact symmetric operators

In the following $H$ denotes an (infinite dimensional) real Hilbert space with scalar product $(..)$ and the norm $\|\cdot\|$. We will consider mappings $K: H \to H$. Unless otherwise noticed the standard assumptions on $K$ are:

i) $K$ is symmetric, i.e. for all $x, y \in H$ it holds $(x, Ky) = (x, y)$

ii) $K$ is compact, i.e. for any (infinite) sequence $\{x_n\}$ bounded in $H$ contains a subsequence $\{x_{n_k}\}$ such that $\{Kx_{n_k}\}$ is convergent,

iii) $K$ is injective, i.e. $Kx = 0$ implies $x = 0$.

A first consequence is

**Lemma:** $K$ is bounded, i.e.

$$\|K\| = \sup_{\|x\|=1} \|Kx\|.$$ 

**Lemma:** Let $K$ be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{\|x\|=1} \frac{\|x, Ky\|}{\|x\|}.$$ 

**Theorem:** There exists a countable sequence $\{\lambda_i, \varphi_i\}$ of eigenelements and eigenvalues $K\varphi_i = \lambda_i \varphi_i$ with the properties

i) the eigenelements are pair-wise orthogonal, i.e. $(\varphi_i, \varphi_j) = \delta_{i,j}$

ii) the eigenvalues tend to zero, i.e. $\lim_{i \to \infty} \lambda_i$

iii) the generalized Fourier sums $S_n := \sum_{i=1}^n (x, \varphi_i) \varphi_i \to x$ with $n \to \infty$ for all $x \in H$

iv) the Parseval equation

$$\|x\|^2 = \sum_{i=1}^\infty (x, \varphi_i)^2$$

holds for all $x \in H$. 

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Let $H$ be a (infinite dimensional) Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, the norm $\| \cdot \|$ and $A$ be a linear operator with the properties

i) $A$ is self-adjoint, positive definite

ii) $A^{-1}$ is compact.

Without loss of generality, possible by multiplying $A$ with a constant, we may assume

$$\langle x, Ax \rangle \geq \|x\|^2 \quad \text{for all } x \in D(A)$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigenelement of $K$ is also an eigenelement of $A$ to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \rightarrow \lambda_i^{-1}$ we have from the previous section

i) there is a countable sequence $\{i, \varphi_i\}$ with

$$A \varphi_i = \lambda_i \varphi_i \quad \text{and} \quad \lim_{i \to \infty} \lambda_i,$$

ii) any $x \in H$ is represented by

$$x = \sum_{i=1}^{\infty} \langle x, \varphi_i \rangle \varphi_i \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^{\infty} \langle x, \varphi_i \rangle^2.$$

**Lemma:** Let $x \in D(A)$, then

$$(**) \quad Ax = \sum_{i=1}^{\infty} \lambda_i \langle x, \varphi_i \rangle \varphi_i \quad \text{and} \quad \|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 \langle x, \varphi \rangle^2,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i \langle x, \varphi \rangle \langle y, \varphi \rangle.$$

Because of (*) there is a one-to-one mapping $\iota$ of $H$ to the space $\hat{H}$ of infinite sequences of real numbers

$$\hat{H} = \{i \hat{x} = (x_1, x_2, \ldots)\}$$

defined by

$$\hat{x} = \iota x \quad \text{with} \quad x_i = \langle x, \varphi_i \rangle.$$

If we equip $\hat{H}$ with the norm

$$\|\hat{x}\| = \sum_{i=1}^{\infty} \langle x, \varphi_i \rangle^2,$$

then $\iota$ is an isometry.
By looking at (**) it is reasonable to introduce for non-negative $\alpha$ the weighted inner products

$$(x, y)_\alpha = \sum_i \alpha^n_i (x, \varphi_i)(y, \varphi_i) = \sum_i \alpha^n_i x_i y_i,$$

and the norms

$$\|x\|_\alpha = (x, x)_\alpha.$$

Let $\hat{H}_\alpha$ denote the set of all sequences with finite $\alpha-$ norm. then $\hat{H}_\alpha$ is a Hilbert space. The proof is the same as the standard one for the space $l_2$.

Similarly one can define the spaces $H_\alpha$: they consist of those elements $x \in H$ such that $Ix \in \hat{H}_\alpha$ with scalar product

$$(x, y)_\alpha = \sum_i \alpha^n_i (x, \varphi_i)(y, \varphi_i) = \sum_i \alpha^n_i x_i y_i,$$

and norm

$$\|x\|_\alpha = (x, x)_\alpha.$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (**) it holds

$$\|x\|_2 = (Ax, Ax)_0 = H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values $\lambda_i$ tend to infinity we would have for $\alpha < 0$: $\lim \lambda^n_i \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, ...)$ with

$$\|\hat{x}\|_2 < \infty \quad |\hat{x}|_\alpha = \infty.$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.
There are certain relations between the spaces \( \{ H_\alpha | \alpha \geq 0 \} \) for different indices:

**Lemma:** Let \( \alpha < \beta \). Then

\[
\| x \|_\alpha \leq \| x \|_\beta
\]

and the embedding \( H_\beta \to H_\alpha \) is compact.

**Lemma:** Let \( \alpha < \beta < \gamma \). Then

\[
\| x \|_\beta \leq \| x \|_\gamma, \text{ for } x \in H_\gamma
\]

with \( \mu = \frac{\gamma - \beta}{\gamma - \alpha} \) and \( \nu = \frac{\beta - \alpha}{\gamma - \alpha} \).

**Lemma:** Let \( \alpha < \beta < \gamma \). To any \( x \in H_\beta \) and \( t > 0 \) there is a \( y = y_t(x) \) according to

i) \( \| x - y \|_\alpha \leq t^{\beta - \alpha} \| x \|_\beta \)

ii) \( \| x - y \|_\beta \leq \| x \|_\beta \), \( \| y \|_\beta \leq \| x \|_\beta \)

iii) \( \| y \|_\gamma \leq t^{(\gamma - \beta)} \| x \|_\beta \)

**Corollary:** Let \( \alpha < \beta < \gamma \). To any \( x \in H_\beta \) and \( t > 0 \) there is a \( y = y_t(x) \) according to

i) \( \| x - y \|_\alpha \leq t^{\beta - \alpha} \| x \|_\beta \) for \( \alpha \leq \rho \leq \beta \)

ii) \( \| y \|_\alpha \leq t^{(\gamma - \rho)} \| x \|_\beta \) for \( \beta \leq \sigma \leq \gamma \).

**Remark:** Our construction of the Hilbert scale is based on the operator \( A \) with the two properties i) and ii). The domain \( D(A) \) of \( A \) equipped with the norm

\[
|Ax|^2 = \sum_{i=1}^n \lambda_i^2 (x, \varphi_i)^2
\]

turned out to be the space \( H_2 \) which is densely and compactly embedded in \( H = H_0 \). It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator \( A \) with the properties i) and ii) such that

\[
D(A) = H_2, \quad R(A) = H_0, \quad \text{and} \quad \| x \|_2 = \| Ax \|.
\]
We give three examples of differential operator and singular integral operators, whereby the integral operators are related to each other by partial integration:

**Example 1:** Let $H = L^2(0,1)$ and

$$Au \equiv -u''$$

with

$$D(A) = W_2^2(0,1) = W_2^1(0,1) \cap W_z^2(0,1).$$

Building on the orthogonal set of eigenpairs $\{\lambda_i, \phi_i\}$ of $A_1$, i.e.

$$-\phi_i'' = \lambda_i \phi_i, \quad \phi_i(0) = \phi_i(1) = 0$$

it holds the inclusion

$$D(A) \subseteq H_1 = W_2^1(0,1) \subseteq L^2(0,1).$$

**Example 2:** Let $H = \mathcal{L}_{\nu}(\Gamma)$ with $\Gamma = S^1(R^2)$, i.e. $\Gamma$ is the boundary of the unit sphere. Then $H$ is the space of integrable periodic function in $R$. Let

$$ (Au)(x) \equiv -\int \log \frac{x - y}{2} u(y)dy = \int k(x - y)u(y)dy $$

and

$$D(A) = H = \mathcal{L}_{\nu}(\Gamma).$$

The Fourier coefficients of this convolution are

$$ (Au)_k = k \mu_k = \frac{1}{2|\nu|} u_k $$

i.e. it holds

$$D(A) \subseteq H_\nu = H_{\nu/2}(\Gamma).$$

A relation of this Fourier representation to the fractional function is given by

$$ x - [x] - \frac{1}{2} = -\sum \sin \frac{2\pi nx}{\pi \nu} $$
Remark: We give some further background and analysis of the even function

\[ k(x) = -\log\left|\frac{2\sin\frac{x}{2}}{2}\right| = -\log\left|\frac{2\sin\frac{x}{2}}{2}\right| \]

Consider the model problem

\[ -\Delta U = 0 \quad \text{in } \Omega \]
\[ U = f \quad \text{on } \Gamma = \partial \Omega , \]

whereby the area \( \Omega \) is simply connected with sufficiently smooth boundary. Let \( y = y(s) - s \in (0,1] \) be a parametrization of the boundary \( \partial \Omega \). Then for fixed \( \varpi \) the functions

\[ U(\varpi) = -\log|\varpi - \varpi| \]

Are solutions of the Laplace equation and for any \( L_2(\partial \Omega) \) - integrable function \( u = u(t) \) the function

\[ (Au)(\varpi) = \int_{\partial \Omega} \log|\varpi - u(t)| \, dt \]

is a solution of the model problem. In an appropriate Hilbert space \( H \) this defines an integral operator, which is coercive for certain areas \( \Omega \) and which fulfills the Garding inequality for general areas \( \Omega \). We give the Fourier coefficient analysis in case of \( H = L_2(\Gamma) \) with \( \Gamma = S^1(R) \), i.e. \( \Gamma \) is the boundary of the unit sphere. Let \( x(s) := (\cos(s), \sin(s)) \) be a parametrization of \( \Gamma = S^1(R) \) then it holds

\[ |x(s) - x(t)|^2 = \left[ \left( \frac{\cos(s) - \cos(t)}{\sin(s) - \sin(t)} \right) \right]^2 \]
\[ = 2^2 - 2 \cos(s - t) = 2(1 - \cos(2\frac{s - t}{2})) = 2 \left[ 2 \sin^2 \frac{s - t}{2} \right] = 4 \sin^2 \frac{s - t}{2} \]

and therefore

\[ -\log|x(s) - x(t)| = -\log\left|\frac{\sin\frac{s - t}{2}}{2}\right| = k(s - t) \cdot \]

The Fourier coefficients \( k_\nu \) of the kernel \( k(x) \) are calculated as follows

\[ k_\nu := \frac{1}{2\pi} \int_{0}^{2\pi} k(x)e^{-\nu x} \, dx = \frac{1}{2\pi} \int_{0}^{2\pi} \log\left|\frac{2\sin\frac{x}{2}}{2}\right| e^{-\nu x} \, dx = \frac{2}{2\pi} \int_{0}^{\frac{\pi}{2}} \log\left|\frac{2\sin\frac{x}{2}}{2}\right| e^{\nu \cos(x)} \, dx = k_{\nu} \]

As \( \nu \log(2\sin \frac{x}{2}) \to 0 \) partial integration leads to

\[ k_\nu = \frac{1}{\nu \pi} \int_{0}^{\nu \pi} \frac{2\sin(\nu \pi) \cos\frac{t}{2}}{2 \sin\frac{t}{2}} \, dt = -\frac{1}{\nu \pi} \int_{0}^{\nu \pi} \frac{2\nu t + 2(\nu - 1)}{2 \sin\frac{t}{2}} \, dt - \frac{1}{\nu \pi} \int_{0}^{\nu \pi} \frac{2\nu t - 2}{2 \sin\frac{t}{2}} \, dt \]

\[ k_\nu = -\frac{1}{\nu \pi} \int_{0}^{\nu \pi} \left[ \frac{1}{2} + \cos(t) \right] dt = -\frac{1}{\nu} \cdot \]

As \( \nu \log(2\sin \frac{x}{2}) \to 0 \) partial integration leads to

\[ k_\nu = -\frac{1}{\nu \pi} \int_{0}^{\nu \pi} \left[ \frac{1}{2} + \cos(t) \right] dt = -\frac{1}{\nu} \cdot \]
Extension and generalizations

For \( t > 0 \) we introduce an additional inner product resp. norm by

\[
(x, y)^{(t)} = \sum_{i=1}^{n} e^{-\sqrt{x_i}} (x, \varphi_i)(y, \varphi_i)
\]

\[
\| x \|^{(t)} = (x, x)^{(t)}.
\]

Now the factor have exponential decay \( e^{-\sqrt{x_i}} \) instead of a polynomial decay in case of \( \lambda_i^{\alpha} \). Obviously we have

\[
\| x \|^{(t)} \leq c(\alpha, t) \| x \|_a \quad \text{for } x \in H_a
\]

with \( c(\alpha, t) \) depending only from \( \alpha \) and \( t > 0 \). Thus the \((t) - \text{norm}\) is weaker than any \( \alpha - \text{norm}\). On the other hand any negative norm, i.e. \( \| x \|_a \) with \( \alpha < 0 \), is bounded by the \( 0 - \text{norm}\) and the newly introduced \((t) - \text{norm}\). It holds:

**Lemma:** Let \( \alpha > 0 \) be fixed. The \( \alpha - \text{norm}\) of any \( x \in H_a \) is bounded by

\[
\| x \|_a^2 \leq \delta^{2\alpha} \| x \|_0^2 + e^{\delta \alpha} \| x \|_o^2,
\]

with \( \delta > 0 \) being arbitrary.

**Remark:** This inequality is in a certain sense the counterpart of the logarithmic convexity of the \( \alpha - \text{norm}\), which can be reformulated in the form \((\mu, \nu > 0, \mu + \nu > 1)\)

\[
\| x \|_a^2 \leq \nu e^{\mu} \| x \|_0^2 + \mu e^{-\nu} \| x \|_a^2
\]

applying Young’s inequality to

\[
\| x \|_a^2 \leq (\| x \|_0^2)^{\mu} (\| x \|_a^2)^{\nu}.
\]

The counterpart of lemma 4 above is

**Lemma:** Let \( t, \delta > 0 \) be fixed. To any \( x \in H_a \) there is a \( y = y_t(x) \) according to

\( i) \) \( \| x - y \| \leq \| x \| \)

\( ii) \) \( \| y \| \leq \delta^{-t} \| x \| \)

\( iii) \) \( \| x - y \|_o \leq e^{-t/\delta} \| x \| \).
Eigenfunctions and Eigendifferentials

Let $H$ be a (infinite dimensional) Hilbert space with inner product $(..)$, the norm $\|..\|$ and $A$ be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that $A^{-1}$ compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of $H$ in the following way:

$$R \rightarrow L(H, H)$$

$$\lambda \rightarrow E_\lambda = \int \phi_\lambda (\varphi_\mu, \varphi_\mu^*) d\mu \ , \ \mu \in [0, \infty),$$

i.e.

$$dE_\lambda = \varphi_\lambda (\varphi_\mu, \varphi_\mu^*) d\lambda \ .$$

The spectrum $\sigma(A) \subset C$ of the operator $A$ is the support of the spectral measure $dE_\lambda$.

The set $E_\lambda$ fulfills the following properties:

i) $E_\lambda$ is a projection operator for all $\lambda \in R$

ii) for $\lambda \leq \mu$ it follows $E_\lambda \leq E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

iii) $\lim_{\lambda \rightarrow \infty} E_\lambda = 0$ and $\lim_{\lambda \rightarrow 0} E_\lambda = Id$

iv) $\lim_{\mu \rightarrow \lambda} \mu E_\mu = E_\lambda$.

**Proposition:** Let $E_\lambda$ be a set of projection operators with the properties i)-iv) having a compact support $[a, b]$. Let $f : [a, b] \rightarrow R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \rightarrow H$ with

$$(A_f x, y) = \int_a^b f(\lambda) d(E_\lambda x, y) \ .$$

Symbolically one writes

$$A = \int \lambda dE_\lambda \ .$$

Using the abbreviation

$$(E_\lambda x, y) = (E_\lambda, y) \quad , \quad (\lambda E_\lambda x, y) = (\lambda x, y)$$

one gets

$$(Ax, y) = \int \lambda d(E_\lambda x, y) = \int \lambda d\mu_{\lambda y}(\lambda) \ , \ |x|^2 = \int \lambda d(E_\lambda x, x) = \int \lambda d\mu_{\lambda x}(\lambda)$$

$$(A^2 x, y) = \int \lambda^2 d(E_\lambda x, y) = \int \lambda^2 d\mu_{\lambda y}(\lambda) \ , \ \|A^2 x\|^2 = \int \lambda^2 d(E_\lambda x, x) = \int \lambda^2 d\mu_{\lambda x}(\lambda) \ .$$
The function \( \sigma(\lambda) := \|E_{\lambda} x\|^2 \) is called the spectral function of \( A \) for the vector \( x \). It has the properties of a distribution function.

It holds the following eigenpair relations

\[
A \varphi_\lambda = \lambda \varphi_\lambda \quad A \varphi_\mu = \lambda \varphi_\mu \quad \|\varphi_\lambda\|^2 = \infty \quad , \quad (\varphi_\lambda, \varphi_\mu) = \delta(\varphi_\lambda - \varphi_\mu)
\]

The \( \varphi_\lambda \) are not elements of the Hilbert space. The so-called eigendifferentials, which play a key role in quantum mechanics, are built as superposition of such eigenfunctions.

Let \( I \) be the interval covering the continuous spectrum of \( A \). We note the following representations:

\[
x = \sum \langle x, \varphi_i \rangle \varphi_i + \int \varphi_\mu (\varphi_\mu, x) d\mu \quad Ax = \sum \lambda_i \langle x, \varphi_i \rangle \varphi_i + \int \lambda \varphi_\mu (\varphi_\mu, x) d\mu
\]

\[
\|\| = \sum |\langle x, \varphi_i \rangle|^2 + \int |\varphi_\mu (\varphi_\mu, x)|^2 d\mu
\]

\[
\|\| = \sum \lambda_i |\langle x, \varphi_i \rangle|^2 + \int \lambda \varphi_\mu (\varphi_\mu, x)|^2 d\mu
\]

\[
\|\| = |\lambda_i| = \sum \lambda_i^2 |\langle x, \varphi_i \rangle|^2 + \int \lambda \varphi_\mu (\varphi_\mu, x)|^2 d\mu
\]

**Example:** The location operator \( Q \) and the momentum operator \( P \) both have only a continuous spectrum. For positive energies \( \lambda \geq 0 \) the Schrödinger equation

\[
H \varphi_\lambda (x) = \lambda \varphi_\lambda (x)
\]

delivers no element of the Hilbert space \( H \), but linear, bounded functional with an underlying domain \( M \subset H \) which is dense in \( H \). Only if one builds wave packages out of \( \varphi_\lambda (x) \) it results into elements of \( H \). The practical way to find Eigen-differentials is looking for solutions of a distribution equation.
Non Linear Problems

Let the problem be given by

\[ F(x,u) = 0 \]

with the (roughly) regularity assumptions:

i) there is a unique solution

ii) \( F, F_u \) are Lipschitz continuous.

The approximation problem is given by:

\[
\text{find } \varphi \in S_h \quad (F(\cdot, \varphi), \chi) = 0 \quad \text{for } \chi \in S_h.
\]

**Error analysis**

Put

\[
f(x) = F_u(x, u(x)) \quad \text{and} \quad \varphi = u - e
\]

Then

\[
(f e, \chi) = (R, \chi)
\]

with a remainder term

\[
R \geq R(e) \geq F(\cdot, u - e) + fe
\]

resp.

\[
(f e, \chi) = (f u - R(e), \chi).
\]

Let \( P_h \) denote the \( L^2 \)–projection related to \( (f \cdot, \cdot) = (R, \cdot) \), then

\[
\varphi = P_h(u - \frac{1}{f} R(e))
\]

resp.

\[
e = (I - P_h)u + P_h \frac{1}{f} R(e) =: T(e).
\]

Therefore the difference \( e = u - e \) is a fix point of \( T \).

Let

\[
B_{\kappa \varepsilon} := \left\{ \|e\|_{L^2} \leq \kappa \varepsilon \right\} \quad \text{and} \quad \varepsilon := \inf_{\chi \in S_h} \|u - \chi\|_{L^2}.
\]

With that some key properties of \( T \) are summaries in the following

**Lemma:**
i) There is a $\kappa > 0$ such that for $\bar{x}$ sufficiently small, then $T$ maps the ball $B_{\kappa \bar{x}}$ into itself.

ii) for $\bar{x}$ sufficiently small, $T$ is a contradiction in $B_{\kappa \bar{x}}$.

**Proof:** i) Because of $P_h$ and $f^{-1}$ are being bounded it holds

$$
|I - P_h|_{\infty} \leq c_1 \inf_{\chi \in \mathcal{S}_h} \|u - \chi\|_{\infty} = \bar{E}.
$$

and

$$
\left\| P_h \left( \frac{1}{f}(R(e)) \right) \right\|_{\infty} \leq c_1 \|R(e)\|_{\infty}.
$$

It is

$$
|F(\cdot, u - e) + fe\|_{\infty} \leq c_1 \|\|e\|_{\infty} = c_1 c_2 \bar{E}^2
$$

with $c_1$ being the Lipschitz constant of $F_u$. Therefore

$$
|F(e)|_{\infty} \leq c_1 \bar{E} + c_1 c_2 \bar{E}^2.
$$

Now fixing $\kappa > c_1$ and choosing $\bar{E}_0$ according to $\kappa = c_1 + c_1 c_2 \bar{E}_0^2$ gives i)

ii) it holds

$$
|F(e_1) - T(e_2)|_{\infty} = \left| P \left( \frac{1}{f}(R(e)) - R(e) \right) \right|_{\infty} \leq c_1 \|R(e_i) - R(e_i)\|_{\infty}
$$

and

$$
R(e_1) - R(e_2) = F(\cdot, u - e_1) - F(\cdot, u - e_2) = (F_u(\cdot, \partial) - F_u(u))(e_1 - e_2).
$$

With

$$
F_u(\cdot, \partial) = F_u(\cdot, u - \partial e_1 - (1 - \partial)e_2)
$$

one gets

$$
\|F_u(\cdot, \partial) - F_u(\cdot, u)\| \leq \kappa \bar{E}_1.
$$

Choosing

$$
\bar{E} < \text{Min}(\varepsilon_0, \frac{1}{c_1 c_2 \kappa})
$$

then proves ii).

**Consequence:** The operator $T$ has a unique fix-point in the ball $B_{\kappa \bar{x}}$.

From this it follows the

**Theorem:** The FEM admits the error estimate

$$
\|u - \varphi\|_{\infty} \leq c \inf_{\chi \in \mathcal{S}_h} \|u - \chi\|_{\infty}.
$$