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HILBERT TRANSFORM OF $\log|f|$

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ABSTRACT. There are two general ways to evaluate the Hilbert transform of a function of real variable $u(x)$. We can extend u to a harmonic function in the upper half plane by the Poisson integral formula. Non-tangential limit of its harmonic conjugate exists almost everywhere and is defined to be the Hilbert transform of $u(x)$. There is also a singular integral formula for the Hilbert transform of $u(x)$. It is fairly difficult to directly evaluate the Hilbert transform of $u(x)$. In this paper we give an explicit formula for the Hilbert transform of $\log|f|$, where f is a function in the Cartwright class.

1. INTRODUCTION

Suppose $u(t)$ is a real valued function such that $\int_{-\infty}^{\infty} \frac{|u(t)|}{1+t^2}\, dt < \infty$. Then the integral

$$\begin{aligned} U(z) + i\,\widetilde{U}(z) &:= \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t}+\frac{t}{1+t^2}\right)u(t)\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\,u(t)\,dt + i\,\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\Re z - t}{|z-t|^2}+\frac{t}{1+t^2}\right)u(t)\,dt \end{aligned}$$

converges absolutely for $\Im z > 0$ and is an analytic function of z. U is the unique harmonic extension of u to the upper half plane and it is given by a Poisson integral formula. $\widetilde{U}$ is the unique harmonic conjugate of U such that $\widetilde{U}(i) = 0$. It is well known that U and $\widetilde{U}$ have non-tangential limits at almost every $t \in \mathbb{R}$. The non-tangential limit of U is u. The non-tangential limit of $\widetilde{U}$ is denoted by $\tilde{u}$ and is called the Hilbert transform of u. $\tilde{u}$ can also be found by the following singular integral [3]:

$$\tilde{u}(t) = \lim_{\varepsilon\to 0}\int_{|x-t|>\varepsilon}\left(\frac{1}{t-x}+\frac{x}{1+x^2}\right)u(x)\,dx.$$

It is fairly difficult to evaluate this integral for $u(x) = \log|\sin x|$ or for $u(x) = \log|p(x)|$ where $p(x)$ is a polynomial. In this paper we give an explicit formula for $\log|f|$ where f belongs to a large class of functions which contains $\sin x$ and $p(x)$.

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In section 2 we discuss the argument of a meromorphic Blaschke product on the real line. The argument is a C^∞ function and it can be given explicitly by a formula. Then in section 3 the Cartwright class is defined, and an important representation theorem in the upper half plane is derived. Section 4 is about the behavior of these functions on the real line. Finally in section 5 we give an explicit formula for $\widetilde{\log|f|}$.

2. Blaschke products

Let $\{z_n\}$ be a sequence of complex numbers in the upper half plane such that $\lim\limits_{n\to\infty}|z_n| = \infty$ and $\sum_n \frac{\Im z_n}{|z_n|^2} < \infty$. Let $b_n(z) = \frac{1-\frac{z}{z_n}}{1-\frac{z}{\bar z_n}}$ and put $B(z) = \prod_n b_n(z)$. $B(z)$ is a meromorphic function with zeros at $\{z_n\}$ and poles at $\{\bar z_n\}$. At each point $t \in \mathbb{R}$, B is analytic and $|B(t)| = 1$. Therefore, there is a real C^∞ function φ such that $B(t) = e^{i\varphi(t)}$ for every $t \in \mathbb{R}$ [2]. We can find an explicit formula for $\varphi(t)$:

$$b_n(t) = \frac{1-\frac{t}{z_n}}{1-\frac{t}{\bar z_n}} = \frac{(1-\frac{\Re z_n}{|z_n|^2}t) + i(\frac{\Im z_n}{|z_n|^2}t)}{(1-\frac{\Re z_n}{|z_n|^2}t) - i(\frac{\Im z_n}{|z_n|^2}t)} = \frac{e^{i\varphi_{z_n}(t)}}{e^{-i\varphi_{z_n}(t)}} = e^{2i\varphi_{z_n}(t)}.$$

The first candidate for $\varphi_{z_n}(t)$ is $\arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1-\frac{\Re z_n}{|z_n|^2}t}\right)$. If $\Re z_n = 0$, then $\varphi_{z_n}(t) = \arctan\left(\frac{t}{\Im z_n}\right)$ is a well defined C^∞ function on $\mathbb{R}$. But if $\Re z_n \neq 0$, $\arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1-\frac{\Re z_n}{|z_n|^2}t}\right)$ is not continuous at $t = \frac{|z_n|^2}{\Re z_n}$. We should use the following branches of arctan.

If $\Re z_n > 0$, then

$$\varphi_{z_n}(t) = \begin{cases} \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1-\frac{\Re z_n}{|z_n|^2}t}\right) + \pi, & t > \frac{|z_n|^2}{\Re z_n}, \\ \frac{\pi}{2}, & t = \frac{|z_n|^2}{\Re z_n}, \\ \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1-\frac{\Re z_n}{|z_n|^2}t}\right), & t < \frac{|z_n|^2}{\Re z_n}, \end{cases}$$

and if $\Re z_n < 0$

$$\varphi_{z_n}(t) = \begin{cases} \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1-\frac{\Re z_n}{|z_n|^2}t}\right), & t > \frac{|z_n|^2}{\Re z_n}, \\ \frac{-\pi}{2}, & t = \frac{|z_n|^2}{\Re z_n}, \\ \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1-\frac{\Re z_n}{|z_n|^2}t}\right) - \pi, & t < \frac{|z_n|^2}{\Re z_n}. \end{cases}$$

Since $b_n(0) = 1$, we defined φ_{z_n} such that $\varphi_{z_n}(0) = 0$. Therefore

$$B(t) = \prod_n b_n(t) = \prod_n e^{2i\,\varphi_{z_n}(t)} = e^{2i\sum_n \varphi_{z_n}(t)}. \tag{2.1}$$

Hence $\varphi(t) = 2\sum_n \varphi_{z_n}(t)$. To prove φ is continuous suppose $t \in [a,b]$ where a and b are arbitrary real numbers. Since $\frac{|z_n|^2}{|\Re z_n|} \to \infty$, there exists N such that for each $n \geqslant N$ and for each $t \in [a,b]$, $\varphi_{z_n}(t) = \arctan\left(\frac{\frac{\Im z_n}{|z_n|^2}t}{1-\frac{\Re z_n}{|z_n|^2}t}\right)$. Thus $|\varphi_{z_n}(t)| \leqslant c\,\frac{\Im z_n}{|z_n|^2}$ for each $n \geqslant N$. Therefore by the Weierstrass M-test φ is continuous. We can even prove that φ is a C^∞ function, but continuity is enough in what follows.

3. Cartwright class

An entire function $f(z)$ is said to be of exponential type if there are constants A and B such that $|f(z)| \leqslant B\, e^{A\,|z|}$ everywhere. $Cart$ denotes the space of entire functions of exponential type which satisfy the boundedness condition

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2}\, dx < \infty.$$

Let $\{z_n\}$ denote the sequence of the upper half plane zeros of f. Since $\sum_n \frac{\Im z_n}{|z_n|^2} < \infty$ and $\lim_{n\to\infty} |z_n| = \infty$, the Blaschke product formed with this sequence $B_+(z) = \prod_n \frac{1-\frac{z}{z_n}}{1-\frac{z}{\bar{z}_n}}$ is a well defined meromorphic function. Let

$$h_+[f] = \limsup_{y\to+\infty} \frac{\log|f(iy)|}{y},$$

$$h_-[f] = \limsup_{y\to+\infty} \frac{\log|f(-iy)|}{y}.$$

Therefore $O(z) = \frac{e^{-ih_+\, z}\; f(z)}{B_+(z)}$ has removable singularities at zeros of f. It is at least an entire function free of zeros in the upper half plane. But since $f \in Cart$, then $O(z) \in Cart$ and for $\Im z > 0$, $O(z) = c \exp\left(\frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{1+t^2}\right) \log|f(t)|\, dt\right)$. c is a constant of absolute value one [1]. Therefore for $\Im z > 0$

(3.1)
$$f(z) = c\, e^{-ih_+\, z}\, B_+(z)\, \exp\left(\frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{1+t^2}\right) \log|f(t)|\, dt\right).$$

B_+ is analytic at each point of $\mathbb{R}$ and by (2.1) for every $t \in \mathbb{R}$

$$B_+(t) = e^{2i\sum_n \varphi_{z_n}(t)}. \tag{3.2}$$

4. Cartwright functions on the real line

Let $f \in Cart$. Since $f \in Cart$, by definition, $\log^+|f| \in L^1(\frac{dt}{1+t^2})$. If $f \neq 0$, by a deep theorem in function theory [1, 3], $\log^-|f| \in L^1(\frac{dt}{1+t^2})$. Therefore for almost every $t \in \mathbb{R}$ the Hilbert transform $\widetilde{\log|f|}(t)$ exists. Let α and β be two consecutive real zeros of f. Consider an arbitrary closed interval $[a,b] \subset (\alpha,\beta)$. On $[a,b]$, $\log|f(t)|$ is Lipschitz. Thus $\widetilde{\log|f|}(t)$ is at least continuous on $[a,b]$ [4]. Therefore, $\widetilde{\log|f|}(t)$ is continuous on (α,β). Hence it exists at every $t \in \mathbb{R}$ except at real zeros of f.

Theorem 4.1. *Let $f \in Cart$ and let $\{x_n\}$ denote the sequence of real zeros of f. Then for every $t \in R \setminus \{x_n\}$*

$$\frac{f(t)}{|f(t)|} = c\; B_+(t)\; e^{-ih_+\, t}\; e^{i\,\widetilde{\log|f|}(t)}$$

where c is a constant of modulus one.

Proof. By (3.1), in the upper half plane $\Im z > 0$

$$\begin{aligned} f(z) &= c\, e^{-ih_+ z}\, B_+(z)\, \exp\left(\frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{1+t^2} \right) \log|f(t)|\, dt \right) \\ &= c\, e^{-ih_+ z}\, B_+(z)\, \exp\left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, \log|f(t)|\, dt \right. \\ &\qquad\qquad \left. + i\, \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\Re z - t}{|z-t|^2} + \frac{t}{1+t^2} \right) \log|f(t)|\, dt \right). \end{aligned}$$

Take the non-tangential limit of both sides as $z \to t$. Thus, for every $t \in R \setminus \{x_n\}$

$$f(t) = c\, e^{-ih_+ t}\, B_+(t)\, \exp\big(\log|f(t)| + i\, \widetilde{\log|f|}(t) \big) = c\, e^{-ih_+ t}\, B_+(t)\, |f(t)|\, e^{i\, \widetilde{\log|f|}(t)}.$$

□

Let $f^*(z) := \bar{f}(\bar{z})$. Thus

$$h_+[f^*] = \limsup_{y \to +\infty} \frac{\log|f^*(iy)|}{y} = \limsup_{y \to +\infty} \frac{\log|f(-iy)|}{y} = h_-[f], \tag{4.1}$$

$$f^*(t) = \bar{f}(\bar{t}) = \bar{f}(t), \qquad \log|f^*(t)| = \log|f(t)|. \tag{4.2}$$

Since $f \in Cart$ by (4.2), $f^* \in Cart$. The upper half plane zeros of f^* are conjugates of the lower half plane zeros of f, say $(\bar{w}_n)$. Put $B_-(z) = \prod_n \frac{1 - \frac{z}{\bar{w}_n}}{1 - \frac{z}{w_n}}$. B_- is analytic at each point of real line and by equation (2.1) for every $t \in \mathbb{R}$

$$B_-(t) = e^{2i \sum_n \varphi_{\bar{w}_n}(t)}. \tag{4.3}$$

Corollary 4.2. *Let $f \in Cart$ and let $\{x_n\}$ denote the sequence of real zeros of f. Then for every $t \in R \setminus \{x_n\}$*

$$\frac{\bar{f}(t)}{|f(t)|} = c\; B_-(t)\; e^{-ih_- t}\; e^{i\, \widetilde{\log|f|}(t)}$$

where c is a constant of modulus one.

Proof. Apply Theorem 4.1 to $f^*(z)$. □

5. Hilbert transform of $\log|f|$

Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{|n| \to \infty} |x_n| = \infty$ and $x_n < x_m$ if $n < m$. Let $\{k_n\}$ be a sequence of integers. The distribution function $\nu_{\{x_n\}}(t)$ is constant between x_{n-1} and x_n and at each point x_n jumps up k_n units. The value of $\nu_{\{x_n\}}(t)$ at x_n is not important. Therefore $\nu_{\{x_n\}}(t)$ is defined up to an additive constant. For example, if we let x_n repeat k_n times, we can define $\nu_{\{x_n\}}(t)$ by the following formula:

$$\nu_{\{x_n\}}(t) = \begin{cases} \#x_n \in [0,t], & t \geqslant 0, \\ (-1)\, \#x_n \in (t,0], & t < 0. \end{cases}$$

Lemma 5.1. *Let $f \in Cart$ and let $\{x_n\}$ denote the sequence of real zeros of f and $\{k_n\}$ be the sequence of orders of f at each zero. Then $\widetilde{\log|f|} + \pi \nu_{\{x_n\}}$ is a continuous function.*

Proof. Let x_{n-1} and x_n be two consecutive real zeros of f. On $[a,b]\subset(x_{n-1},x_n)$, $\log|f|$ is Lipschitz. Therefore, $\widetilde{\log|f|}$ is at least continuous on $[a,b]$. On the other hand $\nu_{\{x_n\}}$ is constant on $[a,b]$. Hence $\widetilde{\log|f|}+\pi\nu_{\{x_n\}}$ is continuous on $[a,b]$ and thus on (x_{n-1},x_n). We can decompose f as $f(z)=(z-x_n)^{k_n}\,g(z)$, where g is analytic and nonzero on a neighborhood of x_n. Thus $\log|f(t)|=k_n\,\log|t-x_n|+\log|g(t)|$. According to the above discussion, since g is not zero in an interval around x_n, $\widetilde{\log|g|}$ is continuous in this interval. The Hilbert transform of $k_n\log|t-x_n|$ is a step function jumping down by $k_n\,\pi$ at x_n. Therefore, $\widetilde{\log|f|}$ is continuous on $(x_n-\delta,x_n)\cup(x_n,x_n+\delta)$ for a small δ, and jumps down by $k_n\,\pi$ at x_n. On the other hand $\pi\nu_{\{x_n\}}$ is also continuous on this neighborhood and jumps up by $k_n\,\pi$ at x_n. Hence $\widetilde{\log|f|}+\pi\nu_{\{x_n\}}$ has a removable discontinuity at x_n. □

Lemma 5.2. *Let $f\in Cart$. Let $\{x_n\}$, $\{z_n\}$ and $\{w_n\}$ be respectively the sequence of real, upper and lower half plane zeros of f. Then for every $t\in R\setminus\{x_n\}$*

$$\widetilde{\log|f|}(t)\equiv\theta+\big(\frac{h_++h_-}{2}\big)\,t-\sum_n\varphi_{z_n}(t)-\sum_n\varphi_{\bar w_n}(t)\pmod{\pi},$$

where θ is a constant.

Proof. By Theorem 4.1 and Corollary 4.2, for every $t\in R\setminus\{x_n\}$ we have

$$c_1\;B_+(t)\;e^{-ih_+\,t}\;e^{i\,\widetilde{\log|f|}(t)}=c_2\;\bar B_-(t)\;e^{ih_-\,t}\;e^{-i\,\widetilde{\log|f|}(t)}.$$

Hence by (3.2) and (4.3)

$$\begin{aligned}\exp\big(2i\,\widetilde{\log|f|}(t)\big)&=c\,e^{i(h_++h_-)\,t}\,\frac{\bar B_-(t)}{B_+(t)}=c\,e^{i(h_++h_-)\,t}\;\frac{e^{-2i\sum_n\varphi_{\bar w_n}(t)}}{e^{2i\sum_n\varphi_{z_n}(t)}}\\&=c\,e^{i(h_++h_-)\,t}\;e^{-2i\sum_n\varphi_{\bar w_n}(t)-2i\sum_n\varphi_{z_n}(t)}\\&=e^{2i\left(\theta+\left(\frac{h_++h_-}{2}\right)\,t-\sum_n\varphi_{\bar w_n}(t)-\sum_n\varphi_{z_n}(t)\right)}.\end{aligned}$$

$c=\frac{c_2}{c_1}=e^{2i\theta}$ is a constant of modulus one. Thus for every $t\in R\setminus\{x_n\}$

$$\widetilde{\log|f|}(t)\equiv\theta+\big(\frac{h_++h_-}{2}\big)\,t-\sum_n\varphi_{\bar w_n}(t)-\sum_n\varphi_{z_n}(t)\pmod{\pi}.$$

□

Main Theorem 5.3. *Let $f\in Cart$. Let $\{x_n\}$, $\{z_n\}$ and $\{w_n\}$ be respectively the sequence of real, upper and lower half plane zeros of f. Then for every $t\in R\setminus\{x_n\}$*

$$\boxed{\widetilde{\log|f|}(t)=-\pi\nu_{\{x_n\}}(t)+\big(\frac{h_++h_-}{2}\big)\,t-\sum_n\varphi_{z_n}(t)-\sum_n\varphi_{\bar w_n}(t)+\theta}$$

where θ is a constant.

Proof. $\nu_{\{x_n\}}(t)$ is a step function which jumps up at $\{x_n\}$ by an integer. Thus for every $t\in R\setminus\{x_n\}$, $\pi\nu_{\{x_n\}}(t)\equiv c_1\pmod{\pi}$, where c_1 is a constant. Hence by Lemma 5.2, for every $t\in R\setminus\{x_n\}$

$$\begin{aligned}\widetilde{\log|f|}(t)+\pi\nu_{\{x_n\}}(t)&\equiv\widetilde{\log|f|}(t)+c_1\\&\equiv c_1+c_2+\big(\frac{h_++h_-}{2}\big)\,t-\sum_n\varphi_{\bar w_n}(t)-\sum_n\varphi_{z_n}(t)\pmod{\pi}.\end{aligned}$$

The right side is a continuous function of t. By Lemma 5.1 the left side is also continuous. Hence there is a constant c_3 such that for every $t \in R \setminus \{x_n\}$

$$\widetilde{\log|f|}(t) + \pi\nu_{\{x_n\}}(t) = c_1 + c_2 + c_3 + \big(\frac{h_+ + h_-}{2}\big)\, t - \sum_n \varphi_{\bar{w}_n}(t) - \sum_n \varphi_{z_n}(t).$$

$\square$

Corollary 5.4. *Let $f \in Cart$. Let $\{x_n\}$ be the sequence of real zeros of f. Suppose f has no other zeros. Then for every $t \in R \setminus \{x_n\}$*

$$\widetilde{\log|f|}(t) = \big(\frac{h_+ + h_-}{2}\big)\, t - \pi\nu_{\{x_n\}}(t) + \theta,$$

where θ is a constant.

Let us consider the function $f(z) = \sin z \in Cart$. $h_+[f] = h_-[f] = 1$ and f has only simple zeros at $\{n\pi\}_{n\in\mathbb{Z}}$. Hence $\nu_{\{x_n\}}(t) = [\frac{t}{\pi}]$. Therefore, for every $t \in R \setminus \{n\pi\}_{n\in\mathbb{Z}}$

$$\widetilde{\log|\sin x|}(t) = t - \pi[\frac{t}{\pi}] + \theta.$$

The following result is due to Cartwright and Levinson.

Corollary 5.5. *Let $f \in Cart$. Suppose f has only real zeros $\{x_n\}$. Then*

$$\lim_{|t|\to\infty} \frac{\nu_{\{x_n\}}(t)}{t} = \frac{h_+ + h_-}{2\pi}.$$

Proof. By Corollary 5.4 for every $t \in R \setminus \{x_n\}$

$$\frac{\nu_{\{x_n\}}(t)}{t} = \frac{h_+ + h_-}{2\pi} - \frac{\widetilde{\log|f|}(t)}{\pi t} + \frac{\theta}{\pi t}.$$

$\frac{\widetilde{\log|f|}(t)}{t} \to 0$ as $|t| \to \infty$ [1]. $\square$

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