Totally Positive Kernels, Pólya Frequency Functions, and Generalized Hypergeometric Series

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ABSTRACT

Recently, K. I. Gross and the author [*J. Approx. Theory* 59:224–246 (1989)] analyzed the total positivity of kernels of the form $K(x,y) = {}_p\mathscr{F}_q(xy)$, $x, y \in \mathbf{R}$, where ${}_p\mathscr{F}_q$ denotes a classical generalized hypergeometric series. There, we related the determinants which define the total positivity of K to the hypergeometric functions of matrix argument, defined on the space of $n \times n$ Hermitian matrices. In the first part of this paper, we apply these methods to derive the total positivity properties of the kernels K for more general choices of the parameters in these hypergeometric series. In particular, we prove that if $a_i > 0$ and k_i is a positive integer $(i = 1, \dots, p)$ then $K(x,y) = {}_p\mathscr{F}_q(a_1, \dots, a_p; a_1 + k_1, \dots, a_p + k_p; xy)$ is strictly totally positive on $\mathbf{R}^2$. In the second part, we use the theory of entire functions to derive some Pólya frequency function properties of the hypergeometric series ${}_p\mathscr{F}_q(x)$. Some of these latter results suggest a curious duality with the total positivity properties described above. For instance, we prove that if $p > 1$ and the a_i and k_i are as above, then there exists a probability density function f on $\mathbf{R}$, such that f is a strict Pólya frequency function, and $1/{}_p\mathscr{F}_p(a_1 + k_1, \dots, a_p + k_p; a_1, \dots, a_p; z) = \mathscr{L}f(z)$, the Laplace transform of f.

1. INTRODUCTION

In a recent article, Gross and the author [4] used techniques from harmonic analysis on the space S_n of $n \times n$ Hermitian matrices to study the total positivity properties of certain kernels, $K(x,y)$, defined in terms of the

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classical generalized hypergeometric functions. Our methods were based on an integral representation [4, Theorem 3.1] for the determinants which define the total positivity of the kernel K, and which we describe as follows.

Let $K(x, y) = f(xy)$, where f is a real-analytic function of a real variable. Let s and t be matrices in S_n having eigenvalues $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$, respectively, and such that all products $s_i t_j$ lie in the domain of f. Let $V(s) = \prod_{1 \leq i < j \leq n}(s_i - s_j)$ denote the Vandermonde determinant. Further let $\mathrm{U}(n)$ denote the group of $n \times n$ unitary matrices and for $u \in \mathrm{U}(n)$ let du denote the Haar measure on $\mathrm{U}(n)$ normalized to have total volume 1. Then it is proven in [4, Theorem 3.1] that there exists a sequence of real-analytic functions ψ_n, defined on the space S_n, such that for all $n = 1, 2, 3, \ldots,$ the $n \times n$ determinant $\det(K(s_i, t_j))$ may be expressed as

$$\frac{\det\big(K(s_i, t_j)\big)}{V(s)V(t)} = \int_{\mathrm{U}(n)} \psi_n(sutu^{-1})\, du. \tag{1.1}$$

In particular, the definition of total positivity (see Section 2) implies that the kernel K is totally positive if and only if the functions ψ_n are all nonnegative on S_n. In the case when $f(x) = {}_p\mathscr{F}_p(x)$ or ${}_{p+1}\mathscr{F}_p(x)$, a classical generalized hypergeometric series, the functions ψ_n in (1.1) may be expressed in terms of the hypergeometric functions of matrix argument [3, 5, 6]. In [4] we applied the theory of the hypergeometric functions of matrix argument, for certain choices of the hypergeometric parameters, to derive the total positivity of the kernel K.

At this point, two advantages of our techniques are worth noting. Firstly, the integral representation (1.1) provides an explicit evaluation of the $n \times n$ determinant $\det(K(s_i, t_j))$, and leads to direct proofs of the total positivity of many kernels; in some cases, the method also determines the sign of the determinant when the kernel is not totally positive (cf. [4, Corollaries 3.4 and 3.5]). Secondly, our method has the advantage, over more classical methods, of yielding the maximal domain of positivity for some hypergeometric kernels (cf. [4, Corollary 3.5 and Theorem 5.5]).

In the first part of this paper, we consider more general choices for the hypergeometric parameters than were treated in [4]. For instance, if $a_i > 0$ $(i = 1, \ldots, p)$, it was proved in [2] and [4] that the kernel $K(x, y) = {}_p\mathscr{F}_p(a_1, \ldots, a_p; a_1 + 1, \ldots, a_p + 1; xy)$, is strictly totally positive (of order infinity) on $\mathbf{R}^2$. We will extend this result by proving (in Theorem 3.2) that for positive integers $k_1, \ldots, k_p$, the kernel $K(x, y) = {}_pF_p(a_1, \ldots, a_p; a_1 + k_1, \ldots, a_p + k_p; xy)$ is strictly totally positive on $\mathbf{R}^2$. Further, we determine the order of total positivity of K if not all the $k_i > 0$ are positive integers.

The second part of the paper, contained in Section 4, is different in character from the results described above. In this part, we derive some

Pólya frequency function properties of the hypergeometric series ${}_p\mathscr{F}_q(x)$ when $p \leqslant q$. These properties exhibit a surprising duality with the total positivity properties of the ${}_p\mathscr{F}_q$ kernels. By applying I. J. Schoenberg's characterizations of the Pólya frequency functions on $\mathbf{R}$ [7, Chapter 7] together with some general results on entire functions [8], we prove that there exist probability density functions f on $\mathbf{R}$ such that the Laplace transform, $\mathscr{L}f(z)$, of f can be expressed in terms of the *reciprocals* of the hypergeometric functions ${}_p\mathscr{F}_q(z)$. In particular we show (in Theorem 4.1) that if $p > 1$, $a_i > 0$, and k_i is a positive integer ($i = 1, \ldots, p$), then there exists a probability density function f on $\mathbf{R}$ such that f is a strict Pólya frequency function of order infinity, and such that $\mathscr{L}f(z) = 1/{}_p\mathscr{F}_p(a_1 + k_1, \ldots, a_p + k_p; a_1, \ldots, a_p; z)$. In the case when $p < q$ we obtain a similar result together with a characterization of the domain on which the probability density f is a strict Pólya frequency function.

2. PRELIMINARIES

2.1. *Totally Positive Kernels [2, 4, 7]*

Suppose that $\mathscr{D} \subseteq \mathbf{R}^2$ and r is a positive integer. A kernel $K : \mathscr{D} \to \mathbf{R}$ is *totally positive of order* r (TP_r) on $\mathscr{D}$ if for all $s_1 < s_2 < \cdots < s_r$ and $t_1 < t_2 < \cdots < t_r$ such that $(s_i, t_j) \in \mathscr{D}$, the $n \times n$ determinant

$$\det\big(K(s_i,t_j)\big) = \begin{vmatrix} K(s_1,t_1) & K(s_1,t_2) & \cdots & K(s_1,t_n) \\ K(s_2,t_1) & K(s_2,t_2) & \cdots & K(s_2,t_n) \\ \vdots & \vdots & & \vdots \\ K(s_n,t_1) & K(s_n,t_2) & \cdots & K(s_n,t_n) \end{vmatrix} \tag{2.1}$$

is nonnegative for all $n = 1, \ldots, r$. If the determinants $\det(K(s_i, t_j))$ in (2.1) are strictly positive for all $n = 1, \ldots, r$, then we say that the kernel K is *strictly totally positive of order* r (STP_r) on $\mathscr{D}$. If K is TP_r (STP_r) for all $r = 1, 2, \ldots$, then we say that K is TP_∞ (STP_∞) on $\mathscr{D}$.

A C^∞ kernel K is *extended totally positive* (ETP) on $\mathscr{D}$ if for all $n = 1, 2, \ldots$, the $n \times n$ determinant

$$\tilde{K}(x,y) = \det\left(\frac{\partial^{i+j-2}}{\partial x^{i-1} \partial y^{j-1}} K(x,y) \right)$$

is positive for all $(x, y) \in \mathscr{D}$. It is known [7, p. 55] that if K is STP_∞ and real-analytic then it is ETP. Conversely, if K is ETP then it is STP_∞.

A standard technique for constructing totally positive kernels is the *basic composition formula* [7, p. 17]: If the kernels K and L are TP_k and TP_l respectively, and $d\mu$ is a nonnegative Borel measure on $\mathbf{R}$ such that the integral

$$M(x,y) = \int_{\mathbf{R}} K(x,w)L(w,y)\, d\mu(w) \tag{2.2}$$

converges absolutely, then the kernel M is TP_r, where $r = \min\{k,l\}$. The proof of this result is by an application of the well-known Binet-Cauchy formula [7, p. 1].

2.2. *Zonal Polynomials and Schur Functions*

Let $m = (m_1, \ldots, m_n)$ be a partition; that is, the m_i are nonnegative integers such that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$. For each partition m define the Schur function χ_m on $\mathbf{R}^n$ by

$$\chi_m(t_1, \ldots, t_n) = \frac{\det\left(t_i^{m_j+n-j}\right)}{V(t)} \tag{2.3}$$

whenever $t_1, \ldots, t_n$ are distinct, and by L'Hospital's rule otherwise. Then χ_m is a polynomial on $\mathbf{R}^n$, homogeneous of degree $|m| = m_1 + \cdots + m_n$, and is invariant under the symmetric group on n symbols. For each $t \in \mathrm{S}_n$ with eigenvalues $t_1, \ldots, t_n$, we define

$$\chi_m(t) = \chi_m(t_1, \ldots, t_n). \tag{2.4}$$

The formula (2.4) extends χ_m to a unitarily invariant polynomial on the space S_n; that is, $\chi_m(utu^{-1}) = \chi_m(t)$, $t \in \mathrm{S}_n$, $u \in \mathrm{U}(n)$.

For each partition m, define the zonal polynomial Z_m on S_n by

$$Z_m(t) = \frac{|m|!\prod_{1 \leqslant j < k \leqslant n}(m_j - m_k - j + k)}{\prod_{j=1}^{n}(m_j + n - j)!}\chi_m(t). \tag{2.5}$$

The normalization (2.5) is chosen so that [3, Equation (5.3.2); 6]

$$(\mathrm{tr}\, t)^j = \sum_{|m|=j} Z_m(t). \tag{2.6}$$

2.3. *Hypergeometric Functions of Matrix Argument*

For many $a \in \mathbf{C}$, let $(a)_k = a(a+1)\cdots(a+k-1)$, $k = 0,1,2,\ldots$, be the usual Pochhammer symbol. Further, for any partition m, define the generalized Pochhammer symbol

$$[a]_m = \prod_{j=1}^{n} (a-j+1)_{m_j}. \tag{2.7}$$

The *hypergeometric function* ${}_pF_q$ of matrix argument is defined to be the real-analytic function on S_n given by the series expansion

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t) = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{|m|=j} \frac{[a_1]_m \cdots [a_p]_m}{[b_1]_m \cdots [b_q]_m} Z_m(t), \tag{2.8}$$

where none of the numbers $-b_i + j - 1$ $(1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant n)$ is a nonnegative integer.

In the case $n = 1$, we denote the classical generalized hypergeometric function by ${}_p\mathscr{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t)$. The connection between the hypergeometric functions ${}_pF_q$ and their classical counterparts ${}_p\mathscr{F}_q$ is [4, Theorem 4.2; 5] that if the eigenvalues of $t \in S_n$ are $t_1,\ldots,t_n$, respectively, then

$$\begin{aligned} &{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t) \\ &\quad = \frac{\det\left(t_j^{n-i}\,{}_p\mathscr{F}_q(a_1-i+1,\ldots,a_p-i+1;b_1-i+1,\ldots,b_q-i+1;t_j)\right)}{V(t)}. \end{aligned} \tag{2.9}$$

The hypergeometric functions of matrix argument were first defined by James [6] and have been widely used in multivariate statistical analysis. The article [3] derives the convergence properties of the series (2.8), and lists some recent applications of these hypergeometric functions to analytic number theory, chemistry, and physics. In the simplest case $(p,q) = (0,0)$, it follows from (2.6) and (2.8) [or from (2.9)] that ${}_0F_0(t) = e^{\operatorname{tr} t}$ for all $t \in S_n$. Moreover from (2.9) it follows that ${}_1F_0(a;t) = \det(1-t)^{-a}$ for $\|t\| := \max\{|t_i| : i = 1,\ldots,n\} < 1$ (the binomial theorem on S_n).

3. TOTAL POSITIVITY PROPERTIES OF GENERALIZED HYPERGEOMETRIC SERIES

In [2, Theorems 6, 7] and [4, Theorem 5.3], it was proved that if $a_i > 0$ for $i = 0, 1, \ldots, p$, then the kernels $K(x,y) = {}_p\mathscr{F}_p(a_1, \ldots, a_p; a_1+1, \ldots, a_p+1; xy)$ and $K(x,y) = {}_{p+1}\mathscr{F}_p(a_0, a_1, \ldots, a_p; a_1+1, \ldots, a_p+1; xy)$ are STP_∞ on the domains $\mathbf{R}^2$ and $\{(x,y) \in \mathbf{R}^2 : |xy| < 1\}$, respectively. Naturally, this raises the problem of whether similar results hold for the kernels $K(x,y) = {}_p\mathscr{F}_p(a_1, \ldots, a_p; a_1+2, \ldots, a_p+2; xy)$ and $K(x,y) = {}_{p+1}\mathscr{F}_p(a_0, a_1, \ldots, a_p; a_1+2, \ldots, a_p+2; xy)$. It follows from Corollary 4.4 in [4] that a necessary and sufficient condition for the kernel $K(x,y) = {}_p\mathscr{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; xy)$ to be STP_∞ on a domain of the form $\{(x,y) \in \mathbf{R}^2 : \alpha < xy < \beta\}$ is that

$$ {}_pF_q(a_1+n-1, \ldots, a_p+n-1; b_1+n-1, \ldots, b_q+n-1; xI_n) > 0 \quad (3.1) $$

for all $n = 1, 2, \ldots,$ and for all $\alpha < x < \beta$, where I_n denotes the $n \times n$ identity matrix. In the case when $q = p-1$ or p and $b_i = a_i + 1$, $i = 1, \ldots, q$, we deduced the inequality (3.1) in [4] from an Euler integral formula (in the case $p = 1$) and induction on p. Here, we apply the criterion (3.1) to generalize the results in [2] and [4].

THEOREM 3.1. *Let $a > 0$, $b > 0$, and k be a positive integer. Then the following results hold*:

(1) *the kernel $K(x,y) = {}_1\mathscr{F}_1(a; a+k; xy)$ is STP_∞ on $\mathbf{R}^2$.*

(2) *the kernel $K(x,y) = {}_2\mathscr{F}_1(a, b; a+k; xy)$ is STP_∞ on the domain $\{(x,y) \in \mathbf{R}^2 : |xy| < 1\}$.*

Proof. To prove (1) we need to show that, for any positive integer n, the function

$$ \phi_n(x) = {}_1F_1(a+n-1; a+k+n-1; xI_n) > 0 $$

for all $x \in \mathbf{R}$.

Suppose that $x \geqslant 0$. Then for any partition m and $a > 0$, it follows from (2.7) that $[a+n-1]_m > 0$. Further, $Z_m(xI_n) = x^{|m|} Z_m(I_n) \geqslant 0$, since $Z_m(I_n) > 0$ [3, Equation (5.3.1)]. Therefore, the positivity of $\phi_n(x)$ follows from the zonal polynomial expansion (2.8) of the function ${}_1F_1$.

If $x < 0$, then by the Kummer identity [4, Equation (5.5); 6]

$$ {}_1F_1(a; c; t) = e^{\mathrm{tr}(t)} {}_1F_1(c-a; c; -t), $$

we find that

$$\phi_n(x) = e^{nx}\,{}_1F_1(k; a+k+n-1; -xI_n).$$

Then the positivity of $\phi_n(x)$ again follows from the zonal polynomial expansion of the function ${}_1F_1$.

In proving (2), we similarly need to show that the function

$$\phi_n(x) = {}_2F_1(a+n-1, b+n-1; a+k+n-1; xI_n) > 0$$

for $|x| < 1$. If $0 \leqslant x < 1$, then we use the zonal polynomial expansion of the function ${}_2F_1$ to deduce the positivity of $\phi_n(x)$. If $-1 < x < 0$, then by the Kummer identity [4, Equation (5.8); 6]

$${}_2F_1(a,b;c;t) = \det(I_n - t)^{-b}\,{}_2F_1\big(c-a, b; c; -t(I_n - t)^{-1}\big),$$

which is valid for $\|t\| < 1$ and $\|t(I_n - t^{-1}\| < 1$, we obtain

$$\phi_n(x) = (1-x)^{-n(b+n-1)}\,{}_2F_1\big(k, b+n-1; a+k+n-1; -x(1-x)^{-1}I_n\big).$$

Hence the positivity of $\phi_n(x)$ also follows from the zonal polynomial expansion of the function ${}_2F_1$.

To extend Theorem 3.1 to the generalized hypergeometric series ${}_p\mathscr{F}_p$ and ${}_{p+1}\mathscr{F}_p$, we need to recall [7, Chapter 7] that a function $f: \mathbf{R} \to \mathbf{R}$ is a *Pólya frequency function of order r* (PF_r) if the kernel $K(x,y) = f(x-y)$ is TP_r on $\mathbf{R}^2$. Now here is the complete extension of Theorem 3.1; note that the method of proof is the same as in [2, Theorem 6].

THEOREM 3.2. *Let $a_j > 0$, $j = 0, 1, \ldots, p$, and k_j be a positive integer, $j = 1, \ldots, p$. Then the following results hold:*

(1) The kernel $K_p(x,y) = {}_p\mathscr{F}_p(a_1, \ldots, a_p; a_1 + k_1, \ldots, a_p + k_p; xy)$ is STP_∞ on $\mathbf{R}^2$.

(2) The kernel $K_p(x,y) = {}_{p+1}\mathscr{F}_p(a_0, a_1, \ldots, a_p; a_1 + k_1, \ldots, a_p + k_p; xy)$ is STP_∞ on the domain $\{(x,y) \in \mathbf{R}^2 : |xy| < 1\}$.

Proof. Since the proofs of (1) and (2) are similar, we only provide details for the proof of (1). For $p = 1$, we have already proved in Theorem 3.1 that the kernel K_1 is ETP (and hence STP_∞) on $\mathbf{R}^2$. By inductive hypothesis, assume that K_{p-1} is STP_∞ on $\mathbf{R}^2$. It is well known that the functions ${}_p\mathscr{F}_q$ satisfy the Euler integral formula

$$
{}_p\mathscr{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)=\frac{\Gamma(b_q)}{\Gamma(a_p)\Gamma(b_q-a_p)}
$$

$$
\times\int_0^1 w^{a_p-1}(1-w)^{b_q-a_p-1}\,{}_{p-1}\mathscr{F}_{q-1}(a_1,\ldots,a_{p-1};b_1,\ldots,b_{q-1};xw)\,dw,
$$

where $\operatorname{Re} b_q > \operatorname{Re} a_p > 0$, and for all x such that the ${}_p\mathscr{F}_q$ series converges. Therefore, for $x>0$,

$$
\begin{aligned}
K_p(x,y)&=\frac{\Gamma(a_p+k_p)}{\Gamma(a_p)\Gamma(k_p)}\int_0^1 w^{a_p-1}(1-w)^{k_p-1}K_{p-1}(xw,y)\,dw\\
&=\frac{\Gamma(a_p+k_p)}{\Gamma(a_p)\Gamma(k_p)}x^{-(a_p+k_p-1)}\int_0^x w^{a_p-1}(x-w)^{k_p-1}K_{p-1}(w,y)\,dw.
\end{aligned}
\tag{3.2}
$$

If we define

$$
d\mu(w)=\frac{\Gamma(a_p+k_p)}{\Gamma(a_p)}w^{a_p-1}\,dw,\qquad w>0,
$$

and

$$
L(x,w)=\begin{cases}\dfrac{x^{-(a_p+k_p-1)}(x-w)^{k_p-1}}{\Gamma(k_p)}, & 0<w<x<\infty,\\ 0, & 0<x\leqslant w<\infty,\end{cases}
\tag{3.3}
$$

then (3.2) becomes

$$
K_p(x,y)=\int_0^\infty L(x,w)K_{p-1}(w,y)\,d\mu(w).
\tag{3.4}
$$

By [7, p. 107, Theorem 2.1], the function

$$f(x) = \frac{x^{k_p - 1}}{\Gamma(k_p)}, \qquad x > 0, \tag{3.5}$$

is PF_∞; hence the kernel L is TP_∞ on $\mathbf{R}_+^2$. Since K_{p-1} is STP_∞ on $\mathbf{R}^2$ by the inductive hypothesis, then on applying the basic composition formula (2.2) to (3.4), we deduce that K_p is STP_∞ on $\mathbf{R}_+ \times \mathbf{R}$. By analyticity, we also find that K_p is ETP on $\mathbf{R}_+ \times \mathbf{R}$.

For $x < 0$, we rewrite (3.4) as

$$K_p(x,y) = \int_0^\infty L(-x,w) K_{p-1}(w,-y)\, d\mu(w). \tag{3.6}$$

Proceeding in a similar manner, we again deduce, by applying the Binet-Cauchy formula to (3.6), that K_p is ETP on $(-\infty, 0) \times \mathbf{R}$. By continuity, K_p is ETP (and hence STP_∞) on $\mathbf{R}^2$.

REMARK 3.3. Proceeding in similar fashion, we can also prove that if $k_1, \ldots, k_p$ are positive but not all integers, then the kernels K_p in Theorems 3.1 and 3.2 are TP_r where $r = \min\{[k_i] : k_i \text{ is nonintegral}\} + 1$, where $[k]$ denotes the greatest integer less than or equal to k. The proof rests on the property [7, p. 107] that if k_p is not a positive integer then the function $f(x)$ in (3.5) is $TP_{[k_p]+1}$.

4. PÓLYA FREQUENCY PROPERTIES OF GENERALIZED HYPERGEOMETRIC SERIES

In [7], PF properties have been derived for some of the generalized hypergeometric series. For instance in [7, Chapter 3], the results include a derivation of the PF properties of the ${}_0\mathscr{F}_1$ hypergeometric series. For other ${}_p\mathscr{F}_q$ series, the methods of [7] together with classical Euler integrals can be used to develop the corresponding PF properties. As an example, let us prove that the function $f(x) = x^{a+b-1}\,{}_1\mathscr{F}_1(a; a+b; x)$, $x > 0$, is PF_∞, where a and b are positive integers.

By (3.2) with $p = 1$,

$$x^{a+b-1}\,{}_1\mathscr{F}_1(a; a+b; x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x w^{a-1}(x-w)^{b-1} e^w\, dw, \qquad x > 0.$$

Therefore we obtain $f(x)$ as a convolution,

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} x^{a+b-1} {}_1\mathscr{F}_1(a; a+b; x) = x^{b-1} * x^{a-1}e^x, \qquad x \leqslant 0. \quad (4.1)$$

Since the class of PF_∞ functions is closed under convolutions, and both functions on the right hand side of (4.1) are PF_∞, then it follows that $f(x)$ is also PF_∞.

In closing, we derive another type of PF_∞ property for the generalized hypergeometric series by showing that for $p \leqslant q$, the reciprocals of the functions ${}_p\mathscr{F}_q$ can be related to PF_∞ probability density functions. To do this we will need some results from the theory of entire functions [8] and I. J. Schoenberg's deep characterizations [7, Chapter 7] of the class of PF_∞ density functions.

THEOREM 4.1. *Suppose that $p \leqslant q$, $a_i > 0$, $i = 1, \ldots, q$, and $k_1, \ldots, k_p$ are positive integers. Then the following results hold*:

(1) *If $p = q > 1$, then for any $\gamma \geqslant 0$ and $\delta \in \mathbf{R}$, there exists a PF_∞ probability density function f on $\mathbf{R}$ such that*

$$e^{-\gamma z^2 + \delta z} {}_p\mathscr{F}_p(a_1 + k_1, \ldots, a_p + k_p; a_1, \ldots, a_p; z) = \frac{1}{\mathscr{L}f(z)}. \quad (4.2)$$

Further, the density function f is actually strictly PF_∞ in the sense that the kernel $K(x, y) = f(x - y)$ is STP_∞ on $\mathbf{R}^2$.

(2) *If $p < q$, then for any $\delta \geqslant 0$ there exists a PF_∞ probability density function f on $\mathbf{R}_+$ such that*

$$e^{\delta z} {}_p\mathscr{F}_q(a_1 + k_1, \ldots, a_p + k_p; a_1, \ldots, a_q; z) = \frac{1}{\mathscr{L}f(z)}. \quad (4.3)$$

Further, for any $n = 1, 2, \ldots$, if $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, then the $n \times n$ determinant $\det(f(x_i - y_j))$ is strictly positive if and only if there holds the interlacing condition

$$-\infty < y_i + \delta < x_i, \qquad i = 1, \ldots, n. \quad (4.4)$$

Proof. If $p \leqslant q$, then the hypergeometric function ${}_p\mathscr{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is an entire function of order $\rho = (q - p + 1)^{-1}$, which equals one if $p = q$. Hence the genus g of ${}_p\mathscr{F}_p(a_1, \dots, a_p; b_1, \dots, b_p; z)$ is zero or one. In the case of the confluent hypergeometric function ${}_1\mathscr{F}_1$, it is known [1, p. 184] that $g = 1$; using this result together with induction on p and the confluence relation between ${}_p\mathscr{F}_p$ and ${}_{p+1}\mathscr{F}_{p+1}$, we deduce that $g = 1$ for all the functions ${}_p\mathscr{F}_p$.

By Hadamard's factorization theorem [8, p. 250],

$$ {}_p\mathscr{F}_p(a_1 + k_1, \dots, a_p + k_p; a_1, \dots, a_p; z) = e^{\alpha z + \beta} \prod_{j=1}^{\infty} (1 + \lambda_j z) e^{-\lambda_j z}, \quad (4.5) $$

where α and β are constants and $z_j = -1/\lambda_j$ are the zeros of ${}_p\mathscr{F}_p$. Setting $z = 0$ in (4.5), we obtain $\beta = 0$. Further, by taking logarithms on both sides of (4.5), differentiating with respect to z, and setting $z = 0$, we obtain $\alpha = (a_1 + k_1) \cdots (a_p + k_p)/(a_1 \cdots a_p)$. Note also that $\lambda_j > 0$ for all j. To prove this, we write

$$ {}_p\mathscr{F}_p(a_1 + k_1, \dots, a_p + k_p; a_1, \dots, a_p; z) = \sum_{j=0}^{\infty} \frac{\psi(j) z^j}{j!}, $$

where

$$ \psi(z) = \prod_{i=1}^{p} \frac{\Gamma(z + a_i + k_i)\Gamma(a_i)}{\Gamma(z + a_i)\Gamma(a_i + k_i)} $$

is an entire function of genus zero. By a theorem of Laguerre [8, p. 270, §8.63], it follows that z_j is real and negative; hence $\lambda_j > 0$ for all j. Since $g = 1$, we also have that $\Sigma\lambda_j$ diverges while $\Sigma\lambda_j^2 < \infty$ [in fact, $\Sigma\lambda_j^2$ can be calculated explicitly by taking logarithms of (4.5), differentiating twice with respect to z, and setting $z = 0$].

Putting all these results together, we have now shown that for any $\gamma \geqslant 0$ and $\delta \in \mathbf{R}$,

$$ e^{-\gamma z^2 + \delta z} {}_p\mathscr{F}_p(a_1 + k_1, \dots, a_p + k_p; a_1, \dots, a_p; z) $$

$$ = e^{-\gamma z^2 + (\alpha + \delta) z} \prod_{j=1}^{\infty} (1 + \lambda_j z) e^{-\lambda_j z}. \quad (4.6) $$

By [7, p. 345, Theorem 3.2], the right hand side of (4.6) equals $1/\mathscr{L}f(z)$, where f is a PF_∞ probability density function on $\mathbf{R}$. By [7, p. 357, Theorem 6.1], f is actually strictly PF_∞, since $\Sigma\lambda_j^2 < \infty$.

To prove (2), we proceed in a similar fashion. In this case, $\rho < 1$ and $g = 0$. The corresponding infinite product representation is that for any $\delta \geqslant 0$,

$$e^{\delta z}\,{}_p\mathcal{F}_q(a_1 + k_1, \dots, a_p + k_p; a_1, \dots, a_q; z) e^{\delta z} \prod_{j=1}^{\infty} (1 + \lambda_j z), \qquad (4.7)$$

where $\lambda_j > 0$ and $\Sigma\lambda_j < \infty$. Since $\rho < 1$, then $\Sigma\lambda_j < \infty$ [in fact, $\Sigma\lambda_j = (a_1 + k_1) \cdots (a_p + k_p)/(a_1 \cdots a_q)$]. Again appealing to [7, loc. cit.], we see that the right hand side of (4.7) equals $1/\mathscr{L}f(z)$, where f is a PF_∞ probability density function on $\mathbf{R}_+$. Finally, the interlacing condition (4.4) for strict positivity follows from [7, p. 358, Equation (6.3)].

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