We say K is *consistent at* p if each $\Gamma(\sigma, p)$ lies in $V(K, p)$ in a one-one manner. This is obviously a necessary condition that K be smoothly imbeddable in E^m.

It may happen that a relation $a_1v_1 + \ldots + a_kv_k = 0$ is not true at p, yet arbitrarily near p a relation arbitrarily near this one holds. This might preclude smooth imbedding.

A necessary and sufficient condition for imbeddability of a general complifold seems highly difficult to obtain. Instead, we shall give a slight restriction on K.

4. *Cellwise homogeneity.* Let K be consistent at each p. If for each σ, and cells $\sigma_1, \ldots, \sigma_s$, with σ as face, the part of $V(K, p)$ over these cells is of constant dimension as p moves over σ, we say K is *cellwise homogeneous.*

THEOREM: *Any cellwise homogeneous complifold of dimension n may be smoothly imbedded in* E^{2n+1}, *and smoothly immersed in* E^{2n}.

[1] Presented to the American Mathematical Society Sept. 8, 1942.

ON TOTALLY POSITIVE FUNCTIONS, LAPLACE INTEGRALS AND ENTIRE FUNCTIONS OF THE LAGUERRE-POLYA-SCHUR TYPE

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Communicated October 17, 1946

The purpose of this note is to furnish a solution of the problem of representing by Laplace integrals the reciprocals of entire functions with real zeros only, which are of genus zero or one up to certain exponential factors. The solution appears in terms of a certain class of functions which we propose to call *totally positive functions.*

1. *Totally Positive Functions.*—A real-valued function $\Lambda(x)$, defined for all real x, is said to be *totally positive* (abbreviated: t. p.) if it satisfies the following three conditions:

(α) $\Lambda(x)$ is measurable.

(β) If $x_1 < x_2 < \ldots < x_n$ and $t_1 < t_2 < \ldots < t_n$, then the determinant of order n, whose element in the ith row and jth column is $\Lambda(x_i - t_j)$, should be non-negative, i.e.,

$$\det\|\Lambda(x_i - t_j)\| \geqq 0. \quad (1)$$

This inequality should be verified for $n = 1, 2, 3, \ldots$.

(γ) $\Lambda(x)$ should be positive for at least two distinct values of x.

For $n = 1$, the inequality (1) is equivalent to $\Lambda(x) \geqq 0$ for all real x, justifying the name of total positivity for our set of conditions.

A trivial example of t. p. functions is furnished by

$$\Lambda(x) = e^{ax+b}. \tag{2}$$

Indeed, all determinants (1) vanish if $n \geqq 2$. Another t. p. function is

$$\Lambda(x) = \begin{cases} e^{-x} \text{ if } x \geqq 0 \\ 0 \quad \text{if } x < 0. \end{cases} \tag{3}$$

Indeed, we readily find in this case that[1]

$$\det\|\Lambda(x_i - t_j)\| = \exp\{-\Sigma x_i + \Sigma t_j\} \cdot \epsilon,$$

where $\epsilon = 1$ or $\epsilon = 0$, depending on whether or not all inequalities $t_1 \leqq x_1 < t_2 \leqq x_2 < t_3 \leqq x_3 < \ldots < t_{n-1} \leqq x_{n-1} < t_n \leqq x_n$ hold simultaneously. Incidentally, with $\Lambda(x)$ defined by (3), the functions $\Lambda(ax + b)$ $(a \neq 0)$ are the only discontinuous t. p. functions. A further noteworthy example is

$$\Lambda(x) = e^{-x^2}. \tag{4}$$

The total positivity of this function is likewise clear because

$$\det\|\exp\{-(x_i - t_j)^2\} = \exp\{-\Sigma x_i^2 - \Sigma t_j^2\} \cdot \det\|\exp\{2x_i t_j\}\|,$$

the last determinant being known to be positive.[2]

It is easy to see that of the conditions (α), (β), (γ), neither one is implied by the other two. Thus, if

$$\Lambda(x) = \exp\{-\psi(x)\}, \tag{5}$$

where $\psi(x)$ is a discontinuous, hence non-measurable, solution of $\psi(x + y) = \psi(x) + \psi(y)$, then the conditions ($\beta$) and ($\gamma$) are fulfilled without (α) holding.

The conditions (α) and (γ), together with the inequality (1), for $n = 1$ and $n = 2$ only, are found to be equivalent with the requirement that the function $\psi(x)$, of (5), is a convex function. This remark readily leads to the following proposition: *For any t. p. function* $\Lambda(x)$ *we have*

$$0 < \int_{-\infty}^{\infty} \Lambda(x)dx \leqq +\infty.$$

A t. p. function $\Lambda(x)$ *is either monotone (in the wide sense), in which case*

$$\int_{-\infty}^{\infty} \Lambda(x)dx = +\infty, \tag{6}$$

or else it is not a monotone function, in which case we have

$$0 < \int_{-\infty}^{\infty} \Lambda(x)dx < +\infty. \tag{7}$$

A non-monotone t. p. function $\Lambda(x)$ may therefore also be thought of as a frequency function. We wish to call them *Polya frequency functions.* Among our examples of t. p. functions we find that (2) is monotone, while

(3) and (4) are Polya frequency functions. A study of the class of t. p. functions reduces essentially to a study of its subclass of Polya frequency functions in view of the following proposition: *If* $\Lambda(x)$ *is t. p. then either* $\Lambda(x) = \exp\{ax + b\}$, *or else there is a number* ω *such that* $\Lambda_0(x) = e^{\omega x}\cdot\Lambda(x)$ *is a Polya frequency function.*

2. *Entire functions of the Laguerre-Polya-Schur type.*—Following Polya and Schur[3] we shall say that an entire rational or transcendental function $\Phi(z)$ is an *entire function of type I*, if its canonical representation is of the form

$$\Phi(z) = Cz^n e^{\gamma z}\prod_{\nu=1}^{\infty}(1 + \delta_\nu z), \ (C \gtrless 0, n \geqq 0, \gamma \geqq 0, \delta_\nu \geqq 0). \tag{8}$$

Likewise we shall say that $\Psi(z)$ is an *entire function of type II* if its canonical representation is of the form

$$\Psi(z) = Cz^n e^{-\gamma z^2+\delta z}\prod_{\nu=1}^{\infty} (1+\delta_\nu z)\, e^{-\delta_\nu z}, \ (C\lessgtr 0, n\geqq 0, \gamma\geqq 0, \gamma, \delta_\nu \text{ real}). \tag{9}$$

It was shown by Laguerre and Polya[4] that the entire functions of type I *and no others* are the uniform limits ($\not\equiv 0$) of real polynomials having all their roots on the half-line $x \leqq 0 (z = x + yi)$. Likewise that the entire functions of type II *and no others* are the uniform limits ($\not\equiv 0$) of real polynomials with only real roots. The coefficients of the power series expansions of the functions $\Phi(z)$ and $\Psi(z)$ about the origin enjoy remarkable algebraic properties which were discovered by Polya and Schur (*loc. cit.*) on the foundation of Laguerre's pioneering work. Finally, Polya[5] investigated the expansion coefficients of the reciprocals $1/\Phi(z)$ and $1/\Psi(z)$ about the origin. His results suggested that these reciprocals should allow of representations by the ordinary Laplace-Stieltjes integral and by the bilateral Laplace-Stieltjes integral, respectively, both with monotone determining functions. This problem was later investigated by H. Hamburger.[6] However, Hamburger's results do not exhibit the exact nature of the Laplace integral representations of these particular classes of functions.

3. *Laplace integrals.*—The relationship between the reciprocals of functions of the Laguerre-Polya-Schur type and the Laplace integral is described by the following theorem.

THEOREM 1. *Let* $\Lambda(x)$ *be a totally positive function which is not of the form* $exp\{ax + b\}$. *Then the bilateral Laplace integral*

$$\int_{-\infty}^{\infty} e^{xz}\Lambda(x)dx \tag{10}$$

converges in a strip $\alpha < Rz < \beta(-\infty \leqq \alpha < \beta \leqq +\infty)$ *and represents there the reciprocal*

$$\frac{1}{\Psi(z)} \tag{11}$$

of an entire function $\Psi(z)$ of type II which is not of the form $C\cdot e^{\delta z}$. The endpoints α, β of the strip of convergence of the integral (10) are zeros of $\Psi(z)$, provided they are finite.

Conversely, let (11) be the reciprocal of a function (9) of type II, not of the form $C\cdot e^{\delta z}$, and let $\alpha < Rz < \beta(-\infty \leqq \alpha < \beta \leqq +\infty)$ be a strip in which (11) is regular. Then $1/\Psi(z)$ (if $\Psi(z) > 0$ in $\alpha < z < \beta$), or else $-1/\Psi(z)$ (if $\Psi(z) < 0$ in $\alpha < z < \beta$), may be there represented by a Laplace integral (10), where $\Lambda(x)$ is a totally positive function.

We, therefore, have a 1-1 correspondence between t. p. functions and reciprocals of functions of type II in a given strip of regularity $\alpha < Rz < \beta$ $(-\infty \leqq \alpha < \beta \leqq \infty)$ by the relation

$$\int_{-\infty}^{\infty} e^{xz}\Lambda(x)dx = \frac{1}{\Psi(z)}\cdot\epsilon, \qquad (\alpha < Rz < \beta), \tag{12}$$

where $\epsilon = sgn\Psi(z)$ in $\alpha < z < \beta$.

Further properties of the correspondence (12) are as follows.

THEOREM 2. *The totally positive function $\Lambda(x)$ is a Polya frequency function if and only if*

$$\alpha < 0 < \beta \tag{13}$$

Otherwise $\Lambda(x)$ is always monotone, namely, non-increasing if $0 \leqq \alpha$ and non-decreasing if $\beta \leqq 0$.

THEOREM 3. *The relation (12) may always be inverted by Mellin's integral to*

$$\Lambda(x) = \frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} \frac{\epsilon}{\Psi(z)}\, e^{-xz}dx, \quad (\alpha < \sigma < \beta,\ -\infty < x < \infty). \tag{14}$$

Unless $\Psi(z)$ is free of zeros, there are several t. p. functions $\Lambda(x)$ associated with the same $\Psi(z)$, corresponding to the various strips of regularity of its reciprocal. The way in which these various $\Lambda(x)$ are connected with each other is as follows.

THEOREM 4. *Let (α, β) and (β, γ) be contiguous intervals of regularity of $1/\Psi(z)$ $(-\infty \leqq \alpha < \beta < \gamma \leqq \infty)$ and let $\Lambda(x)$ and $\Lambda_1(x)$ be the corresponding totally positive functions, i.e.,*

$$\frac{\epsilon}{\Psi(z)} = \int_{-\infty}^{\infty} e^{xz}\Lambda(x)dx \ (\alpha < Rz < \beta),\ \frac{\epsilon_1}{\Psi(z)} = \int_{-\infty}^{\infty} e^{xz}\Lambda_1(x)dx \quad (\beta < Rz < \gamma),$$

where $\epsilon = \pm 1$, $\epsilon_1 = \pm 1$, such that $\epsilon\Psi(z) > 0$ if $\alpha < z < \beta$, and $\epsilon_1\Psi(z) > 0$ if $\beta < z < \gamma$. Then

$$\epsilon_1\Lambda_1(x) - \epsilon\Lambda(x) = \text{The residue of } e^{-xz}/\Psi(z) \text{ at } z = \beta, \tag{15}$$

for all real values of x.

Functions $\Phi(z)$ of type I being at the same time also functions of type II, the theorems just stated apply to such functions as well. In this particular case, however, we may add to Theorem 1 the following additional information.

THEOREM 5. *Let $\Lambda(x)$ be a Polya frequency function such that*

$$\Lambda(x) = 0 \text{ if } x > 0. \tag{16}$$

Then

$$\int_{-\infty}^{0} e^{xz}\Lambda(x)dx = \frac{1}{\Phi(z)}, \ (Rz > \alpha, \text{ for some } \alpha < 0) \tag{17}$$

where $\Phi(z)$ is an entire function of type I, with $\Phi(0) > 0$, and not of the form $C \cdot e^{\delta z}$. Conversely, if $\Phi(z)$ is of type I, $\Phi(0) > 0$, not of the form, $C \cdot e^{\delta z}$, then it may be represented by the ordinary Laplace integral (17), where $\Lambda(x)$, if defined as $= 0$ for $x > 0$, is a Polya frequency function.

4. *Examples.*—A few classical integral representations will serve to illustrate the theory. The following formulae are due to Euler:

$$\textstyle\int_{-\infty}^{\infty} e^{xz}e^{-e^x}dx = \Gamma(z) \qquad (Rz > 0), \tag{18}$$

$$\int_{-\infty}^{\infty} e^{xz}\frac{1}{1+e^x}dx = \frac{\pi}{\sin \pi z} \qquad (0 < Rz < 1), \tag{19}$$

$$\int_{-\infty}^{0} e^{xz}(1 - e^x)^{n-1}e^x dx = \frac{(n-1)!}{(z+1)(z+2)\dots(z+n)} \quad (Rz > -1). \tag{20}$$

The right-hand sides are reciprocals of functions of type II, hence the determining functions $\Lambda(x)$ are all totally positive by Theorem 1[7] (converse part). The direct part of Theorem 1 may at times also be applied and that should be the direction of the more significant applications, if any. To illustrate this point, let us prove (the well-known fact) that $1/\Gamma(z)$ is of type II. By (18) it is sufficient to show that $\Lambda(x) = \exp\{-e^x\}$ is totally positive. But this is clear since

$$\det\|\Lambda(x_i - t_j)\| = \det\|\exp\{-e^{x_i}\cdot e^{-t_j}\}\| = \det\|\exp\{\alpha_i\beta_j\}\| > 0,$$

since both sequences of numbers

$$\alpha_i = e^{x_i},\ \beta_i = -e^{-t_i} \qquad (i = 1, 2, \dots, n)$$

are monotone increasing. A similar, equally simple argument applies to (19) showing that $\sin(\pi z)$ is of type II.

In order to illustrate Theorem 4 we inquire into the integral representation

$$\int_{-\infty}^{\infty} e^{xz}\Lambda_n(x)dx = (-1)^{n+1}\Gamma(z),\ (-n-1 < Rz < -n),\ (n > 0), \quad (21)$$

which is assured by Theorem 1. Starting from the representation (18), the various transitional residues appearing in (15) are readily determined with the result that

$$\Lambda_n(x) = (-1)^{n+1}\{e^{-e^x} - (1 - \frac{1}{1!}e^x + \frac{1}{2!}e^{2x} - \ldots \pm \frac{1}{n!}e^{nx})\},$$
$$(-\infty < x < \infty). \quad (22)$$

This function is totally positive by Theorem 1.

Formula (20) illustrates Theorem 5. Another example to Theorem 5 is furnished by the formula

$$\int_0^{\infty} e^{xz}\, dL(x) = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{z}{n^2}}, \qquad (Rz < 1), \qquad (23)$$

where the distribution function $L(x)$ is defined by

$$L(x) = \vartheta_0\left(0 \,\middle|\, \frac{ix}{\pi}\right) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-xn^2}, \qquad (x > 0,\ L(0) = 0).$$

Its derivative

$$\Lambda(x) = \begin{cases} \sum_{-\infty}^{\infty} (-1)^{n+1} n^2 e^{-xn^2} & \text{if } x > 0, \\ 0 & \text{if } x \leqq 0, \end{cases} \qquad (24)$$

is, by Theorem 5, a Polya frequency function.

5. *Concluding remarks.*—1. Various other properties of a t. p. function $\Lambda(x)$ are reflected in corresponding properties of the associated function $\Psi(z)$ of (12). Thus $\Lambda(x)$ is of class C^{∞} if, and only if, in (9) we have either $\gamma > 0$, or else $\gamma = 0$ and $\delta_\nu > 0$ for infinitely many ν. Thus the $\Lambda(x)$ of (24) is of class C^{∞}, i.e., all derivatives of $\Lambda(x)$ vanish at the origin. However, if $\gamma = 0$ and $\Psi(z)$ is a polynomial of degree $m(> 0)$ multiplied by an exponential factor, then $\Lambda(x)$ is exactly of class C^{m-2}. Thus the $\Lambda(x)$ of (20) has exactly $n - 2$ continuous derivatives.

2. It is expected that Polya frequency functions will be useful in descriptive statistics for the purpose of curve-fitting to empirical statistical data. One of the reasons for expecting such use is the extreme smoothness of these curves as a whole. By this we mean the following property: Let $\Lambda(x)$ be a Polya frequency function of class C^{∞}. By a repeated application of Rolle's theorem we conclude that $\Lambda^{(n)}(x)$ has *at least n* distinct real zeros. Actually $\Lambda^{(n)}(x)$ *has exactly n simple real zeros and this is true for all values of n.* An approach to Polya frequency functions from an entirely different

point of view will be discussed in a joint paper of H. B. Curry and the author.

3. A proof of Theorem 1 is essentially based on the results and methods developed by Polya and Schur. The only additional element required is a set of sufficient conditions insuring that a linear transformation be variation-diminishing.[8] Such conditions will serve to establish the following essential lemma: *If* $\Lambda(x)$ *is a Polya frequency function and* $t_1 < t_2 < \ldots < t_n$, *then the linear combination*

$$F(x) = a_1\Lambda(x - t_1) + a_2\Lambda(x - t_2) + \ldots + a_n\Lambda(x - t_n)$$

obeys the rule of signs of Descartes, i.e., the number of variations of sign of $F(x)$, *for all real* x, *cannot exceed the number of variations of signs in the sequence* $a_1, a_2, \ldots, a_n$, *of its coefficients.*

4. The discreet analogue of a t. p. function is a totally positive sequence $\{a_n\}$ $(n = 0, \pm 1, \pm 2, \ldots)$ with the property that the 4-way infinite matrix $||a_{i-j}||$ has only non-negative minors. The author expects to discuss the nature and properties of such sequences on a future occasion.

[1] Determinants related to the one appearing here were considered by A. Bloch and G. Polya, "Abschätzungen des Betrages einer Determinante," *Vierteljahrschrift Zürich*, **78**, 27–33 (1933).

[2] Polya, G., and Szegö, G., *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Problem 76, p. 49.

[3] Polya, G., and Schur, I., "Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen," *Journal für Math.*, **144**, 89–113 (1914), especially p. 93.

[4] Polya, G., "Über Annäherung durch Polynome mit lauter reellen Wurzeln," *Rendiconti di Palermo*, **36**, 1–17 (1913). Laguerre assumes uniform convergence in every finite domain. Polya assumes such convergence only in a neighborhood of the origin.

[5] Polya, G., "Algebraische Untersuchungen über ganze Funktionen vom Geschlechte Null und Eins," *Journal für Math.*, **145**, 224–249 (1915).

[6] Hamburger, H., "Bemerkungen zu einer Fragestellung des Herrn Polya," *Math. Z.*, **7**, 302–322 (1920).

[7] We are actually using here the following addition to Theorem 1: If the reciprocal of a function of type II is represented by a Laplace integral (12), where $\Lambda(x)$ is *continuous*, then $\Lambda(x)$ is totally positive.

[8] Schoenberg, I., "Über variationsvermindernde lineare Transformationen," *Math. Z.*, **32**, 321–328 (1930), especially Satz 1.