TOTAL POSITIVITY OF A CAUCHY KERNEL

THOMAS SIMON

Abstract. We study the total positivity of the kernel $1/(x^2 + 2\cos(\pi\alpha)xy + y^2)$ over $\mathbb{R}^+ \times \mathbb{R}^+$. The case of infinite order is characterized by an application of Schoenberg's theorem. We give necessary conditions for the case of finite order with the help of Chebyshev polynomials of the second kind. Sufficient conditions for the case of finite order are obtained thanks to the Izergin-Korepin determinant and its expression in terms of alternating sign matrices. A conjecture on the generating function of alternating sign matrices with a fixed number of negative entries arises in a natural way from our study. As a by-product, we give a partial answer to a question of Karlin on positive stable semigroups.

1. Introduction

Let I be a real interval and K a real kernel defined on $I \times I$. The kernel K is called totally positive of order n (TP_n) if

$$\det\left[K(x_i, y_j)\right]_{1\le i,j\le m} \;\ge\; 0$$

for every $m \in \{1, \ldots, n\}$, $x_1 < \ldots < x_m$ and $y_1 < \ldots < y_m$. If these inequalities hold for all n one says that K is TP_∞. The kernel K is called sign-regular of order n (SR_n) if there exists $\{\varepsilon_m\}_{1\le m\le n} \in \{-1, 1\}$ such that

$$\varepsilon_m \det\left[K(x_i, y_j)\right]_{1\le i,j\le m} \;\ge\; 0$$

for every $m \in \{1, \ldots, n\}$, $x_1 < \ldots < x_m$ and $y_1 < \ldots < y_m$. In the case $\varepsilon_m = (-1)^{m(m-1)/2}$ for all $m \in \{1, \ldots, n\}$, the kernel K is called reverse-regular of order n (RR_n). If these inequalities hold for all n one says that K is SR_∞ resp. RR_∞. The above properties are said to be strict, with corresponding notations STP, SSR, and SRR, when all involved inequalities are strict. We refer to [13] for the classic account on this field and its various connections with analysis, especially Descartes' rule of signs. Let us also mention the survey paper [14] for a statistical point of view on total positivity, and the recent monograph [21] for a linear algebraic point of view with updated and historical references.

A function $f : \mathbb{R} \to \mathbb{R}^+$ is called a Pólya frequency function of order $n \le \infty$ (PF_n) if the kernel $K(x, y) = f(x - y)$ is TP_n on $\mathbb{R} \times \mathbb{R}$. When this kernel is STP_n, we will use the notation $f \in \mathrm{SPF}_n$.

2010 *Mathematics Subject Classification.* 15A15, 15B48, 33C45, 60E07, 62E10.

Key words and phrases. Alternating sign matrix - Cauchy kernel - Chebyshev polynomial - Izergin-Korepin determinant - Positive stable semi-group - Rectangular matrix - Six-vertex model - Total positivity.

Probability densities belonging to the class PF_∞ have been characterized by Schoenberg - see e.g. Theorem 7.3.2 (i) p. 345 in [13] - through the meromorphic extension of their Laplace transform, whose reciprocal is an entire function having the following Hadamard factorization:

$$\frac{1}{\mathbb{E}[e^{sX}]} \;=\; e^{-\mu s^2+\delta s}\prod_{n=0}^{\infty}(1+a_n s)e^{-a_n s} \tag{1.1}$$

with X the associated random variable, $\mu \geq 0$, $\delta \in \mathbb{R}$, and $\sum a_n^2 < \infty$. The classical example is the Gaussian density. Pólya frequency functions of order 2 are easily characterized by the log-concavity on their support - see Theorems 4.1.8 and 4.1.9 in [13]. But except for the cases $n = 2$ or $n = \infty$ there is no handy criterion for testing the PF_n character of a given function resp. the TP_n character of a given kernel, and such questions might turn out to be difficult. In this paper we consider the simple kernel

$$K_\alpha(x,y) \;=\; \frac{1}{x^2+2\cos(\pi\alpha)xy+y^2}$$

over $(0,+\infty)\times(0,+\infty)$, with $\alpha \in [0,1)$. Because of its proximity to the standard Cauchy density we may and will call K_α a Cauchy kernel, although no confusion should be made with the traditional Cauchy kernel from complex analysis. Up to prefactors, the kernel K_α is the multiplicative convolution kernel associated with a Cauchy distribution or the additive convolution kernel associated with a generalized logistic distribution. The monograph [13] displays total positivity properties of many density functions, but it seems that in this respect the two above very classical distributions have escaped investigation so far. As will be seen below, the kernel K_α has also close connections to some aspects of the statistical mechanical six-vertex model with domain-wall boundary conditions. Our main result is the following

Theorem. (a) *One has*

$$K_\alpha \in \mathrm{STP}_\infty \iff K_\alpha \in \mathrm{SR}_\infty \iff \alpha \in \{1/2, 1/3, \ldots, 1/n, \ldots, 0\}.$$

(b) *For every positive integer n one has*

$$K_\alpha \in \mathrm{TP}_n \iff K_\alpha \in \mathrm{SR}_n \implies \alpha \in \{1/2, 1/3, \ldots, 1/n\} \;\text{ or }\; \alpha \leq 1/n.$$

(c) *One has $K_\alpha \in \mathrm{STP}_n$ if $n \in \{1, \ldots, 6\}$ and $\alpha \leq 1/n$ or $n \geq 7$ and $\alpha \leq 1/(n^2-n-6)$.*

The fact that $K_{1/2}$ and K_0 are STP_∞ is a direct consequence of classical closed formulæ for the determinants involved, due respectively to Cauchy and Borchardt - see e.g. (2.7) and (3.9) in [16] - and which will be recalled later in the paper. The characterization obtained in Part (a) follows without much difficulty from Schoenberg's theorem and will be given in Section 2. The proofs of

Parts (b) and (c) both rely on an analysis of the derivative determinant

$$\Delta_\alpha^n(x,y) \;=\; \det{}_n \left[\frac{\partial^{i+j-2} K_\alpha(x,y)}{\partial x^{i-1}\partial y^{j-1}}\right]$$

where, here and throughout, we set $\det_n$ for the determinant of a matrix whose rows and columns are indexed by $(i,j) \in \{1,\ldots,n\}^2$. To obtain Part (b) we establish a closed expression in terms of Chebyshev polynomials of the second kind for $\Delta_\alpha^n(1,0+) = \lim_{y\downarrow 0}\Delta_\alpha^n(1,y)$, an expression which implies that $\Delta_\alpha^k(1,0+)$ is negative for some $k \in \{2,\ldots,n\}$ whenever $\alpha \notin \{1/2, 1/3, \ldots, 1/n\}$ or $\alpha > 1/n$. The computations are performed in Section 3, in three different ways. The first one is a direct analysis, whereas the two other ones involve more elaborate combinatorial tools, alternating sign matrices resp. rectangular matrices. These two other approaches are introduced because they provide a deeper functional insight on Δ_α^n, whose everywhere positivity is crucial in order to apply Karlin's ETP criterion and show the STP_n character of K_α which is required in Part (c).

This last part of the Theorem is proved in Section 4, with the help of a certain form of the Izergin-Korepin formula. More precisely, $\Delta_\alpha^n(x,y)$ is obtained as a certain limit transformation of the following determinant

$$\bar{\mathrm{D}}_q(X,Y) \;=\; \det{}_n \left[\frac{1}{(x_i+y_j)(qx_i+y_j)}\right]$$

where q is some parameter and $X = \{x_1 < \ldots < x_n\}$ resp. $Y = \{y_1 < \ldots < y_n\}$ are two sets of variables converging to x resp. to y. There is an extensive literature on this latter determinant generalizing Cauchy's double-alternant, which was introduced by Izergin and Korepin in the context of the six-vertex model - see [15, 11, 12, 6, 2, 1] among other references, and also Chapter 7 in [5]. As mentioned to us by a referee, $\Delta_\alpha^n(x,y)$ is up to a prefactor the Izergin-Korepin determinant for the partition function of the homogeneous six-vertex model with domain-wall boundary conditions on an $n\times n$ grid, with row spectral parameter x, column spectral parameter y and Boltzmann weights $a(x,y) = e^{\mathrm{i}\pi\alpha/2}x + e^{-\mathrm{i}\pi\alpha/2}y$, $b(x,y) = e^{-\mathrm{i}\pi\alpha/2}x + e^{\mathrm{i}\pi\alpha/2}y$ and $c(x,y) = 2\sin(\pi\alpha)\sqrt{xy}$. See for example Equations (6) in [11], (7.12-7.14) in [12] or (48) in [2]. The fact that $a(x,y)^2 + b(x,y)^2 - c(x,y)^2 = 2a(x,y)b(x,y)\cos(\pi\alpha)$ means that the region of interest here is the disordered phase - see Section 1 in [4]. On the other hand, $\det_n[K_\alpha(x_i,y_j)]$ is the Izergin-Korepin determinant for the partition function of the inhomogeneous model on an $n\times n$ grid with row spectral parameters $x_1,\ldots,x_n$, column spectral parameters $y_1,\ldots,y_n$, and the same Boltzmann weights as previously. See for example Equations (5) in [11], (5.1-5.2) in [12] or (38) in [2].

To obtain Part (c) we express $\Delta_\alpha^n(x,y)$ in terms of alternating sign matrices, using a correspondence with the six-vertex model with domain-wall boundary conditions, which seems to have been first exploited in full force by Kuperberg [17] and which we learned in reading Chapter 7 of [5]. It is somehow puzzling that we need to appeal to such elaborate combinatorial tools in order to prove such a simple property for such an innocuous kernel. Nevertheless, all other attempts to obtain sufficient conditions for the total positivity of finite order of K_α, some of which will be mentioned throughout the text, turned out fruitless. We also stress that Part (c) of the Theorem is only a partial result, because our upper bound on α is probably not optimal.

Conjecture. *For every integer n, one has*

$$\alpha \leq 1/n \implies K_\alpha \in \mathrm{STP}_n. \tag{1.2}$$

This would show that the implication in Part (b) is actually an equivalence, which is true for $n \leq 6$ by Part (c). The proof of this conjecture seems however to require heavy computations. In Sections 4 and 5, we state three other conjectures of a combinatorial nature whose fulfilment would imply (1.2). The most natural one is Conjecture 4.1 which is a criterion for the strict positivity of the generating function

$$f_{n,k}(z) = \sum_{\substack{A \in \mathcal{A}_n \\ \mu(A)=k}} z^{\nu(A)} \bar{z}^{\nu(A^Q)}$$

evaluated at a certain complex number z, where $\mathcal{A}_n$ is the set of alternating sign matrices of size n, A^Q is the anticlockwise quarter-turn rotation of A, $\mu(A)$ is the number of negative entries and $\nu(A)$ the inversion number (see the precise notations below). Whereas we can easily show this criterion for $k = 0$ or $k =$ equal to the maximum number of -1's, unfortunately we cannot do so for all other k since there is no sufficiently explicit general formula for $f_{n,k}$. Let us stress that the single evaluation of $f_{n,k}(1) = \sharp\{A \in \mathcal{A}_n, \, \mu(A) = k\}$ is a difficult open problem, solved only for certain values of k - see [19] and the references therein. On the other hand, our conjecture on $f_{n,k}$ is of an analytical nature and might not require exact enumerative formulæ. We hope that this conjecture will produce interest among specialists of alternating sign matrices and the six-vertex model.

The present paper was initially motivated by an old question of S. Karlin - see Section 6 below for its precise statement - on the total positivity in space-time of the positive stable semi-group $(t,x) \mapsto p_\alpha(t,x)$ on $(0,+\infty) \times (0,+\infty)$, which we recall to be defined by the Laplace transform

$$\int_0^\infty p_\alpha(t,x) e^{-\lambda x} \, dx = e^{-t\lambda^\alpha}, \qquad \lambda \geq 0. \tag{1.3}$$

See Section 2 in [23] and the references therein for more details on the kernel p_α. Recall that this kernel is not explicit in general, except in the case $\alpha = 1/2$ where

$$p_{1/2}(t,x) = \frac{t}{2\sqrt{\pi x^3}} \, e^{-\frac{t^2}{4x}}.$$

As a simple consequence of Part (b), the following result is obtained in Section 6.

Corollary. *For every $n \geq 2$, one has*

$$p_\alpha \in \mathrm{SR}_n \implies \alpha \in \{1/2, 1/3, \ldots, 1/n\} \text{ or } \alpha \leq 1/n.$$

The reverse implication $\alpha \leq 1/n \Rightarrow p_\alpha \in \mathrm{TP}_n$ was proved in [23] for $n = 2$ and we believe that it is true in general. This would give a complete answer to Karlin's question and also imply the above conjecture - see Section 6 for an explanation. However, we believe that the proof of this reverse implication for all n is a very difficult problem.

2. Proof of Part (a) of the Theorem

This is the easy part, which will follow as an immediate consequence of the following Proposition 2.1 and Corollary 2.1.

Proposition 2.1. *One has*

$$K_\alpha \in \mathrm{STP}_\infty \iff K_\alpha \in \mathrm{TP}_\infty \iff \alpha \in \{1/2, 1/3, \ldots, 1/n, \ldots, 0\}.$$

Proof. We begin with the second equivalence. Consider the generalized logistic distribution with density

$$g_\alpha(x) = \frac{\sin(\pi\alpha)}{2\pi\alpha(\cosh(x) + \cos(\pi\alpha))}$$

over $\mathbb{R}$, and observe the relationship

$$g_\alpha(x - y) = \frac{\sin(\pi\alpha)e^{x+y}}{\pi\alpha} K_\alpha(e^x, e^y)$$

for all $x, y \in \mathbb{R}$. By Theorem 1.2.1. in [13], the TP_∞ resp. STP_∞ character of K_α amounts to the fact that $g_\alpha \in \mathrm{PF}_\infty$ resp. $g_\alpha \in \mathrm{SPF}_\infty$. For every $s \in (-1, 1)$, compute

$$\int_{\mathbb{R}} e^{sx} g_\alpha(x)\, dx = \frac{\sin(\pi\alpha)}{\pi\alpha} \int_0^\infty \frac{u^s}{u^2 + 2u\cos(\pi\alpha) + 1} du = \frac{\sin(\pi\alpha s)}{\alpha \sin(\pi s)} \tag{2.1}$$

where the right-hand side follows at once from the residue theorem and is meant as a limit for $\alpha = 0$. If $\alpha \notin \{1/2, 1/3, \ldots, 1/n, \ldots, 0\}$, the function

$$h_\alpha(s) = \frac{\alpha \sin(\pi s)}{\sin(\pi\alpha s)}$$

has a pole at $1/\alpha$ so that $g_\alpha \notin \mathrm{PF}_\infty$ by the aforementioned Theorem 7.3.2 (i) in [13]. If $\alpha = 0$, the Eulerian formula

$$\frac{\sin(\pi s)}{\pi s} = \prod_{k\geq 1}\left(1 - \frac{s^2}{k^2}\right) = \prod_{k\geq 1}\left(1 + \frac{(-1)^k s}{[(k+1)/2]}\right) e^{\frac{(-1)^{k-1}s}{[(k+1)/2]}},$$

where $[x]$ denotes the integer part of a positive real number x, shows that h_0 is of the type (1.1), in other words that $g_0 \in \mathrm{PF}_\infty$. If $\alpha = 1/n$ for some $n \geq 2$, the conclusion is the same in writing

$$\frac{\sin(\pi s)}{n\sin(\pi s/n)} \;=\; \prod_{\substack{k\,\geq\,1\\ n \text{ does not divide } k}} \left(1 - \frac{s^2}{k^2}\right).$$

This finishes the proof of the second equivalence. To show the first one, it remains to prove that $K_\alpha \in \mathrm{STP}_\infty$ whenever $\alpha \in \{1/2, 1/3, \ldots, 1/n, \ldots, 0\}$. Cauchy's double alternant formula (see e.g. (2.7) in [16] or Example 4.3 in [21])

$$\det_n\left[\frac{1}{x_i^2 + y_j^2}\right] \;=\; \frac{\prod_{1\leq i<j\leq n}(y_j^2 - y_i^2)(x_j^2 - x_i^2)}{\prod_{1\leq i,j\leq n}(x_i^2 + y_j^2)}$$

implies immediately that $K_{1/2} \in \mathrm{STP}_\infty$. Analogously, Borchardt's formula (see e.g. (3.9) in [16])

$$\det_n\left[\frac{1}{(x_i + y_j)^2}\right] \;=\; \det_n\left[\frac{1}{x_i + y_j}\right] \times \operatorname{perm}_n\left[\frac{1}{x_i + y_j}\right] \tag{2.2}$$

yields $K_0 \in \mathrm{STP}_\infty$. An alternative way to prove these two latter facts is to use the Laplace representation

$$\frac{1}{x^2 + y^2} \;=\; \int_0^\infty e^{-x^2 u} e^{-uy^2}\, du \quad \text{and} \quad \frac{1}{(x+y)^2} \;=\; \int_0^\infty e^{-xu} e^{-uy}\, udu.$$

Indeed, the composition formula - see Lemma 3.1.1 in [13] - implies that K_0 and $K_{1/2}$ are STP_∞ because the kernel e^{-ux} is SRR_∞ on $(0, +\infty) \times (0, +\infty)$ - see [13] p. 18.

The remaining cases $\alpha = 1/n$ for some $n \geq 3$ are more involved. First, it follows from the complement formula and the Legendre-Gauss multiplication formula for the Gamma function that

$$\frac{n\sin(\pi s/n)}{\sin(\pi s)} \;=\; \frac{\Gamma(1-s)\Gamma(1+s)}{\Gamma(1-s/n)\Gamma(1+s/n)} \;=\; \prod_{k=1}^{n-1} \frac{\Gamma(\frac{k}{n} - \frac{s}{n})\Gamma(\frac{k}{n} + \frac{s}{n})}{\Gamma(\frac{k}{n})\Gamma(\frac{k}{n})} \tag{2.3}$$

for all $s \in (-1, 1)$. For all $t > 0$, introduce now the Gamma random variable $\mathbf{G}_t$ with density

$$\frac{x^{t-1}e^{-x}}{\Gamma(t)} \mathbf{1}_{\{x>0\}}$$

and with fractional moments

$$\mathbb{E}[\mathbf{G}_t^s] \;=\; \frac{\Gamma(t+s)}{\Gamma(t)}, \qquad s > -t.$$

Setting $f_{1/n}(x) = x^{-1} g_{1/n}(\log x)$ for all $x > 0$ and putting (2.1) and (2.3) together imply after a change of variable that

$$\int_0^\infty x^s f_{1/n}(x) dx \;=\; \prod_{k=1}^{n-1} \frac{\Gamma(\frac{k}{n} - \frac{s}{n})\Gamma(\frac{k}{n} + \frac{s}{n})}{\Gamma(\frac{k}{n})\Gamma(\frac{k}{n})}$$

for all $s \in (-1,1)$. Identifying the fractional moments shows that $f_{1/n}$ is the density of the independent product

$$\prod_{k=1}^{n-1} \mathbf{G}_{\frac{k}{n}} \times \prod_{k=1}^{n-1} \tilde{\mathbf{G}}_{\frac{k}{n}}^{-1}$$

where $\tilde{\mathbf{G}}_t$ is meant as an independent copy of $\mathbf{G}_t$. Hence, the generalized logistic density function $g_{1/n}$ is the density of the independent sum

$$\sum_{k=1}^{n-1} \log(\mathbf{G}_{\frac{k}{n}}) - \sum_{k=1}^{n-1} \log(\tilde{\mathbf{G}}_{\frac{k}{n}}). \tag{2.4}$$

Moreover, the density of $\log(\mathbf{G}_t)$ reads

$$\frac{e^{tx-e^x}}{\Gamma(t)}$$

over $\mathbb{R}$ and is readily seen to be a SPF_∞ function - see Example (2.1) in [13] p.15. This property transfers then to $g_{1/n}$ thanks to the factorization (2.4) and the composition formula. From the above, this shows that $K_{1/n}$ is STP_∞ for all $n \geq 3$.

□

Remark 2.1. (a) The above argument based on the composition formula yields the STP_∞ character of all kernels $(ax+by+c)^{-d}$ for $a,b,d>0$ and $c \geq 0$ (this is the main result of [7] - see Theorem 3.1 therein), because

$$\frac{\Gamma(d)}{(ax+by+c)^d} = \int_0^\infty e^{-axu} e^{-byu} u^{d-1} e^{-cu}\, du.$$

Notice that there does not seem to exist any general closed formula for

$$\det_n \left[\frac{1}{(x_i+y_j)^d} \right].$$

When d is a positive integer, it has been conjectured in [20] that this determinant factorizes through the two Vandermonde determinants appearing in Cauchy's double alternant and some polynomial with non-negative coefficients.

(b) The discrete set characterizing the TP_∞ property of K_α is in one-to-one correspondence with the closure of the set $\{\cos(\pi/(n+1)), n \geq 1\}$ of the largest roots of the Chebyshev polynomials of the second kind. Recall - see e.g. Section 10.11 in [8] - that these polynomials are defined on $(-1,1)$ by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

and have generating function

$$\sum_{n\geq 0} (-1)^n z^n U_n(\cos \pi\alpha) = K_\alpha(1,z), \qquad |z|<1. \tag{2.5}$$

We will encounter these polynomials quite often in the remainder of this paper. Consider now the Gegenbauer polynomials $\{C_n^d, n \geq 0\}$ of order $d > 0$, having generating functions (see e.g. Section 10.9 in [8])

$$\sum_{n\geq 0}(-1)^n z^n \, C_n^d(t) \;=\; \frac{1}{(z^2+2tz+1)^d}, \qquad |t| \leq 1, |z| < 1.$$

Set $\{\lambda_n^d, n \geq 1\}$ for the sequence of their largest roots, which is known, by the interlacing property of classical orthogonal polynomials, to increase and converge to 1. In view of Proposition 2.1 and the above remark, the following is natural

Conjecture 2.1. *For every $d > 0$ and every $t \in [-1, 1]$,*

$$\frac{1}{(x^2+2txy+y^2)^d} \;\in\; \mathrm{TP}_\infty \iff t \;\in\; \{\lambda_n^d, n \geq 1\} \cup \{1\}.$$

Observe, however, that the set $\{\lambda_n^d, n \geq 1\}$ is not explicit in general.

(c) It is easy to see that

$$x \;\mapsto\; K_\alpha(\sqrt{x}, \sqrt{y}) \;=\; \frac{1}{x + 2\cos(\pi\alpha)\sqrt{xy} + y}$$

is a completely monotone function for every $y > 0$ and $\alpha \in [0, 1/2]$. Indeed, the function $x \mapsto x + 2\cos(\pi\alpha)\sqrt{xy} + y$ is then a Bernstein function, whose reciprocal is completely monotone - see Theorem 3.6 (ii) in [22] and the whole chapter 3 therein for an account on Bernstein functions. Hence, by Bernstein's theorem, there exists a positive finite kernel $L_\alpha(x, y)$ on $(0, +\infty) \times (0, +\infty)$ such that

$$K_\alpha(x, y) \;=\; \int_0^\infty e^{-x^2 z}\, L_\alpha(y, z) dz.$$

One has $L_{1/2}(y, z) = e^{-y^2 z}$, a SRR_∞ kernel which implies the STP_∞ character of $K_{1/2}$, as we saw above. When $\alpha \in (0, 1/2)$ the explicit computation mentioned in the introduction

$$\int_t^\infty \frac{ct}{2\sqrt{\pi(z-t)^3}} e^{-\frac{c^2t^2}{4(z-t)}}\, e^{-xz}\, dz \;=\; e^{-t(x+c\sqrt{x})}, \qquad c, t, x > 0$$

shows after some change of variable that

$$L_\alpha(y, z) \;=\; \frac{\cos(\pi\alpha) y\sqrt{z}}{2\sqrt{\pi}} \int_0^1 e^{-t(\frac{1-\sin^2(\pi\alpha)t}{1-t})y^2 z} \frac{t\, dt}{\sqrt{(1-t)^3}}.$$

However, it seems difficult to study how the sign-regularity of the kernel

$$\int_0^1 e^{-t(\frac{1-\sin^2(\pi\alpha)t}{1-t})yz} \frac{t\, dt}{\sqrt{(1-t)^3}},$$

which is a certain integral transform of a SRR_∞ kernel, depends on α. Such a study would provide direct information on the total positivity of K_α, thanks to the composition formula.

Proposition 2.2. *The function g_α is positive-definite for every $\alpha \in [0,1)$.*

Proof. The beginning of the proof of Proposition 2.1 implies by analytic continuation that

$$\int_{\mathbb{R}} e^{\mathrm{i}sx} g_\alpha(x)\, dx \;=\; \frac{\sinh(\pi\alpha s)}{\alpha\sinh(\pi s)}$$

for every $s \in \mathbb{R}$. Since $\alpha \in [0,1)$, we can apply the Fourier inversion formula to obtain

$$g_\alpha(x) \;=\; \int_{\mathbb{R}} e^{\mathrm{i}sx} \frac{\sinh(\pi\alpha s)}{2\pi\alpha\sinh(\pi s)}\, ds, \qquad x \in \mathbb{R}.$$

The conclusion follows from Bochner's theorem.

□

Remark 2.2. The above proposition is well-known - see e.g. [24] p.83 - but we gave a proof for the reader's benefit. It is also true that g_α^r is positive-definite for every $\alpha \in [0,1)$ and $r > 0$ - see e.g. Exercise 5.6.22 (ii) in [3]. This means that g_α is of the form $e^{-\psi_\alpha}$ where ψ_α is a Lévy-Khintchine exponent - see Proposition 4.4 and Theorem 4.12 in [22].

Corollary 2.1. *For every $\alpha \in [0,1)$ and $n \ge 2$, one has*

$$K_\alpha \in \mathrm{SR}_n \iff K_\alpha \in \mathrm{TP}_n.$$

Proof. We only need to prove the direct inclusion. Proposition 2.1 implies that for every $\alpha \in [0,1)$ the function

$$x \;\mapsto\; \frac{1+\cos(\pi\alpha)}{\cosh(x)+\cos(\pi\alpha)}$$

is the characteristic function of the random variable X_α with density

$$t \;\mapsto\; \frac{(1+\cos(\pi\alpha))\sinh(\pi\alpha t)}{\sin(\pi\alpha)\sinh(\pi t)}.$$

For every $n \ge 2, s_1, \ldots, s_n \in \mathbb{R}$ and $0 < x_1 < \ldots < x_n$, this yields

$$\sum_{1\le i,j\le n} s_i s_j K_\alpha(x_i,x_j) \;=\; \frac{1}{2(1+\cos(\pi\alpha))}\,\mathbb{E}\left[\left|\sum_{k=1}^n t_k e^{\mathrm{i}y_k X_\alpha}\right|^2\right] \;>\; 0,$$

with the notation $t_i = s_i/x_i$ and $y_i = \log(x_i)$. Hence, the quadratic form $[K_\alpha(x_i,x_j)]_{1\le i,j\le n}$ is positive definite for every $n \ge 2$ and $0 < x_1 < \ldots < x_n$. In particular, one has

$$\det\left[K_\alpha(x_i,x_j)\right]_{1\le i,j\le n} \;>\; 0,$$

for every $n \ge 2$ and $0 < x_1 < \ldots < x_n$. Supposing now that K_α is SR_n, the above implies that all terms in the sequence $\{\varepsilon_m\}_{1\le m\le n}$ defined in the introduction must be equal to 1, in other words that K_α is TP_n.

□

3. Proof of Part (b) of the Theorem

Let $X = \{x_1 < \ldots < x_n\}$ and $Y = \{y_1 < \ldots < y_n\}$ be two sets of positive variables and

$$V_X = \prod_{1\le i<j\le n} (x_j - x_i) \qquad \text{and} \qquad V_Y = \prod_{1\le i<j\le n} (y_j - y_i)$$

be the usual Vandermonde determinants. Consider $\mathrm{D}^n_\alpha(X,Y) = \det_n [K_\alpha(x_i, y_j)]$ and the aforementioned derivative determinant

$$\Delta^n_\alpha(x,y) = \det_n \left[\frac{\partial^{i+j-2} K_\alpha}{\partial x^{i-1} \partial y^{j-1}}(x,y)\right], \quad x, y > 0.$$

Using repeatedly the formula

$$\sum_{k=0}^p (-1)^{p-k} \binom{p}{k} f(z + k\varepsilon) \sim f^{(p)}(z)\varepsilon^p, \quad \varepsilon \to 0+$$

which is valid for any smooth real function f, elementary operations on rows and columns show that

$$\Delta^n_\alpha(x,y) = \mathrm{sf}(n-1)^2 \lim_{\varepsilon,\rho\to 0+} \frac{\mathrm{D}^n_\alpha(X_\varepsilon, Y_\rho)}{V_{X_\varepsilon} V_{Y_\rho}} \tag{3.1}$$

where we have set $x_i^\varepsilon = x + (i-1)\varepsilon, y_i^\rho = y + (i-1)\rho$ for $i = 1 \ldots n$, $X_\varepsilon = (x_1^\varepsilon, \ldots, x_n^\varepsilon)$, $Y_\rho = (y_1^\rho, \ldots, y_n^\rho)$, and

$$\mathrm{sf}(k) = \prod_{i=0}^k i!$$

for the superfactorial number. By Proposition 2.2, this implies that $\Delta^n_\alpha(x,x) \ge 0$ for any $x > 0$ and below in Remark 3.1 it will be established that the inequality is actually everywhere strict.

On the other hand, if $\Delta^k_\alpha(1, 0+) < 0$ for some $k \in \{2, \ldots, n\}$, then (3.1) implies that K_α is not TP_n and hence not SR_n by Corollary 2.1. We will prove that this is the case as soon as $\alpha > 1/n$ and $\alpha \notin \{1/2, 1/3, \ldots, 1/n\}$. More precisely, setting

$$U_k^\alpha = \frac{\sin k\pi\alpha}{\sin \pi\alpha} \qquad \text{and} \qquad V_n^\alpha = \prod_{k=1}^n U_k^\alpha$$

for every $k, n \ge 1$, we will show that

$$\Delta^n_\alpha(1, 0+) = \mathrm{sf}(n-1)^2 V_n^\alpha. \tag{3.2}$$

We will give three different proofs of (3.2). The first proof is a direct computation and was kindly communicated to us by a referee. The two other proofs rely on more elaborate combinatorial results on the Izergin-Korepin determinant. These approaches, though more involved, provide deeper insight on conditions for the everywhere positivity of the functional determinant $\Delta^n_\alpha(x,y)$, which is the key-argument for Part (c) of the Theorem. They will be discussed in further detail in Section 4 resp. Section 5 of this paper.

3.1. **Direct proof.** In the above notation, let $\varepsilon > 0$ be fixed. Similarly as for (3.1) we have

$$\det{}_n \left[\frac{\partial^{j-1} K_\alpha}{\partial y^{j-1}}(x_i^\varepsilon, 0+) \right] \; = \; \mathrm{sf}(n-1) \lim_{\rho, y \to 0} \frac{\mathrm{D}_\alpha^n(X_\varepsilon, Y_\rho)}{V_{Y_\rho}}.$$

On the other hand, setting $q = e^{\mathrm{i}\pi\alpha}$, we have for all $x > 0$

$$\begin{aligned} \frac{\partial^{j-1} K_\alpha}{\partial y^{j-1}}(x, 0+) \; &= \; \frac{\partial^{j-1}}{\partial y^{j-1}} \left[\frac{1}{x^2 + (q+q^{-1})xy + y^2} \right] (x, 0+) \\ &= \; \frac{1}{(q-q^{-1})x} \times \frac{\partial^{j-1}}{\partial y^{j-1}} \left[\frac{1}{q^{-1}x + y} - \frac{1}{qx+y} \right] (x, 0+) \\ &= \; \frac{(-1)^{j-1}(j-1)!}{(q-q^{-1})x} \times \left(\frac{1}{q^{-j}x^j} - \frac{1}{q^j x^j} \right) \; = \; (j-1)! U_j^\alpha (-x^{-1})^{j+1}. \end{aligned}$$

Hence,

$$\begin{aligned} \det{}_n \left[\frac{\partial^{j-1} K_\alpha}{\partial y^{j-1}}(x_i^\varepsilon, 0+) \right] \; &= \; \mathrm{sf}(n-1) V_n^\alpha \times \det{}_n \left[(-x_i^\varepsilon)^{-(j+1)} \right] \\ &= \; \mathrm{sf}(n-1) V_n^\alpha \times \frac{V_{X_\varepsilon}}{\prod_{i=1}^n (x_i^\varepsilon)^{n+1}} \end{aligned}$$

by the Vandermonde identity. Putting everything together shows that

$$\lim_{\substack{\varepsilon, \rho \to 0 \\ x \to 1, y \to 0}} \frac{\mathrm{D}_\alpha^n(X_\varepsilon, Y_\rho)}{V_{X_\varepsilon} V_{Y_\rho}} \; = \; V_n^\alpha$$

and the required formula (3.2) follows directly from (3.1).

□

3.2. **Proof with alternating sign matrices.** This second argument hinges upon an expression for $\bar{\mathrm{D}}_q(X, Y)$ (as defined in Section 1) in terms of alternating sign matrices, which is given in Exercise 7.2.13 p. 244 in [5] as a variation on Izergin-Korepin's original formula [11, 15]. This expression, which leads to the closed formula (3.3) for $\Delta_\alpha^n(x, y)$ and will be used in Section 4 for proving Part (c) of the Theorem, reads

$$\bar{\mathrm{D}}_q(X, Y) \; = \; \frac{V_X V_Y}{\mathrm{P}_q(X, Y)} \sum_{A \in \mathcal{A}_n} (-1)^{\mu(A)} (1-q)^{2\mu(A)} q^{n(n-1)/2 - \mu(A) - \nu(A)} \prod_{i=1}^n x_i^{\mu_i(A)} y_i^{\mu^i(A)} \prod_{\substack{1 \le i, j \le n \\ A_{ij} = 0}} (\alpha_{ij} x_i + y_j)$$

where $\mathcal{A}_n$ stands for the set of $n \times n$ alternating sign matrices (ASM) viz. those matrices made out of 0's, 1's and -1's for which the sum of the entries in each row and each column is 1, and the non-zero entries in each row and column alternate in sign. We refer to [5] for a comprehensive account on this topic, and also to Section 3 in [1] for updated results. Above, the following notations are used: $\mu_i(A)$ resp. $\mu^i(A)$ is the number of -1's in the i-th row resp. i-th column of A, $\mu(A)$ the total number of -1's in A, $\nu(A)$ the generalized inversion number of A viz.

$$\nu(A) \;=\; \sum_{\substack{1\le i<i'\le n\\ 1\le j'\le j\le n}} A_{ij}A_{i'j'},$$

and

$$\alpha_{ij} \;=\; \begin{cases} q & \text{if } \sum_{k\le i} A_{kj} \;=\; \sum_{l\le j} A_{il},\\ 1 & \text{otherwise.}\end{cases}$$

Notice that ASM matrices without negative entries are permutation matrices, $\nu(A)$ being then the standard number of inversions in A. In particular, one recovers Borchardt's formula (2.2) in setting $q = 1$ in Izergin-Korepin's formula - see Exercise 7.2.14 in [5]. Setting $q = e^{2\mathrm{i}\pi\alpha}$, we see that (3.1) implies after some simplifications the following closed expression for the derivative determinant:

$$\frac{\Delta^n_\alpha(x,y)}{K_\alpha(x,y)^{n^2}} \;=\; \mathrm{sf}(n-1)^2\sum_{A\in\mathcal{A}_n}(P_\alpha(x,y))^{\mu(A)}(Q_\alpha(x,y))^{2\nu(A)}(\bar{Q}_\alpha(x,y))^{n(n-1)-2\mu(A)-2\nu(A)} \tag{3.3}$$

with the notations

$$P_\alpha(x,y) \;=\; 4\sin^2(\pi\alpha)xy \qquad \text{and} \qquad Q_\alpha(x,y) \;=\; e^{\mathrm{i}\pi\alpha/2}x + e^{-\mathrm{i}\pi\alpha/2}y.$$

This yields

$$\Delta^n_\alpha(1,0+) \;=\; \mathrm{sf}(n-1)^2 e^{-\mathrm{i}\pi n(n-1)\alpha/2}\sum_{A\in\mathcal{S}_n} e^{2\mathrm{i}\pi\alpha\nu(A)},$$

where $\mathcal{S}_n$ stands for the set of permutation matrices of size n. Using the generating function of $\nu(A)$ for permutation matrices given e.g. in Corollary 3.5 of [5], further trigonometric simplifications imply

$$\Delta^n_\alpha(1,0+) \;=\; \mathrm{sf}(n-1)^2 V^\alpha_n,$$

as required by (3.2).

□

Remark 3.1. The formula (3.3) also shows that

$$\Delta^n_\alpha(x,x) \;=\; \frac{\mathrm{sf}(n-1)^2}{(2x\cos(\pi\alpha/2))^{n(n+1)}}\sum_{A\in\mathcal{A}_n}(4\sin^2(\pi\alpha/2))^{\mu(A)} \;>\; 0$$

for all $x > 0$ and $\alpha \in [0,1)$. It seems difficult to simplify this expression, which might explain why ASM matrices are the appropriate combinatorial objects for investigating thoroughly the everywhere positivity of $\Delta^n_\alpha(x,y)$.

3.3. **Proof with rectangular matrices.** This third proof of (3.2) uses an evaluation of the Izergin-Korepin determinant in terms of rectangular matrices separating the variables, which is due to Lascoux - see Theorem q in [18]. The interest of this approach is the polynomial expression (3.6) with coefficients given by (3.7), which will be discussed in greater detail in Section 5. Lascoux's evaluation reads

$$\bar{\mathrm{D}}_q(X,Y) \;=\; \frac{V_X V_Y}{\mathrm{P}_q(X,Y)}\,\det_n\left[H_X E_Y^q\right]$$

where $\mathrm{P}_q(X,Y) \;=\; \prod_{1\le i,j\le n}(x_i+y_j)(qx_i+y_j)$, and

$$H_X \;=\; \left[h_{k-i}(X)\right]_{1\le i\le n,1\le k\le 2n-1}, \qquad E_Y^q \;=\; \left[\frac{q^{k-j+1}-q^{j-1}}{q-1}\,e_{n-k+j-1}(Y)\right]_{1\le k\le 2n-1,1\le j\le n}$$

are two rectangular matrices involving the complete resp. elementary symmetric functions, which we recall to be defined through the generating functions

$$\sum_{k\ge 0} h_k(X)t^k \;=\; \prod_{r=1}^{n}(1-x_r t)^{-1} \qquad \text{and} \qquad \sum_{k\ge 0} e_k(Y)t^k \;=\; \prod_{r=1}^{n}(1+y_r t). \tag{3.4}$$

Setting $q=e^{2\mathrm{i}\pi\alpha}$, $X_x^q=(xq^{-1/2},\ldots,xq^{-1/2})$, and $Y_y=(y,\ldots,y)$, Lascoux's formula and (3.1) yield

$$\Delta_\alpha^n(x,y) \;=\; \mathrm{sf}(n-1)^2\, q^{-n(n-1)/4} K_\alpha(x,y)^{n^2}\,\det_n\left[H_{X_x^q}E_{Y_y}^q\right].$$

By (3.4), we have $h_r(X_x^q)=\binom{n+r-1}{r}x^r q^{-r/2}$, whence

$$H_{X_x^q} \;=\; \left[\binom{n-1+k-i}{k-i}q^{(i-k)/2}x^{k-i}\right]_{1\le i\le n,1\le k\le 2n-1}.$$

On the other hand, (3.4) implies $e_r(Y_y)=\binom{n}{r}y^r$, so that after some simplifications

$$E_{Y_y} \;=\; \left[\binom{n}{n-k+j-1}q^{(k-1)/2}U_{k+2-2j}^{\alpha}y^{n-k+j-1}\right]_{1\le k\le 2n-1,1\le j\le n},$$

with our previous notation for U_r^α. The i-th row of the product $H_{X_x^q}E_{Y_y}$ having a factor $q^{(i-1)/2}$, we finally obtain

$$\Delta_\alpha^n(x,y) \;=\; \mathrm{sf}(n-1)^2\, K_\alpha(x,y)^{n^2}\,\det_n\left[A_n(x)B_n^\alpha(y)\right] \tag{3.5}$$

with the notations

$$A_n(x) \;=\; \left[\binom{n-1+k-i}{k-i}x^{k-i}\right]_{1\le i\le n,1\le k\le 2n-1}$$

and

$$B_n^\alpha(y) \;=\; \left[\binom{n}{n-k+j-1}U_{k+2-2j}^{\alpha}y^{n-k+j-1}\right]_{1\le k\le 2n-1,1\le j\le n}.$$

Setting $L_n^\alpha(x,y) = \det_n[A_n(x)B_n^\alpha(y)]$, the Cauchy-Binet formula implies

$$L_n^\alpha(x,y) \;=\; \sum_{1\le\sigma_1<\ldots<\sigma_n\le 2n-1} \mathrm{A}_\sigma(x)\mathrm{B}_\sigma^\alpha(y) \tag{3.6}$$

where $\mathrm{A}_\sigma(x)$ is the $n\times n$ minor obtained from the columns $\sigma_1,\ldots,\sigma_n$ in $A_n(x)$ and $\mathrm{B}_\sigma^\alpha(y)$ is the $n\times n$ minor obtained from the rows $\sigma_1,\ldots,\sigma_n$ in $B_n^\alpha(y)$. By Leibniz's formula, one has

$$\mathrm{A}_\sigma(x) \;=\; \mathrm{A}_\sigma(1)x^{n_\sigma} \quad\text{and}\quad \mathrm{B}_\sigma^\alpha(y) \;=\; \mathrm{B}_\sigma^\alpha(1)y^{n(n-1)-n_\sigma}$$

with the notation

$$n_\sigma \;=\; \sum_{i=1}^n(\sigma_i - i).$$

This shows that $L_n^\alpha(x,y)$ is a homogeneous polynomial of degree $n(n-1)$ with coefficient

$$\sum_{n_\sigma=k}\mathrm{A}_\sigma(1)\mathrm{B}_\sigma^\alpha(1) \tag{3.7}$$

for the term $x^k y^{n(n-1)-k}$. Besides, by (3.5) and symmetry, we have $L_n^\alpha(x,y) = L_n^\alpha(y,x)$ so that these coefficients are palindromic. We compute

$$L_n^\alpha(1,0+) \;=\; L_n^\alpha(0+,1) \;=\; \mathrm{A}_{\tilde{\mathrm{Id}}}(1)\mathrm{B}_{\tilde{\mathrm{Id}}}^\alpha(1)$$

with the notation $\tilde{\mathrm{Id}} = (n,\ldots,2n-1)$. One finds immediately $\mathrm{B}_{\tilde{\mathrm{Id}}}^\alpha(1) = V_n^\alpha$ and

$$\mathrm{A}_{\tilde{\mathrm{Id}}}(1) \;=\; \det_n\left[\binom{2n-2-i+j}{n-1-i+j}\right] \;=\; 1.$$

Since $K_\alpha(0+,1)=1$, we finally deduce (3.2).

□

Remark 3.2. Using (2.5) and Schur function arguments similar to those developed in [10], it is possible to derive yet another formula for the derivative determinant:

$$\Delta_\alpha^n(x,y) \;=\; \frac{1}{x^{n(n+1)}}\sum_{k=0}^\infty(-yx^{-1})^k \sum_{\substack{\mu_1\,\le\,\cdots\,\le\,\mu_n\\ \mu_1\,+\,\cdots\,+\,\mu_n\,=\,k}} \left(\prod_{i=1}^n U_{i+\mu_i}^\alpha\right)\prod_{1\le i<j\le n}(\mu_j-\mu_i+j-i)^2,$$

which again boils down to (3.2) when $x=1$ and $y\to 0$. Because of the alternating signs, this expression does not seem however very helpful to study the everywhere positivity of $\Delta_\alpha^n(x,y)$.

4. Proof of Part (c) of the Theorem

This last part of the theorem relies on Karlin's ETP criterion on the derivative determinant - see Theorem 2.2.6 in [13] - which states that if $\Delta_\alpha^k(x,y)>0$ for every $k\in\{2,\ldots,n\}$ and $x,y>0$, then K_α is STP_n. In order to apply this criterion we will appeal to certain considerations on ASM matrices, all to be found in [5] and Section 2.1 of [1]. Set $\mu_n = \max\{\mu(A),\ A\in\mathcal{A}_n\}$ and notice that $\mu_n = (n-1)^2/4$ for n odd and that $\mu_n = n(n-2)/4$ for n even - see p. 444 in [1]. From (3.3), it is clear that if

$$F^\alpha_{n,k}(x,y) \;=\; \sum_{\substack{A\in\mathcal{A}_n\\ \mu(A)=k}} Q_\alpha(x,y)^{2\nu(A)}\bar{Q}_\alpha(x,y)^{n(n-1)-2\nu(A)-2k} \;>\; 0$$

for all $k = 0\ldots\mu_n$, then $\Delta^n_\alpha(x,y) > 0$. Notice first that $F^\alpha_{n,k}(x,y)$ is real since it can be written

$$\frac{1}{2}\sum_{\substack{A\in\mathcal{A}_n\\ \mu(A)=k}} (Q_\alpha(x,y)^{2\nu(A)}\bar{Q}_\alpha(x,y)^{n(n-1)-2\nu(A)-2k} + \bar{Q}_\alpha(x,y)^{2\nu(A)}Q_\alpha(x,y)^{n(n-1)-2\nu(A)-2k}).$$

Indeed, setting A^Q for the anticlockwise quarter-turn rotation of A, one has $A^Q \in \mathcal{A}_n$ with $\mu(A^Q) = \mu(A)$ and $2\nu(A^Q) = n(n-1) - 2\nu(A) - 2\mu(A)$ - see p. 444 in [1]. Introducing the function

$$G_{n,k}(\theta) \;=\; \sum_{\substack{A\in\mathcal{A}_n\\ \mu(A)=k}} e^{\mathrm{i}\theta(\nu(A)-\nu(A^Q))},$$

it follows that

$$F^\alpha_{n,k}(x,y) \;=\; K_\alpha(x,y)^{k-n(n-1)/2}G_{n,k}\left(2\arctan(\frac{|x-y|}{x+y}\tan(\pi\alpha/2))\right)$$

and we are hence reduced to a positivity criterion on $G_{n,k}$. Observe that for every $\delta \in [0,1/2]$ the condition

$$2\arctan(\frac{|x-y|}{x+y}\tan(\pi\alpha/2)) \;<\; \pi\delta \quad \text{for all } x,y>0$$

amounts to $\alpha \le \delta$. For $k = 0$, a straightforward computation using the generating function of permutation matrices - see again Corollary 3.5 in [5] - shows that

$$G_{n,0}(\theta) \;=\; V_n^{\theta/\pi}, \tag{4.1}$$

so that from the above we obtain $F^\alpha_{n,0}(x,y) > 0$ for all $x,y>0$ as soon as $\alpha \le 1/n$. Supposing now that $k \ge 1$, then necessarily one has $\nu(A) \ge 1$ and $\nu(A^Q) \ge 1$ - see p. 444 in [1]. We write

$$G_{n,k}(\theta) \;=\; \Re\,(G_{n,k}(\theta)) \;=\; \sum_{\substack{A\in\mathcal{A}_n\\ \mu(A)=k}} \cos\left(\theta(\nu(A)-\nu(A^Q))\right) \;=\; \sum_{i=1}^{\nu_{n,k}} a^i_{n,k}\,\cos\left(\theta(n^2-n-2k-4i)/2\right)$$

where $a^i_{n,k}$ are non-negative integer coefficients and $\nu_{n,k} = \max\{\nu(A),\ A\in\mathcal{A}_n, \mu(A)=k\}$. Notice in passing that no closed expression for $a^i_{n,k}$ or even $\nu_{n,k}$ seem available in the literature. For $n=3$ we have $G_{3,1} = 1$. For $n \ge 4$, it is clear that $G_{n,k}(\theta) > 0$ as soon as $\theta \le \pi/(n^2-n-6)$. Since the function $n \mapsto n^2-n-6$ increases, putting everything together we deduce that if $n \in \{1,2,3\}$ and $\alpha \le 1/n$ or $n \ge 4$ and $\alpha \le 1/(n^2-n-6)$, then K_α is STP_n.

To conclude Part (c), it remains to prove that the following conjecture is true for $n \in \{4,5,6\}$.

Conjecture 4.1. *For every $n \ge 2$ and every $k = 0,\ldots,\mu_n$, one has $G_{n,k}(\theta) > 0$ if $\theta \in [0,\pi/n)$.*

Notice that the validity of this conjecture for all n, k would imply that for all n the kernel K_α is STP_n if $\alpha \le 1/n$, which is the conjecture stated in the introduction. The above formula (4.1) shows that this conjecture is true for all n and $k = 0$. We can also handle the case $k = \mu_n$.

Proposition 4.1. *For every $n \ge 3$ one has $G_{n,\mu_n}(\theta) > 0$ whenever $\theta \in [0, \pi/n)$.*

Proof. First, suppose that $n = 2p+1$ is odd. Then $\mu_n = p^2$ and this concerns one single matrix with inversion number $\nu = p(p+1)/2$ - see p. 444 in [1]. We deduce $G_{n,\mu_n}(\theta) = 1$. Second, suppose that $n = 2p$ is even. Then $\mu_n = p(p-1)$ and this concerns two matrices with inversion numbers $\nu = p(p+1)/2$ resp. $\nu = p(p-1)/2$ - see again p. 444 in [1]. We deduce $G_{n,\mu_n}(\theta) \;=\; 2\cos(n\theta/2)$, which is positive if $\theta \in [0, \pi/n)$.

□

Consider now the bivariate generating function

$$Z_n(u,v) \;=\; \sum_{A \in \mathcal{A}_n} u^{\nu(A)} v^{\mu(A)},$$

which is itself a certain functional determinant - see formula (57) p. 459 in [1] and the proof of this formula in Propositions 1-3 of [2]. The function $G_{n,k}(\theta)$ is expressed in terms of the generating function

$$Z_{n,k}(u) \;=\; \frac{1}{k!}\frac{\partial^k Z_n}{\partial v^k}(u,0) \;=\; \sum_{\substack{A \in \mathcal{A}_n \\ \mu(A)=k}} u^{\nu(A)}.$$

Unfortunately, there is no general closed formula for the latter. But the package written for [1], available on R. Behrend's webpage at Cardiff University, allows us to compute these functions for n, k fixed. Using the notations $[m]_u = 1 + \ldots + u^{m-1}$, $[m]_u! = 1 \times \cdots \times [m]_u$, $\varphi = \theta/\pi$ and U_p^φ, V_p^φ as in Section 3, let us list the functions $Z_{n,k}$ and the associated remaining $G_{n,k}$ for $n = 4, 5, 6$. Recall that $\mu_4 = 2, \mu_5 = 4$ and $\mu_6 = 6$.

- $Z_{4,1}(u) = 2u(1 + 3u + 3u^2 + u^3) = 2u(1+u)^3$.
  $G_{4,1}(\theta) = 16\cos^3(\theta) \;>\; 0$ if $\theta \in [0, \pi/4)$.
- $Z_{5,1}(u) = u(3 + 14u + 34u^2 + 49u^3 + 49u^4 + 34u^5 + 14u^6 + 3u^7) = 3u[3]_u![5]_u + 5u^2[3]_u[4]_u + 9u^3[4]_u + 7u^4[2]_u$.
  $G_{5,1}(\theta) = 3V_3^\varphi U_5^\varphi + 5U_3^\varphi U_4^\varphi + 9U_4^\varphi + 7U_2^\varphi \;>\; 0$ if $\theta \in [0, \pi/5)$.
- $Z_{5,2}(u) = u(2 + 12u + 21u^2 + 24u^3 + 21u^4 + 12u^5 + 2u^6) = 2u(1+u^2)^3 + 12u^2[5]_u + 3u^3[3]_u + 9u^4$.
  $G_{5,2}(\theta) = 16\cos^3(2\theta) + 12U_5^\varphi + 3U_3^\varphi + 9 \;>\; 0$ if $\theta \in [0, \pi/5)$.

- $Z_{5,3}(u) = u(1+6u^2+6u^3+u^5) = u(1+u)(3u^2+(1+u^2)^2-u(1-u)^2)$.
  $G_{5,3}(\theta) = 2\cos(\theta)(3+4\cos^2(2\theta)+4\sin^2(\theta)) > 0$ if $\theta \in [0,\pi/5)$.

- $Z_{6,1}(u) = u(4+24u+78u^2+176u^3+298u^4+400u^5+440u^6+400u^7+298u^8+176u^9+78u^{10}+24u^{11}+4u^{12}) = 4u[4]_u[5]_u[6]_u + 12u^2[2]_u[5]_u[6]_u + 18u^3[4]_u[6]_u + 40u^4[2]_u[6]_u + 24u^5[5]_u + 48u^6[3]_u + 24u^7$.
  $G_{6,1}(\theta) = 4U_4^\varphi U_5^\varphi U_6^\varphi + 12U_2^\varphi U_5^\varphi U_6^\varphi + 18U_4^\varphi U_6^\varphi + 40U_2^\varphi U_6^\varphi + 24U_5^\varphi + 48U_3^\varphi + 24 > 0$ if $\theta \in [0,\pi/6)$.

- $Z_{6,2}(u) = u(3+27u+111u^2+252u^3+415u^4+534u^5+534u^6+415u^7+252u^8+111u^9+27u^{10}+3u^{11}) = 3u[4]_u![6]_u + 15u^2[3]_u[4]_u[5]_u + 39u^3[3]_u[6]_u + 39u^4[6]_u + 64u^5[4]_u + 80u^6[2]_u$.
  $G_{6,2}(\theta) = 3V_4^\varphi U_6^\varphi + 15U_3^\varphi U_4^\varphi U_5^\varphi + 39U_3^\varphi U_6^\varphi + 39U_6^\varphi + 64U_4^\varphi + 80U_2^\varphi > 0$ if $\theta \in [0,\pi/6)$.

- $Z_{6,3}(u) = u(2+20u+56u^2+140u^3+260u^4+328u^5+260u^6+140u^7+56u^8+20u^9+2u^{10}) = 2u[5]_u! + 12u^2[2]_u[4]_u[5]_u + 2u^3[4]_u! + 44u^4[5]_u + 82u^5[3]_u + 40u^6$.
  $G_{6,3}(\theta) = 2V_5^\varphi + 12U_2^\varphi U_4^\varphi U_5^\varphi + 2V_4^\varphi + 44U_5^\varphi + 82U_3^\varphi + 40 > 0$ if $\theta \in [0,\pi/6)$.

- $Z_{6,4}(u) = u(1+21u^2+55u^3+78u^4+78u^5+55u^6+21u^7+u^9) = u[2]_u((1+u^4)^2 - u(1-u)^2(1+u^2)[3]_u + 20u^4) + 21u^3[6]_u + 34u^4[4]_u$.
  $G_{6,4}(\theta) = 2\cos(\theta)(4\cos^2(4\theta) + 8\sin^2(\theta)\cos(2\theta)U_3^\varphi + 20) + 21U_6^\varphi + 34U_4^\varphi > 0$ if $\theta \in [0,\pi/6)$.

- $Z_{6,5}(u) = 6u^3(1+2u+2u^3+u^4) = 6u^3(u(1+u^2)+(1+u)(1+u^3))$.
  $G_{6,5}(\theta) = 12(\cos(2\theta)+2\cos(\theta)\cos(3\theta)) > 0$ if $\theta \in [0,\pi/6)$.

We see from this list that Conjecture 4.1 is true for $n = 4,5,6$, which completes the proof of Part (c).

□

Remark 4.1. (a) It seems that

$$Z_{n,\mu_n-1}(u) = \begin{cases} u^{p(p-1)/2}(p(p-1)+p(p+1)u+p(p+1)u^p+p(p-1)u^{p+1}) & \text{if } n=2p \\ u^{p(p-1)/2}(1+p(p+1)u^p+p(p+1)u^{p+1}+u^{2p+1}) & \text{if } n=2p+1 \end{cases}$$

for every $n \geq 4$. If these formulæ are true, then one can show Conjecture 4.1 for all n and $k = \mu_n - 1$ as well.

(b) The generating functions $Z_{5,3}(u), Z_{6,4}(u), Z_{6,5}(u)$ have vanishing coefficients and for the first two functions we need to introduce negative coefficients in u in order to get a sum of positive terms for $G_{n,k}(\theta)$. Testing the positivity of $G_{n,k}$ when the corresponding $Z_{n,k}$ has vanishing coefficients seems to require bizarre rearrangements, and we are currently unable to find a general procedure.

(c) Defining the real polynomial

$$p_{n,k}(t) \;=\; \sum_{\substack{A\in\mathcal{A}_n\\ \mu(A)=k}} (1+\mathrm{i}t)^{\nu(A)}(1-\mathrm{i}t)^{\nu(A^Q)},$$

it follows that $G_{n,k}(\theta) = (\cos(\theta))^{n(n-1)/2-k} p_{n,k}(\tan\theta)$ and $p_{n,k}(0) = \sharp\{A \in \mathcal{A}_n,\ \mu(A) = k\} > 0$. Therefore, if either $p_{n,k}(t)$ has no positive roots or its smallest positive root satisfies $t_0 > \tan(\pi/n)$ for a particular n and k, then Conjecture 4.1 is valid for that n and k. Using this approach, and computing $p_{n,k}(t)$ with the determinantal formula (57) of [1], it is possible to verify Conjecture 4.1 numerically on Mathematica for all cases with $n \le 9$.

5. A tentative approach with rectangular matrices

In this section we display other heuristic reasons supporting the validity of the implication

$$\alpha \le 1/n \implies \Delta^n_\alpha(x,y) > 0 \quad \text{for all } x, y > 0. \tag{5.1}$$

We will discuss more thoroughly Lascoux's factorization introduced in Paragraph 3.3 and settle two problems on the positivity on certain minors of rectangular matrices involving Chebyshev polynomials, whose solution would imply (5.1). Using the previous notation for $\mathrm{A}_\sigma(1)$ and $\mathrm{B}^\alpha_\sigma(1)$, we first observe that formulæ (3.5), (3.6) and (3.7) show that the positivity of Δ^n_α is ensured by that of $\mathrm{A}_\sigma(1)$ and $\mathrm{B}^\alpha_\sigma(1)$. The analysis for $\mathrm{A}_\sigma(1)$ is easy.

Proposition 5.1. *For every $\sigma : \{1,\ldots,n\} \to \{1,\ldots,2n-1\}$ increasing, one has $\mathrm{A}_\sigma(1) > 0$.*

Proof. The generating function

$$\sum_{r\ge 0} \binom{n-1+r}{r} x^r \;=\; \frac{1}{(1-x)^n}$$

and Edrei's criterion - see Theorem 1.2 p. 394 in [13] - show that the sequence

$$\left\{ \binom{n-1+r}{r},\, r \ge 0 \right\}$$

is TP_∞. In other words, the matrix $A_n(1)$ is TP_∞ viz. all its minors are non-negative. In particular, all coefficients $\mathrm{A}_\sigma(1)$ are non-negative. We now compute its exact positive value with the help of a formula of Gessel and Viennot on binomial determinants. Subtracting the i-th row from the $(i-1)$-th row successively for $i = n \ldots 2$ and then repeating this operation for $i = (n-1)\ldots 2$, $i = (n-2)\ldots 2$, $\ldots$ we see from the Pascal relationships on binomial coefficients that the minor $\mathrm{A}_\sigma(1)$ is actually taken from the matrix

$$\left[\binom{k-1}{k-i}\right]_{1\le i\le n, 1\le k\le 2n-1} \;=\; \left[\binom{k}{i}\right]_{0\le i\le n-1, 0\le k\le 2n-2}.$$

The alternative formula mentioned at the bottom of p. 308 in [9] implies

$$\mathrm{A}_\sigma(1) \;=\; \frac{\prod_{1\le i<j\le n}(\sigma_j-\sigma_i)}{\mathrm{sf}(n-1)}.$$

In particular $\mathrm{A}_\sigma(1)$ is always a positive integer, equal to one if and only if $\sigma_n-\sigma_1=n-1$.

□

The analysis for $\mathrm{B}^\alpha_\sigma(1)$ is however more delicate. After transposition, we see that it is the $n\times n$ minor obtained from the columns $\sigma_1,\ldots,\sigma_n$ in the horizontal matrix

$$\left[\binom{n}{k-i}U^\alpha_{2i-k}\right]_{1\le i\le n,1\le k\le 2n-1}.$$

Again, the generating function

$$\sum_{r\ge 0}\binom{n}{r}x^r \;=\; (1+x)^n$$

and Edrei's criterion show that the matrix

$$\left[\binom{n}{k-i}\right]_{1\le i\le n,1\le k\le 2n-1}$$

is TP_∞. But since some U^α_{2i-k} are negative, nothing can be said a priori about the non-negativity of $\mathrm{B}^\alpha_\sigma(1)$. Transforming the rows $(L_1,\ldots,L_n)$ through the simultaneaous linear operations

$$L_i \;\to\; \sum_{j=i}^{n}\binom{n}{j}L_j$$

multiplies the $n\times n$ minors by a constant positive factor and yields a matrix whose (i,k) entry is given by

$$\begin{cases}\displaystyle\sum_{j=i}^{n}\binom{n}{j}\binom{n}{k-j}U^\alpha_{2j-k} & \text{if } 1\le i\le n-1, i\le k\le n+i,\\[2mm] \displaystyle\binom{n}{k-n}U^\alpha_{2n-k} & \text{if } i=n, n\le k\le 2n-1,\end{cases}$$

and 0 otherwise. Observe that the $(i,n+i)$ entry equals

$$\sum_{j=i}^{n}\binom{n}{j}\binom{n}{n+i-j}U^\alpha_{2j-n-i} \;=\; \sum_{j=i}^{[(n+i)/2]}\binom{n}{j}\binom{n}{n+i-j}\left(U^\alpha_{2j-n-i}+U^\alpha_{n+i-2j}\right) \;=\; 0,$$

so that we have a band-matrix of width n, whose (i,k) entry is given by

$$\sum_{j=i}^{n}\binom{n}{j}\binom{n}{k-j}U^\alpha_{2j-k}$$

if $1\le i\le n, i\le k\le n-1+i$, and 0 otherwise. We set $B_{\alpha,n}$ for the above matrix, whose entries are all non-negative when $\alpha<1/n$. We also remark that $B_{\alpha,n}$ is persymmetric, viz. $B_{\alpha,n}(i,k)=$

$B_{\alpha,n}(n+1-i,2n-k)$ for any (i,k). Indeed, $B_{\alpha,n}(i,k) - B_{\alpha,n}(n+1-i,2n-k)$ equals

$$\sum_{j=i}^{n}\binom{n}{j}\binom{n}{k-j}U^{\alpha}_{2j-k} + \sum_{j=n+1-i}^{n}\binom{n}{j}\binom{n}{2n-k-j}U^{\alpha}_{2n-2j-k} = \sum_{j=0}^{n}\binom{n}{j}\binom{n}{j+k-n}U^{\alpha}_{2n-2j-k}$$
$$= \sum_{j=n-k}^{n}\binom{n}{j}\binom{n}{j+k-n}U^{\alpha}_{2n-2j-k}$$

and the last sum clearly vanishes. The validity of the following conjecture would imply (5.1).

Conjecture 5.1. For every $n \geq 2$, all minors $\mathrm{B}_\sigma(1)$ are positive whenever $\alpha < 1/n$.

The difficulty in proving this conjecture comes from the rectangular shape of the matrices $\{B_{\alpha,n}\}$, which makes it seemingly impossible to use any kind of induction argument. Let us display the five first elements of the sequence $\{B_{\alpha,n}\}$. When if $\alpha < 1/n$, these matrices have positive entries and display some spatial unimodality around the middle.

$$B_{\alpha,1} = (U_1^\alpha),\ B_{\alpha,2} = \begin{pmatrix} 2U_1^\alpha & U_2^\alpha & 0 \\ 0 & U_2^\alpha & 2U_1^\alpha \end{pmatrix},\ B_{\alpha,3} = \begin{pmatrix} 3U_1^\alpha & 3U_2^\alpha & U_3^\alpha & 0 & 0 \\ 0 & 3U_2^\alpha & U_3^\alpha + 9U_1^\alpha & 3U_2^\alpha & 0 \\ 0 & 0 & U_3^\alpha & 3U_2^\alpha & 3U_1^\alpha \end{pmatrix},$$

$$B_{\alpha,4} = \begin{pmatrix} 4U_1^\alpha & 6U_2^\alpha & 4U_3^\alpha & U_4^\alpha & 0 & 0 & 0 \\ 0 & 6U_2^\alpha & 4U_3^\alpha + 24U_1^\alpha & U_4^\alpha + 16U_2^\alpha & 4U_3^\alpha & 0 & 0 \\ 0 & 0 & 4U_3^\alpha & U_4^\alpha + 16U_2^\alpha & 4U_3^\alpha + 24U_1^\alpha & 6U_2^\alpha & 0 \\ 0 & 0 & 0 & U_4^\alpha & 4U_3^\alpha & 6U_2^\alpha & 4U_1^\alpha \end{pmatrix},$$

and

$$B_{\alpha,5} = \begin{pmatrix} 5U_1^\alpha & 10U_2^\alpha & 10U_3^\alpha & 5U_4^\alpha & U_5^\alpha & \cdots & 0 \\ 0 & 10U_2^\alpha & 10U_3^\alpha + 50U_1^\alpha & 5U_4^\alpha + 50U_2^\alpha & U_5^\alpha + 25U_3^\alpha & \cdots & 0 \\ 0 & 0 & 10U_3^\alpha & 5U_4^\alpha + 50U_2^\alpha & U_5^\alpha + 25U_3^\alpha + 100U_1^\alpha & \cdots & 0 \\ 0 & 0 & 0 & 5U_4^\alpha & U_5^\alpha + 25U_3^\alpha & \cdots & 0 \\ 0 & 0 & 0 & 0 & U_5^\alpha & \cdots & 5U_1^\alpha \end{pmatrix}.$$

Using some elementary trigonometry, for these first five values of n we could check that $B_{\alpha,n}$ is TP (viz. all its minors are non-negative) if $\alpha < 1/n$.

Conjecture 5.2. For every $n \geq 1$, the matrix $B_{\alpha,n}$ is TP whenever $\alpha < 1/n$.

This conjecture is stronger than Conjecture 2 since it involves all minors. Notice that several criteria have appeared in recent years to prove the total positivity of a given matrix without checking every minor. See all the results mentioned in Section 2.5 of [21] and also Theorem 2.16 therein for a criterion only on 2×2 minors, interestingly also related to the zeroes of Chebyshev polynomials of the second kind. Unfortunately, none of these criteria seems particularly helpful in our situation.

6. Proof of the Corollary

Setting $f_\alpha(x) = p_\alpha(1,x)$, we see from (1.3) that $p_\alpha(t,x) = t^{-1/\alpha} f_\alpha(xt^{-1/\alpha})$, so that by Theorem 1.2.1. p.18 in [13] the kernel $p_\alpha(t,x)$ has the same sign-regularity over $(0,+\infty)\times(0,+\infty)$ as the kernel $f_\alpha(e^{x-y})$ over $\mathbb{R}\times\mathbb{R}$. In Paragraph 7.12.E p.390 of [13] it is shown that

$$f_\alpha(e^{x-y}) \in \mathrm{STP}_\infty \iff f_\alpha(e^{x-y}) \in \mathrm{TP}_\infty \iff \alpha \in \{1/2, 1/3, \ldots, 1/n, \ldots\}$$

and the question is raised whether $f_\alpha(e^{x-y})$ should be TP of some finite order when α is not the reciprocal of an integer. In [23], we obtained the equivalence $f_\alpha(e^{x-y}) \in \mathrm{TP}_2 \iff \alpha \le 1/2$.

Let us now prove the Corollary. We need to show that if α is not the reciprocal of an integer and $\alpha > 1/n$, then $f_\alpha(e^{x-y}) \notin \mathrm{SR}_n$. If this were true, then the kernels $f_\alpha(e^{y-x})$ and

$$e^{-(x+y)} \int_{\mathbb{R}} f_\alpha(e^{u-x}) f_\alpha(e^{u-y})\, e^{2u}\, du$$

would also be SR_n by Theorem 1.2.1. and Lemma 3.1.1. in [13]. However, a well-known fractional moment identification - see (3.1) in [23] and the references therein - shows that the latter kernel equals $g_\alpha(x-y)$ with the notations of Section 2. Hence, we get a contradiction to Part (b) of the theorem.

□

We finish this paper with yet another conjecture on the total positivity of the positive stable kernel p_α, which reformulates Karlin's question in a more precise manner. By Lemma 3.1.1. in [13] and the same argument as in the proof of the Corollary, this would also show the open problem stated in the introduction. But as mentioned in the introduction we believe that this last conjecture is harder because the kernel p_α is not explicit in general.

Conjecture 6.1. *For every $n \ge 2$ one has*

$$p_\alpha \in \mathrm{STP}_n \iff p_\alpha \in \mathrm{SR}_n \iff \alpha \in \{1/2, 1/3, \ldots, 1/n, \ldots\} \text{ or } \alpha < 1/n.$$

Acknowledgements. I am grateful to Roger Behrend for several useful comments on the first draft of this paper, and to Arno Kuijlaars for having lent me his personal copy of [5]. I would also like to thank both referees for their careful reading and suggestions which led to a considerable improvement in the presentation of this article. In particular, I am indebted to one referee for a short proof of Part (b), a better outline of Part (c), several further references on ASM matrices and the six-vertex model, and some help with the English language. Ce travail a bénéficié d'une aide de l'Agence Nationale de la Recherche portant la référence ANR-09-BLAN-0084-01.

References

[1] R. E. Behrend. Multiply-refined enumeration of alternating sign matrices. *Adv. Math.* **245**, 439-499, 2013.

[2] R. E. Behrend, P. Di Francesco and P. Zinn-Justin. On the weighted enumeration of alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A* **119** (2), 331-363, 2012.

[3] R. Bhatia. *Positive definite matrices.* Princeton University Press, Princeton, 2007.

[4] P. Bleher and V. Fokin. Exact calculation of the six-vertex with domain wall boundary condition. Disordered phase. *Comm. Math. Phys.* **268** (1), 223-284, 2006.

[5] D. M. Bressoud. *Proofs and confirmations. The story of the alternating sign matrix conjecture.* Cambridge University Press, Cambridge, 1999.

[6] F. Colomo and A. Pronko. On the partition function of the six-vertex with domain wall boundary conditions. *J. Phys. A.* **37** (6), 1987-2002, 2004.

[7] H. Dette and A. Munk. Sign regularity of a generalized Cauchy kernel with applications. *J. Statist. Plann. Inference* **52** (2), 131-142, 1996.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *Higher transcendental functions Vol II.* McGraw-Hill, New-York, 1954.

[9] I. Gessel and X. Viennot. Binomial determinants, paths, and hook-length formulæ. *Adv. Math.* **58**, 300-321, 1985.

[10] K. I. Gross and D. St. P. Richards. Total positivity, spherical series, and hypergeometric functions of matrix argument. *J. Approx. Theory* **59**, 224-246, 1989.

[11] A. Izergin. Partition function of the six-vertex model in a finite volume. *Soviet Phys. Dokl.* **32** (11), 878-879, 1987.

[12] A. Izergin, D. Coker and V. Korepin. Determinant formula for the six-vertex model. *J. Phys. A* **25** (16), 4315-4334, 1992.

[13] S. Karlin. *Total positivity. Vol. I.* Stanford University Press, Stanford, 1968.

[14] J.-S. Kim and F. Proschan. Total positivity. In S. Kotz et al. (eds.) *Encyclopedy of Statistical Sciences* **14**, 8665-8672. Wiley, New-York, 2006.

[15] V. Korepin. Calculation of norms of Bethe wave functions. *Comm. Math. Phys.* **86** (3), 391-418, 1982.

[16] C. Krattenthaler. Advanced determinant calculus. *Sém. Lothar. Combin.* **42**, Art. B42q, 1999.

[17] G. Kuperberg. Another proof of the alternating sign matrix conjecture. *Internat. Math. Res. Notices* **3**, 139-150, 1996.

[18] A. Lascoux. Square ice enumeration. *Sém. Lothar. Combin.* **42**, Art. B42p, 1999.

[19] F. Le Gac. Quelques problèmes d'énumération autour des matrices à signe alternant. PhD Thesis, Université Bordeaux 1, 2011.

[20] O. Levêque and E. Preissmann. A generalization of Cauchy's determinant identity. Available at `http://ipg.epfl.ch/~leveque/Conjectures/cauchy.pdf`.

[21] A. Pinkus. *Totally positive matrices.* Cambridge University Press, Cambridge, 2010.

[22] R. L. Schilling, R. Song and Z. Vondraček. *Bernstein functions.* De Gruyter, Berlin, 2010.

[23] T. Simon. Multiplicative strong unimodality for positive stable laws. *Proc. Amer. Math. Soc.* **139** (7), 2587-2595, 2011.

[24] E. M. Stein and R. Shakarchi. *Complex analysis.* Princeton University Press, Princeton, 2003.

Laboratoire Paul Painlevé, Université Lille 1, Cité Scientifique, F-59655 Villeneuve d'Ascq Cedex. Laboratoire de physique théorique et modèles statistiques, Université Paris Sud, Bâtiment 100, 15 rue Georges Clémenceau, F-91405 Orsay Cedex. *Email* : `simon@math.univ-lille1.fr`