A Local Riemann Hypothesis, I

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PRELIMINARY VERSION

In Tate's thesis [**30**], Hecke L-functions are studied by means of the local integrals

$$\zeta(s,\nu,f) = \int_F f(x)\,\nu(x)\,|x|^s\,d^\times x,$$

where f is an element of the Schwartz space $S(F)$ on a local field F, and ν is a character of $F^\times$. Weil [**35**] defined a representation $\omega = \omega_\psi$ of the metaplectic group $\widetilde{SL}(2,F)$ on $S(F)$. We consider the restriction of ω to the special orthogonal group $SO(2)$ of $\widetilde{SL}(2,F)$, corresponding to the quadratic form x^2+y^2. If -1 is not a square in F, this representation is multiplicity free, and $S(F)$ decomposes into a direct sum of one-dimensional invariant subspaces. The *Local Riemann Hypothesis* is the assertion that if f lies in one of these spaces, then the zeros of the local integral $\zeta(s,\nu,f)$ lie on the line $\mathrm{re}(s)=\frac{1}{2}$. (We refer to the text for the correct statement if -1 is a square.) This is proved in a substantial number of cases, in this paper and its companion piece by Kurlberg [**19**].

If $F=\mathbb{R}$, we will prove an extension of this result to the harmonic oscillator in n-dimensions. This result may be formulated in a way that makes sense over a p-adic field, though we have not investigated this yet. In this connection, we also have a *reciprocity law* for the values at negative integers of the Laguerre polynomials, and a geometrical interpretation of these values.

We will also state a certain conjecture, that if the spherical Whittaker function of a spherical representation of $GL(n,\mathbb{R})$ which is a functorial lift from $GL(2,\mathbb{R})$ vanishes anywhere on the group, then the representation is tempered. This generalizes a theorem of Pólya on the zeros of Bessel functions.

Up-to-date information on this topic may be found on the world-wide-web at:

`http://match.stanford.edu/rh/` .

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1. The zeros of the Mellin transforms of Hermite polynomials. For the quantum mechanical harmonic oscillator see Weyl [**36**], and Cartier [**7**].

We recall the result of Bump and Ng [**5**], showing that the Mellin transforms of the Hermite functions have their zeros on the line $\mathrm{re}(s) = \frac{1}{2}$. (At first Bump and Ng considered the case of H_n with n even, and Vaaler pointed out that the case n odd could be added.)

Our normalizations will be different than in [**5**]. Let

$$f_n(x) = 2^{-n/2}\, H_n(\sqrt{2\pi}\, x)\, e^{-\pi x^2},$$

where the Hermite polynomials are defined by

$$H_n(x) = (-1)^n\, e^{x^2}\, \frac{d^n}{dx^n}\, e^{-x^2}.$$

The f_n are the eigenfunctions of the Hamiltonian $x^2 - \frac{1}{4\pi^2}\frac{d^2}{dx^2}$ of the quantum mechanical harmonic oscillator. That is, they satisfy the Schrödinger equation

$$\left(x^2 - \frac{1}{4\pi^2}\frac{d^2}{dx^2}\right) f_n = \frac{2n+1}{2\pi}\, f_n.$$

Define polynomials p_n by

$$M_n(s) = \begin{cases} \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\, p_n(s) & \text{if } n \text{ is even;} \\ \pi^{-(s+1)/2}\,\Gamma\left(\frac{s+1}{2}\right)\, \sqrt{2\pi}\, p_n(s) & \text{if } n \text{ is odd.} \end{cases}$$

where the Mellin transform

$$M_n(s) = \int_0^\infty f_n(x)\, x^s\, \frac{dx}{x}.$$

We have

$$f_{n+1}(x) = \left(\sqrt{2\pi}\, x - \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\right) f_n(x),$$

and consequently, integrating by parts, we have

$$M_{n+1}(s) = \sqrt{2\pi}\, M_n(s+1) + \frac{s-1}{\sqrt{2\pi}}\, M_n(s-1).$$

This implies that

$$p_{n+1}(s) = \begin{cases} p_n(s+1) + p_n(s-1) & \text{if } n \text{ is even;} \\ s\, p_n(s+1) + (s-1)\, p_n(s-1) & \text{if } n \text{ is odd.} \end{cases}$$

The polynomials p_n have certain properties in common with the Riemann zeta function. We have the functional equation

$$p_n(1-s) = \begin{cases} p_n(s) & \text{if } n \equiv 0,\, 1 \bmod 4; \\ -p_n(s) & \text{if } n \equiv 2,\, 3 \bmod 4. \end{cases}$$

Moreover

Theorem 1. *The zeros of p_n lie on the line* $\mathrm{re}(s) = \frac{1}{2}$.

We give two proofs of this. Another proof may be found in Bump and Ng [**5**].

FIRST PROOF. We recall a familiar classical fact, that *orthogonal polymomials have real zeros.* More precisely, let μ be a positive Borel measure on $\mathbb{R}$, and assume that μ is not supported on any finite set. We may apply Gram-Schmidt process to the sequence $\{1, x, x^2, \cdots\}$ and obtain a sequence of polynomials $P_0, P_1, P_2, \cdots$ such that the degree of P_n is n, which are orthogonal with respect to μ. The zeros of these are real and simple. Indeed, after multiplying the polynomials P_n by suitable constants, they'll have real coefficients. If $r_1, \cdots, r_k$ are the zeros of P_n which have odd multiplicity, if $k < n$ we could expand $Q(x) = \prod_i (x - r_i)$ in terms of P_i with $i < n$, so Q would be orthogonal to P_n; but patently $QP_n \geq 0$, so this is a contradiction.

Let us show that the polynomials $p_{2n}\big(\frac{1}{2} + it\big)$ form an orthogonal family with respect to a suitable measure. Indeed, the even Hermite functions f_{2n} are eigenfunctions of a self-adjoint differential operator (the oscillator Hamiltonian), so they form an orthogonal family on the half-line $\mathbb{R}^+$, which we parametrize exponentially. Thus, consider the functions $\phi_n(x) = f_{2n}(e^{2\pi x})\, e^{\pi x}$. These are orthogonal with respect to Lebesgue measure on $\mathbb{R}$. The Fourier transform of ϕ_n is $2\pi M_{2n}\big(\frac{1}{2} + it\big)$, so by the Plancherel theorem these are orthogonal:

$$\int_{-\infty}^{\infty} M_{2n}\big(\tfrac{1}{2} + it\big)\, M_{2m}\big(\tfrac{1}{2} + it\big)\, dt = 0$$

if $m \neq n$. Thus the polynomials $p_{2n}\big(\frac{1}{2} + it\big)$ form an orthonomral family, with respect to the measure $|\Gamma\big(\frac{1}{4} + \frac{it}{2}\big)|^2\, dt$.

Similarly, the polynomials p_{2n+1} are orthogonal with respect to $|\Gamma\big(\frac{3}{4} + \frac{it}{2}\big)|^2\, dt$. They must therefore all have real zeros. ■

SECOND PROOF. Let f be an eigenfunction of the oscillator Hamiltonian. Thus, f satisfies the Schrödinger equation

$$\left(x^2 - \frac{1}{4\pi^2}\frac{d^2}{dx^2}\right) f = \frac{\lambda}{2\pi}\, f$$

for some value of λ. Define the Mellin transform

$$M(s) = \int_0^{\infty} f(x)\, x^s\, \frac{dx}{x}.$$

Integrating the above Schrödinger equation by parts gives

$$M(s+2) - \frac{1}{4\pi^2}\,(s-1)\,(s-2)\,M(s-2) = \frac{\lambda}{2\pi}\,M(s).$$

We have either

$$M(s) = \begin{cases} \pi^{-s/2}\,\Gamma\big(\frac{s}{2}\big)\; p(s) & \text{or} \\ \pi^{-(s+1)/2}\,\Gamma\big(\frac{s+1}{2}\big)\;\sqrt{2\pi}\,p(s), & \end{cases}$$

with $p(s)$ a polynomial, according as $\hat{f} = \pm f$ or $\hat{f} = \pm i f$ (i.e., according as $f = f_n$ with n even or n odd.) We have therefore either

$$\lambda\,p(s) = s\,p(s+2) - (s-1)\,p(s-2),$$

or

$$\lambda\,p(s) = (s+1)\,p(s+2) - (s-2)\,p(s-2).$$

The situation will be more symmetrical if we make the substitution $q(s) = p\big(s + \frac{1}{2}\big)$. Thus, we wish to show the zeros of q are purely imaginary, and we have

$$\lambda\,q(s) = (s+a)\,q(s+2) - (s-a)\,q(s-2),$$

with $a = \frac{1}{2}$ or $a = \frac{3}{2}$. The theorem now follows from the following

Lemma. *Let $q(s)$ be a polynomial, and assume that the zeros of $q(s)$ lie in the closed strip $\{\mathrm{re}(s) \in [-c, c]\}$ with $c > 0$. Then if $a > 0$, the zeros of*

$$r(s) = (s+a)\,q(s+2) - (s-a)\,q(s-2)$$

lie in the open strip $\{\mathrm{re}(s) \in (-c, c)\}$.

To prove this, suppose that $\mathrm{re}(s) \geq c$, yet $r(s) = 0$. We will obtain a contradiction. (The case $\mathrm{re}(s) \leq -c$ may be handled similarly.) Let $q(s) = c\prod_{i=1}(s - r_i)$. If $r(s) = 0$, then

$$|(s+a)\,q(s+2)| = |(s-a)\,q(s-2)|,$$

so

$$|s+a|\,\prod |s+2-r_i| = |s-a|\,\prod |s-2-r_i|.$$

Now since $\mathrm{re}(s) > 0$, $a > 0$, we have $|s+a| > |s-a|$; moreover, since $|\mathrm{re}(r_i)| \leq c$, $\mathrm{re}(s) > c$, we have $\mathrm{re}(s - r_i) \geq 0$, and so $|s+2-r_i| > |s-2-r_i|$. Multiplying these inequalities together, we obtain a contradiction. ■

The preceeding proof is similar to the original proof of Pólya of an interesting property of the K-Bessel functions, namely, his theorem that if $y > 0$ and $K_\nu(y) = 0$, then ν is purely imaginary. Pólya's proof [**23**] depends on the recurrence identity (Watson [**34**], 3.71)

$$2\nu\, K_v(x) = x\left(K_{\nu+1}(x) - K_{\nu-1}(x)\right).$$

The operator which takes an even function $q(\nu)$ and replaces it by $\nu^{-1}\big(q(\nu+1)-q(\nu-1)\big)$ has the property (like the operator $q \mapsto r$ in the Lemma) of moving the zeros of a function closer to the imaginary axis, and so an eigenfunction of this operator should have its zeros on the imaginary axis. Since $\nu \mapsto K_\nu(x)$ is not a polynomial function, making this argument rigorous requires care. An easier (but arguably less insightful) proof may be found in Titchmarsh [**31**], Section 10.23.

Pólya connects his result with the Riemann hypothesis by arguing that

$$\pi^2\left(K_{\frac{9}{4}+\frac{it}{2}}(2\pi) + K_{\frac{9}{4}-\frac{it}{2}}(2\pi)\right)$$

has analytic properties similar to $\frac{1}{2}s(s-1)\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s)$, with $s = \frac{1}{2} + it$. (Actually this value, taken from Titchmarsh [**31**], seems to us to be off by a constant, but this is unimportant.) This function also has its zeros on the line $\mathrm{re}(s) = \frac{1}{2}$.

It is worth pointing out that there is another more "philosophical" way of connecting Pólya's result on the Bessel functions with the Riemann hypothesis. We begin by noting that it implies a Riemann hypothesis for the Fourier coefficients of Eisenstein series. Consider the classical $SL(2,\mathbb{Z})$ Eisenstein series

$$E(z,s) = \tfrac{1}{2}\,\pi^{-s}\,\Gamma(s)\sum \frac{y^s}{|cz+d|^{2s}},$$

where the summation is over nonzero pairs of integers (c,d). It is well known that if $n \neq 0$, then the n-th Fourier coefficient

$$\int_0^1 E(x+iy)\,e^{2\pi inx}\,dx = 2\,|n|^{s-1/2}\,\sigma_{1-2s}(|n|)\,\sqrt{y}\,K_{s-1/2}(2\pi|n|y).$$

(See Bump [**2**] Section I.6.) Both the divisor function $\sigma_{1-2s}(|n|)$ and the K-Bessel function $K_{s-1/2}$ have their zeros on the line $\mathrm{re}(s) = \frac{1}{2}$. Now if, on the other hand, we consider the Eisenstein series of half-integral weight (see Maass [**20**], Shimura [**28**] and Goldfeld and Hoffstein [**13**]), the Fourier coefficients are quadratic L-functions. So the analogous assertion—that the Fourier coefficients of the Eisenstein series satisfy a Riemann hypothesis—in the case of the Eisenstein series of half-integral weight, should reduce to the classical Riemann hypothesis.

One may be a bit more careful here. Actually the Fourier coefficients of these Eisenstein series are the products of quadratic L-functions with certain finite Dirichlet polynomials, and one would like to assert that these polynomials themselves have their zeros on the line $\mathrm{re}(s) = 1/2$. David Cardon has looked at the case of Eisenstein series on the double cover of $GL(2)$ over a rational function field, and his work suggests that the correct formulation is that *the Whittaker coefficients in the modified sense of Gelbart, Howe and Piatetski-Shapiro* [**11**] *should satisfy the Riemann hypothesis.*

We propose here a conjectural generalization of Pólya's result on the zeros of the Bessel function K_ν. Let π be a spherical principal series representation of $PGL(2,\mathbb{R})$, and let W be the $SO(2)$-fixed vector (determined up to constant multiple) in its Whittaker model with respect to the additive character $\psi(x) = e^{2\pi ix}$ of $\mathbb{R}$. Then

$$W\left(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} k\right) = \sqrt{y}\, K_\nu(2\pi y)\, e^{2\pi ix},$$

when $k \in SO(2)$, for some complex number ν. So Pólya's result may be formulated as saying that *if the $SO(2)$-fixed Whittaker vector in a spherical principal series representation vanishes anywhere on $PGL(2,\mathbb{R})$, then the representation is tempered.*

More generally, let π be a spherical principal series representation of $PGL(n,\mathbb{R})$, and assume that π is a symmetric $n-1$-st power lifting of a spherical principal series representation of $PGL(2,\mathbb{R})$. This means that there is a quasicharacter χ of $\mathbb{R}^\times/\{\pm 1\}$ such that π is obtained by normalized parabolic induction from the character

$$\begin{pmatrix} y_1 & * & \cdots & * \\ & y_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & y_n \end{pmatrix} \mapsto \chi(y_1)^{n-1}\,\chi(y_2)^{n-3}\cdots\chi(y_n)^{1-n}.$$

Let W be the $SO(n)$-fixed vector in the Whittaker model of π, determined up to constant multiple.

Conjecture. *In this setting, if W vanishes anywhere on $GL(n,\mathbb{R})$, then π is tempered (i.e. χ is unitary).*

We will offer three pieces of evidence for this statement.

Firstly, it is true when $n = 2$ by Pólya's result.

Secondly, for one particular nontempered spherical Whittaker function (which is a symmetric square lift from $GL(2)$) on $GL(3,\mathbb{R})$ we can verify this claim—we recall that the spherical Whittaker functions on $GL(3,\mathbb{R})$ and $GL(3,\mathbb{C})$ are the same, and that for one particular principal series representation, corresponding to the cubic theta function on $GL(3,\mathbb{C})$, the Whittaker function can be expressed in terms of the Bessel function $K_{1/3}$, so the asserted nonvanishing follows from Pólya's result. See Bump and Friedberg [**3**] and Bump and Huntley [**4**].

And thirdly, an analogous statement is true for spherical Whittaker functions on $PGL(n,F)$, when F is a nonarchimedean local field. Let π be a spherical principal series representation with Satake parameters $\alpha_1, \cdots, \alpha_n$. Let

$$h = \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{pmatrix}$$

be a dominant element of the diagonal subgroup, so that if λ_i is the valuation of y_i, we have $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Let s_λ be the Schur polynomial corresponding to the partition $\lambda = (\lambda_1, \cdots, \lambda_n)$, a symmetric polynomial in n variables (Macdonald [**21**]). According to Shintani [**29**] and Casselman and Shalika [**8**], the value $W(h)$ of the normalized Whittaker function with respect to an additive character ψ whose conductor is the ring $\mathfrak{o}$ of integers in F equals $\delta(h)^{1/2}\, s_\lambda(\alpha_1, \cdots, \alpha_n)$, where δ is the modular quasicharacter of the Borel subgroup of $GL(n, F)$. Now suppose that π is a symmetric $n-1$-st power lift from $GL(2)$. Thus we assume that there exists a complex number α such that

$$(\alpha_1, \cdots, \alpha_n) = (\alpha^{n-1}, \alpha^{n-3}, \cdots, \alpha^{1-n}).$$

Proposition. *In this situation, if $W(h) = 0$ for h dominant, then π is tempered.*

PROOF. We have $s_\lambda(\alpha^n, \alpha^{n-2}, \cdots, \alpha^{-n}) = 0$, and we will show that $|\alpha| = 1$. Indeed, by homogeneity of the Schur polynomial, we have $s_\lambda(\alpha^{2n-2}, \alpha^{2n-4}, \cdots, 1) = 0$. We recall that

$$s_\lambda(\alpha_1, \cdots, \alpha_n) = \frac{\begin{vmatrix} \alpha_1^{\lambda_1+n-1} & \alpha_2^{\lambda_1+n-1} & \cdots & \alpha_n^{\lambda_1+n-1} \\ \alpha_1^{\lambda_2+n-2} & \alpha_2^{\lambda_2+n-2} & \cdots & \alpha_n^{\lambda_2+n-2} \\ \vdots & & & \vdots \\ \alpha_1^{\lambda_n} & \alpha_2^{\lambda_n} & \cdots & \alpha_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \cdots & \alpha_n^{n-2} \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}}.$$

Substituting $(\alpha^{2n-2}, \alpha^{2n-4}, \cdots, 1)$ for $(\alpha_1, \cdots, \alpha_n)$, the numerator here becomes

$$\begin{vmatrix} \beta_1^{n-1} & \beta_1^{n-2} & \cdots & 1 \\ \beta_2^{n-1} & \beta_2^{n-2} & \cdots & 1 \\ \vdots & & & \vdots \\ \beta_n^{n-1} & \beta_n^{n-2} & \cdots & 1 \end{vmatrix} = \prod_{i<j} (\beta_i - \beta_j),$$

where $\beta_i = \alpha^{2(\lambda_i+n-i)}$. If this is zero, then some $\beta_i = \beta_j$, which implies that α is a root of unity. Thus $|\alpha| = 1$, so π is tempered. ■

2. The metaplectic representation. Witten, Brekke, Freund and Olsen in [**1**], [**10**] and [**9**] considered p-adic analogs of bosonic string theory. This led Ruelle, Thiran, Verstegen and Weyers [**27**] to consider the p-adic harmonic oscillator, also studied in the recent book of Vladimirov, Volovich and Zelenov [**32**]. The p-adic harmonic oscillator may be understood in terms of the restriction of the metaplectic representation of the double cover of $SL(2,\mathbb{R})$ on $L^2(\mathbb{R})$ to the group $SO(2)$ of symmetries of the Hamiltonian of a single particle moving in a quadratic potential field. In this formulation, there is no obstacle to replacing $\mathbb{R}$ by an arbitrary local field, and this is the point of view we will take.

Let F be a local field of characteristic not equal to 2. Let $(\ ,\ )$ denote the Hilbert symbol of F. Let ψ denote a nontrivial additive character of F. Let dx denote the measure on F which is self-dual with respect to the Fourier transform; thus if

$$\hat{f}(x) = \int_F f(y)\,\psi(2xy)\,dy,$$

dx is self-dual if $\hat{\hat{f}}(x) = f(-x)$. If $t \in F^\times$, let

$$\gamma(t) = |t|^{1/2} \int_F \psi(tx^2)\,dx.$$

This oscillatory integral is conditionally convergent in an obvious sense. The absolute value of γ equals 1—indeed it is an eight-th root of unity—and

$$\gamma(a)\,\gamma(b) = (a,b)\,\gamma(ab)\,\gamma(1).$$

Furthermore, we have

$$\gamma(b^2\,a) = \gamma(a), \qquad \gamma(-a) = \gamma(a)^{-1}.$$

Let $G = SL(2,F)$, and let $\widetilde{G}$ be the metaplectic double cover of $SL(2,F)$ defined by Kubota's cocycle $\sigma : G \times G \to \mu_2 = \{\pm 1\}$. Thus in terms of the Hilbert symbol,

$$\sigma(g_1,g_2) = \left(\frac{X(g_1)}{X(g_1g_2)}, \frac{X(g_2)}{X(g_1g_2)}\right),$$

where

$$X\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0; \\ d & \text{otherwise.} \end{cases}$$

Let $\mathbf{s} : G \to \widetilde{G}$ be the standard section, so that

$$\mathbf{s}(g_1)\,\mathbf{s}(g_2) = \sigma(g_1,g_2)\,\mathbf{s}(g_1g_2).$$

We will also use the notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{s}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \widetilde{G}.$$

The metaplectic representation $\omega = \omega_\psi$ is an action of $\widetilde{G}$ on the Schwartz space $S(F)$. It is given on generators by

$$\left(\omega \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} f\right)(x) = \psi(tx^2)\, f(x),$$

$$\left(\omega \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} f\right)(x) = \gamma(1)\, \hat{f}(x),$$

$$\left(\omega \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} f\right)(x) = |a|^{1/2}\, \frac{\gamma(1)}{\gamma(a)}\, f(ax).$$

See Weil [**35**] and Gelbart and Piatetski-Shapiro [**12**].

Let

$$H = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a,\, b \in F,\, a^2 + b^2 = 1 \right\},$$

and let $\widetilde{H}$ be the preimage of H in $\widetilde{G}$. Let H' be the unique maximal compact subgroup of H, $\widetilde{H}'$ its preimage in $\widetilde{H}$. If -1 is not a square in F, then H is compact, so actually $H' = H$ and $\widetilde{H}' = \widetilde{H}$. On the other hand, if -1 is a square, then $H \cong F^\times$, so H' is a proper subgroup. The action of $\widetilde{H}$ on the Schwartz space by means of the metaplectic representation is given by the following formula:

$$\left(\omega \begin{bmatrix} a & -b \\ b & a \end{bmatrix} f\right)(x) = |b|^{-1/2}\, \gamma(b)^{-1} \int_F \psi\left(\frac{1}{b}(ax^2 - 2xy + ay^2)\right) f(y)\, dy.$$

If -1 is not a square, so that $\widetilde{H}$ is compact, then the restriction of ω to $\widetilde{H}$ is multiplicity-free. If $F = \mathbb{R}$, this follows from our proof of Theorem 2 below (though it was known long before by Howe). If F is p-adic, this follows from the Howe duality principle for the dual pair $U(1) \times U(1)$ in $SL(2)$. (Our group $SO(2)$ is the same as $U(1)$.) See Howe [**16**] and Waldspurger [**33**] for Howe duality, which is a theorem except in residual characteristic two. Other papers concerned specifically with the character of the metaplectic representation restricted to $SO(2)$ in the case of odd residual characteristic are Moen [**22**] and Prasad [**24**]. Tonghai Yang [**37**] has formulas for the actual eigenfunctions of $U(1)$ acting on the Schwartz space.

In the case of residue characteristic two, the fact that the restriction of the metaplectic representation to compact $SO(2)$ is multiplicity-free is still known. This is implicit in the work of Rogawski [**26**], which uses global to local methods, and a purely local proof may be found in Harris, Kudla and Sweet [**14**]. Also P. Ruelle, E. Thiran, D. Verstegen and J. Weyers [**27**] have calculated the character of the restriction of the metaplectic representation to tori in the fields $\mathbb{Q}_p$, including $\mathbb{Q}_2$, and their result implies this multiplicity one statement for $\mathbb{Q}_2$.

On the other hand if -1 is a square in F, the restriction of ω to $\widetilde{H}$ does not decompose into a direct sum of constituents (though its dual space of distributions does so decompose). Instead we will consider the group $\widetilde{H}'$. The restriction of ω to this group is not multiplicity free.

The metaplectic cover splits over $\widetilde{H}$. Indeed, if -1 is not a square, $\widetilde{H}$ is contained in $SL(2,\mathfrak{o})$, and an explicit splitting over this maximal compact subgroup was given by Kubota [**18**]. If we define

$$\kappa\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{cases} -1 & \text{if } v(b) \text{ is odd and } a \equiv -1 \text{ modulo } \mathfrak{p}; \\ 1 & \text{otherwise,} \end{cases}$$

then

$$\sigma(g_1, g_2) = \frac{\kappa(g_1)\,\kappa(g_2)}{\kappa(g_1\,g_2)}$$

when g_1, $g_2 \in H$. (It is worth mentioning that if the valuation $v(b) > 0$, then $a \equiv \pm 1$ modulo $\mathfrak{p}$ since $a^2 + b^2 = 1$.) We may therefore define a representation of the abelian group H by

$$\left(\omega\begin{pmatrix} a & -b \\ b & a \end{pmatrix} f\right)(x) = \kappa\begin{pmatrix} a & -b \\ b & a \end{pmatrix} |b|^{-1/2}\,\gamma(b)^{-1} \int_F \psi\left(\frac{1}{b}(ax^2 - 2xy + ay^2)\right) f(y)\,dy.$$

On the other hand, if -1 is a square in F, then H is conjugate to the diagonal torus in $SL(2)$, and it is well known (and easy to prove from Kubota's cocycle formula) that the metaplectic cover splits over this subgroup. Since the cover splits over H', we may regard ω as giving a representation of this group.

Local Riemann Hypothesis. *Suppose that F is a local field. Assume that F is not complex, and that the characteristic of F is not equal to 2. Let $f \in S(F)$ be an eigenfunction of this action of $H \cap K$, and let ν be a character of $F^\times$. Then the Mellin transform*

$$\int_F f(x)\,\nu(x)\,|x|^s\,d^\times x,$$

if not identically zero, has its only zeros on the line $\mathrm{re}(s) = \frac{1}{2}$.

This assertion is largely proved, in this paper and its companion piece, Kurlberg [**19**].

Lt us study what happens when we change the additive character. If $\lambda \in F^\times$, let ψ_λ be the character $x \mapsto \psi(\lambda x)$. Let $d_\psi x$ denote the additive Haar measure which is self-dual with respect to ψ. Then $d_{\psi_\lambda} x = |\lambda|^{1/2}\, d_\psi x$. Let ω_ψ denote the metaplectic representation parametrized by ψ. If $f \in S(F)$, let $f_\lambda(x) = f(\lambda x)$. Then it is easy to see that

$$\omega_{\psi_{\lambda^2}}\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} f_\lambda\right)(x) = \omega_\psi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} f\right)(\lambda x).$$

Thus if f is an eigenfunction of $\widetilde{H}$ under the representation ω_ψ, then f_λ is an eigenfunction of $\widetilde{H}$ under $\omega_{\psi_{\lambda^2}}$. The zeros of $\zeta(s,\nu,f)$ and $\zeta(s,\nu,f_\lambda)$ are at the same places, so we have the freedom to change ψ to ψ_{λ^2} for any square λ^2.

Theorem 2. *The Local Riemann Hypothesis is true if $F = \mathbb{R}$.*

PROOF. We reduce this to Theorem 1. Since we have the freedom to change ψ by a square, we may assume that $\psi(x) = e^{\pm i\pi x}$. We will assume that $\psi(x) = e^{i\pi x}$; the other case is obtained by replacing i by $-i$ throughout the following discussion.

In this case, the self-dual measure on $\mathbb{R}$ coincides with Lebesgue measure, and

$$\gamma(1) = \int_{-\infty}^{\infty} e^{i\pi x^2}\, dx = \lim_{t\to 0+} \int_{-\infty}^{\infty} e^{-\pi(t-i)x^2}\, dx = \lim_{t\to 0+} (t-i)^{-1/2} = \frac{1}{\sqrt{2}}(1-i).$$

Let $\mathfrak{g}$ be the Lie algebra of $SL(2,\mathbb{R})$. The exponential map $\mathfrak{g} \to SL(2,\mathbb{R})$ lifts to a map $\widetilde{\exp} : \mathfrak{g} \to \widetilde{G}$. We then have a representation $d\omega$ of $\mathfrak{g}$ on $S(\mathbb{R})$ by

$$\big((d\omega\, X)(f)\big)(x) = \frac{d}{dt}\, (\widetilde{\exp}(tX)\, f)(x)|_{t=0}.$$

Let $\mathfrak{F} : S(\mathbb{R}) \to S(\mathbb{R})$ denote the Fourier transform $\mathfrak{F} f = \hat{F}$, and let $\mathfrak{F}^{-1}$ be its inverse:

$$(\mathfrak{F}^{-1} f)(x) = \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i xy}\, dy.$$

Define "momentum" and "position" operators P and Q on the Schwartz space by

$$(Pf)(x) = \frac{1}{2\pi i}\,\frac{df}{dx}(x), \qquad (Qf)(x) = x\, f(x).$$

We have

$$\mathfrak{F}^{-1}\, Q^2\, \mathfrak{F} = P^2.$$

Indeed, $(\mathfrak{F}^{-1}\, Q^2\, \mathfrak{F}\, f)(x)$ equals

$$\int_{-\infty}^{\infty} y^2\, \hat{f}(y)\, e^{-2\pi ixy}\, dy = -\frac{1}{4\pi^2}\frac{d^2}{dx^2}\int_{-\infty}^{\infty} \hat{f}(y)\, e^{-2\pi ixy}\, dy = -\frac{1}{4\pi^2}\frac{d^2 f}{dx^2}(x).$$

We now prove that

$$d\omega\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = i\pi Q^2, \qquad d\omega\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = i\pi P^2.$$

The first identity follows directly from the definitions:

$$\left(d\omega\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f\right)(x) = \frac{d}{dt}\left(\omega\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} f\right)(x)\bigg|_{t=0} = \frac{d}{dt}e^{i\pi x^2 t} f(x)\bigg|_{t=0} = i\pi x^2\, f(x).$$

Since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

we have

$$d\omega\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \left(\omega\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^{-1}\left(d\omega\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\left(\omega\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = i\pi\, \mathfrak{F}^{-1}Q^2\mathfrak{F},$$

and so the second identity follows from the first.

Now suppose that f is an eigenfunction of $\widetilde{H}$. Since

$$\widetilde{H} = \widetilde{\exp}\left(\mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right),$$

f is also an eigenfunction of

$$d\omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = d\omega\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + d\omega\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = i\pi(P^2 + Q^2),$$

which is (up to constant) the oscillator Hamiltonian. Hence f is one of the functions f_n.

There are two possibilities for ν: $\nu(x) = \mathrm{sgn}(x)^\delta$, where $\delta = 0$ or 1. Depending on whether f is even or odd, exactly one of the integrals $\int f(x)\,\nu(x)\,|x|^s\,dx/x$ will be nonzero, and this one will be just twice the Mellin transform of f. Consequently, Theorem 2 follows from Theorem 1. ■

We turn now to the case of a p-adic field F. In this case, following some preliminary investigation by Bump and Hoffstein, Kurlberg [**19**] has proved:

Theorem 3. *The Local Riemann Hypothesis is true if F is a nonarchimedean local field of odd residue characteristic.*

On the other hand, Kurlberg has also shown that the Local Riemann Hypothesis is false if $F = \mathbb{C}$.

3. Laguerre polynomials, the n-dimensional harmonic oscillator and a reciprocity law. The *Laguerre polynomials* (cf. Rainville [**25**]) are defined by:

$$L_n^\alpha(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \sum_{k=0}^{n} \frac{(1+\alpha)_n(-x)^k}{k!\,(n-k)!\,(1+\alpha)_k},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. They satisfy the differential equation

$$x\,\frac{d^2}{dx^2}\,L_n^{(\alpha)}(x) + (1+\alpha-x)\,\frac{d}{dx}L_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0,$$

and the orthogonality relation:

$$\int_0^\infty x^\alpha\, e^{-x}\, L_n^{(\alpha)}(x)\, L_m^{(\alpha)}(x)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{\Gamma(1+\alpha+n)}{n!} & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}_n^{(\alpha)}(x) = x^{\alpha/2}\, e^{-x/2}\, L_n^{(\alpha)}(x)$. Then the *Laguerre functions* $\mathcal{L}_n^{(\alpha)}$ are orthogonal with respect to Lebesgue measure on $[0,\infty)$. Their Mellin transforms

$$\mathcal{M}_n^{(\alpha)}(s) = \int_0^\infty \mathcal{L}_n^{(\alpha)}(x)\, x^{s-1}\, dx = 2^{s+\frac{\alpha}{2}}\,\Gamma\big(s+\tfrac{\alpha}{2}\big)\, P_n^{(\alpha)}(s),$$

where

$$P_n^{(\alpha)}(s) = \sum_{k=0}^{n} 2^k \binom{n+\alpha}{n-k} \binom{-s-\frac{\alpha}{2}}{k}.$$

Theorem 4. *The zeros of $P_n^{(\alpha)}(s)$ lie on the line* $\mathrm{re}(s) = \frac{1}{2}$.

PROOF. The first proof of Theorem 1 is easily adapted. Using the orthogonality of the Laguerre functions, we see that the polynomials $P_n^{(\alpha)}\big(\frac{1}{2}+it\big)$ are orthogonal with respect to the measure $2^{1+\alpha}\,|\Gamma\big(\frac{1}{2}+\frac{\alpha}{2}+it\big)|^2\,dt$, and their zeros are therefore real. ∎

The polynomials $P_n^{(\alpha)}(s)$ satisfy a functional equation:

$$P_n^{(\alpha)}(s) = (-1)^n \, P_n^{(\alpha)}(1-s).$$

We may prove this as follows. We start with the generating function for the Laguerre polynomials (Rainville [**25**], p. 202):

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \, t^n = (1-t)^{-1-\alpha} \, e^{-xt/(1-t)}.$$

Taking the Mellin transform in this identity yields

$$\sum_{n=0}^{\infty} P_n^{(\alpha)}(s) \, t^n = (1-t)^{s-1-\alpha/2} \, (1+t)^{-s-\alpha/2},$$

whence the functional equation.

Now let us investigate the harmonic oscillator in n-dimensions. If $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $r = |x| = \sqrt{\sum_i x_i^2}$ be the radial distance from the origin, and let Δ be the n-dimensional Laplacian $\sum_i \partial^2/\partial x_i^2$. Then consider the Schrödinger equation corresponding to a quadratic potential $V(r) = r^2$:

$$(-\Delta + r^2)\phi = \epsilon\phi. \tag{4}$$

The eigenvalue ϵ is the energy level. The potential is rotationally symmetric and the Hamiltonian $-\Delta + r^2$ commutes with the orthogonal group. We may thus restrict ourselves to ϕ which lie in an irreducible subspace of $O(n)$.

Theorem 5. *Let ϕ be a solution to (4) lying in an irreducible subspace of $O(n)$. Let X be any radially symmetric function on $\mathbb{R}^n$, so that $X(tx) = X(x)$. Then the Mellin transform*

$$\int_{\mathbb{R}^n} \phi(x) \, X(x) \, |x|^{2s-\frac{n}{2}-1} \, dx \tag{5}$$

has its zeros on the line $\mathrm{re}(s) = 1/2$.

PROOF. We make use of spherical coordinates. Thus if $x \in \mathbb{R}^n$ is given, we take $r = |x| \in \mathbb{R}^+$ and $\xi = x/|x| \in S^{n-1}$ as basic coordinates. The group $O(n)$ acts on $L^2(S^{n-1})$, which decomposes as a direct sum of irreducible subspaces, each with multiplicity one. Because of this, our assumption that ϕ lies in an irreducible subspace of $O(n)$ implies that ϕ may be written in the form $\phi_0(r) \, \Phi(\xi)$, where Φ lies in one of these irreducible subspaces of $L^2(S^{n-1})$. Since $dx = r^{n-1} \, dr \, d\xi$, the integral equals

$$\int_0^{\infty} \phi_0(r) \, r^{2s+\frac{n}{2}-1} \, \frac{dr}{r} \tag{6}$$

times the inner product on S^{n-1} of X and Φ. In spherical coordinates, the Laplacian in n dimensions has the form:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda,$$

where Λ is the Laplacian on S^{n-1} (Helgason, *Groups and Geometric Analysis* p.16). Moreover, the eigenvalue of Λ on an element of an irreducible subspace of S^{n-1} is equal to the eigenvalue of the Casimir operator on the corresponding irreducible representation, which Helgason shows has the form $-l(l+n-2)$, where $l \in \mathbb{Z}$. We thus have the differential equation (with eigenvalue λ for Λ):

$$\phi_0'' + \frac{n-1}{r}\phi_0' + \left(\frac{-l(l+n-2)}{r^2} - r^2 + \epsilon\right)\phi_0 = 0.$$

In order for $\phi_0 = e^{-r^2/2}\, r^l\, L(r^2)$ to satisfy this differential equation, we need

$$r\, L'' + \left(l + \frac{n}{2} - r\right) L' + \left(\frac{\epsilon}{4} - \frac{l}{2} - \frac{n}{4}\right) L = 0.$$

This differential equation has a regular singular point at the origin, and a solution that is well-behaved there must be a constant multiple of $L = L_k^{(l+\frac{n}{2}-1)}$, where k is an integer, and $\epsilon = 4k + 2l + n$. The result now follows from Theorem 4. ■

We note that this setup can be adapted to the metaplectic group by means of the Weil representation. The eigenfunctions at hand live in irreducible subspaces for the group $O(2) \times O(n)$, which is a maximal compact subgroup of the dual pair $SL(2,\mathbb{R}) \times O(n)$ in $Sp(2n,\mathbb{R})$, acting on $L^r(\mathbb{R}^n)$ via the standard polarization in the Weil representation. Expressed this way, the integrals of Theorem 5 have p-adic analogs, and though we haven't had a chance to investigate whether these satisfy a Riemann hypothesis, we hazard to conjecture that they do, at least in the case of anisotropic $O(n)$.

The polynomials $P_n^{(\alpha)}$ satisfy a *reciprocity law* relating their values at negative integers. We will show that

$$\binom{m+\alpha}{m} P_n^{(\alpha)}\Big(-m - \frac{\alpha}{2}\Big) = \binom{n+\alpha}{n} P_m\Big(-n - \frac{\alpha}{2}\Big)\,. \tag{7}$$

Indeed, the left side equals

$$\sum_{k=0}^{n} 2^k \binom{m+\alpha}{m}\binom{n+\alpha}{n-k}\binom{m}{k},$$

and the reciprocity law follows from the identity

$$\binom{m+\alpha}{m}\binom{n+\alpha}{n-k}\binom{m}{k}=\binom{n+\alpha}{n}\binom{m+\alpha}{m-k}\binom{n}{k}.$$

We note the special case

$$P_n^{(0)}(-m)=P_m^{(0)}(-n). \tag{8}$$

This identity has an interesting *combinatorial interpretation*.

Theorem 6. *$P_n^{(0)}(-m)$ equals the number of lattice points $(x_1,\cdots,x_n)\in\mathbb{Z}^n$ such that $\sum|x_i|\le m$.*

Proof. We can count the number of lattice points in $\mathbb{Z}^n$ satisfying $\sum|x_i|\le m$ as follows. The number of lattice points having exactly k nonzero entries is $2^k\binom{n}{k}\binom{m}{k}$ if $0\le k\le\min(m,n)$, because there are $\binom{n}{k}$ choices for which coordinates shall be nonzero; and once this choice is fixed, there are 2^k possible distibutions of signs, and $\binom{m}{k}$ possible distributions of absolute values. Hence the number of lattice points is

$$\sum_{k=0}^{\min(m,n)}2^k\binom{n}{k}\binom{m}{k}=P_n^{(0)}(-m).$$

This completes the proof. ∎

We derive a generating function for $P_n^{(0)}(-m)$. Let $a(m,n)$ be the number of lattice points satisfying the condition on the theorem. Then $a(m,n)-a(m,n-1)$ is the number of lattice points satisfying exactly $\sum|x_i|\le m$ having a nonzero last component. If the last component is $\pm m-k$, with $0\le k\le m-1$, then the number of possibilities for the first $n-1$ components is $a(k,n-1)$, and so we have

$$a(m,n)-a(m,n-1)=2\sum_{k=0}^{m-1}a(k,n-1).$$

Hence (assuming m, $n>0$) we have

$$a(m,n)-a(m,n-1)-a(m-1,n)+a(m-1,n-1)=2\,a(m-1,n-1),$$

which leads to the recursion

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a(m,n)\,x^m\,y^n=(1-x-y-xy)^{-1}.$$

The reciprocity law (8) is reflected by the symmetry of the generating function.

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