A Local Riemann Hypothesis, II

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1. Introduction

In [3], Bump and Ng made the remarkable discovery that the zeroes of the Mellin transform of Hermite functions of even level lie on the critical line $Re(s) = 1/2$. Hermite functions are eigenfunctions of the Hamiltonian $H = x^2 - \frac{1}{4\pi^2}\frac{d^2}{dx^2}$ of the quantum mechanical harmonic oscillator. H may be given a group theoretical interpretation via the Weil representation of $\widetilde{SL_2(\mathbf{R})}$, the two fold metaplectic cover of $SL_2(\mathbf{R})$, as follows: Let $SO_2 \subset SL_2$ be the subgroup of transformations preserving the form $x^2 + y^2$. The cover splits over SO_2 and thus we may consider the Weil representation restricted to SO_2. By differentiation at identity it can be shown (Bump, Choi, Kurlberg and Vaaler [2]) that $H(f) = \lambda_f f$ is equivalent to f lying in a one dimensional SO_2-invariant subspace of $L^2(\mathbf{R})$.

Differential operators does not make sense p-adically, but the Weil representation exists for all $\mathbb{Q}_p$. We may thus define the p-adic Hermite functions as the set of functions lying in one dimensional SO_2-invariant subspaces. (We need to be careful in how to define SO_2 over $\mathbb{Q}_p$ since the stabilizer of the quadratic form x^2+y^2 is not compact for all p, see the text for details.)

We may relate this to the study of classical zeta functions as follows: Let $A, A^\times$ denote the adeles respectively ideles over $\mathbb{Q}$, and let ν be an idele class character (assumed to be even for simplicity of notation.) Let $f : A^\times \to \mathbf{C}$ be defined as the product

$$f(x) = f_\infty(x_\infty) \times \prod_p f_p(x_p),$$

where $f_\infty(x_\infty) = e^{-\pi x_\infty^2}$ and f_p is the characteristic function on the p-adic integers at the unramified places. For the ramified places we take f_p to be the characteristic function of the local conductor of ν. As in Tate [6] we have

$$\zeta(s,\nu,f) = \int_{A^\times} f(x)\nu(x)|x|^s d^\times x = L(s,\nu)\Gamma(s/2)\pi^{-s/2}\gamma(s,\nu) = \Lambda(s,\nu),$$

where $L(s,\nu)$ is a Dirichlet L-series, $\gamma(s,\nu)$ is a product of local ramified factors, and $\Lambda(s,\nu)$ is the "completed" L-function that satisfies a functional equation. (In classical language the integral representation amounts to expressing $\Lambda(s,\nu)$ as the Mellin transform of a theta function.)

Now, at all the unramified places v, the f_v's are examples of Hermite functions. It is thus natural to ask what happens when we replace a finite number of factors f_v by arbitrary Hermite functions. Since we are modifying the Euler product at a finite number of places, the question can be settled by local calculations. Bump conjectured that the "new" local factors,

$$\zeta(s, \nu_v, f_v) = \int_{\mathbb{Q}_v^\times} |x|^s \nu_v(x) f_v(x)\, d^\times x$$

should satisfy a *local Riemann hypothesis*, i.e. that their zeroes lie on the critical line $Re(s) = 1/2$. The case $\mathbb{Q}_v = \mathbf{R}$ is of course Bump and Ng's discovery, a proof for $\mathbb{Q}_v = \mathbb{Q}_p$, $p = 3(4)$, and $\nu = 1$ is due to Bump and Hoffstein (unpublished.) For more details on this conjecture, along with a generalization to the n-dimensional harmonic oscillator, see Bump, Choi, Kurlberg and Vaaler [2].

In this paper we prove the conjecture for local fields of odd residue characteristic and we also show why it does not hold for $F = \mathbb{C}$. (Theorems 4 and 5.)

2. Preliminaries

Let F be a nonarchimedean local field of odd characteristic. $\mathfrak{O} = \mathfrak{O}_F$ will denote the ring of integers in F, $\mathfrak{P} = (\pi)$ will be the unique maximal ideal in $\mathfrak{O}$, and finally $q = |\mathfrak{O}/P|$. We will use Weil's "module normalization" of the absolute value, i.e., $|\pi| = 1/q$.

Let ψ be an additive character on F with conductor $\mathfrak{P}^n$. Let ν be a *unitary* multiplicative character on $F^\times$ such that $\nu(\pi) = 1$ (see lemma 2.)

We let the *level* of ν be the smallest integer m such that $\nu|_{1+\mathfrak{P}^m} = 1$. If $\nu|_{\mathfrak{O}^\times} = 1$, then we say the level of ν is zero.

It will be convenient to normalize the additive and multiplicative Haar measures on F and $F^\times$ so that the Fourier transform with respect to ψ is self dual, and $\mu^\times(\mathfrak{O}^\times) = 1$. Since the Fourier transforms maps

$$1_{\mathfrak{P}^n} \to \mu(\mathfrak{P}^n)1_{\mathfrak{O}} \to \mu(\mathfrak{P}^n)\mu(\mathfrak{O})1_{\mathfrak{P}^n}$$

we have $\mu(\mathfrak{O}) = q^{n/2}$. With $d^\times x = C\frac{dx}{|x|}$ we see that $1 = \int_{\mathfrak{O}^\times} d^\times x$ implies

$$C = \frac{q^{1-n/2}}{q-1},$$

since $\int_{\mathfrak{O}^\times} d^\times x = C\int_{\mathfrak{O}^\times} \frac{dx}{|x|} = C\mu(\mathfrak{O}^\times) = C\mu(\mathfrak{O})\frac{q-1}{q}$.

$S(F)$ will be the Schwartz space of F, i.e., the space of compactly supported locally constant complex valued functions on F. For any function or character ϕ on F, let $\phi_a(x) = \phi(ax)$, and let the "dilation operator" $T_a : S(F) \to S(F)$ be defined by $T_a(\phi) = \phi_a$. If X is a union of cosets of $\mathfrak{P}^k$, then $S(X, \mathfrak{P}^k)$ will be the space of functions supported on X and constant on cosets of $\mathfrak{P}^k$.

2.1. **The local Tate integrals.** The zeta functions that we are interested in are:

Definition 1. *Let*

$$\zeta(s,\nu,f) = \int_{F^\times} |x|^s \nu(x) f(x) d^\times x.$$

Remark: Sometimes we will write $\zeta(s, f)$ for $\zeta(s, 1, f)$.

For the reader's convenience we recall Tate's local functional equation (see Tate [6] for details).

Theorem 1. *Let $f \in S(F)$, and let ψ, ν be characters on F, $F^\times$ respectively. Then there exists a meromorphic function $c(s)$, depending only on ψ and ν, such that*

$$\zeta(s, \nu, f) = c(s)\zeta(1 - s, \nu^{-1}, \hat{f}),$$

$\hat{f}$ being the ψ-Fourier transform of f.

Remark: An easy calculation with $f = 1_{1+\mathfrak{P}^n}$ shows that $c(s)$ is a function of exponential type (and hence nowhere vanishing) when $\nu \neq 1$.

The following lemmas show that the real parts of the zeroes of $\zeta(s, \nu, f)$ are unchanged when f is replaced by f_a, or when ν is twisted by an unramified character. (Thus we may make the assumption that $\nu(\pi) = 1$ without loss of generality.)

Lemma 1.

$$\zeta(s, \nu, f_a) = \nu(a^{-1})|a|^{-s}\zeta(s, \nu, f).$$

Proof: Change of variables. □

Lemma 2. *The real parts of the zeroes of $\zeta(s, \nu, f)$ depend only on $\nu|_{\mathfrak{O}^\times}$.*

Proof:

$$\zeta(s, \nu, f) = \int_{F^\times} f(x)\nu(x)|x|^s d^\times x$$

$$= \sum_k q^{-ks} \int_{\mathfrak{O}^\times} f(\pi^k x)\nu(\pi^k x) d^\times x = \sum_k (q^{-s}\nu(\pi))^k \int_{\mathfrak{O}^\times} f(\pi^k x)\nu(x) d^\times x.$$

Since $|\nu(\pi)| = 1$, we are done. □

2.2. **The Weil representation.** In this section we develop properties of the Weil representation that we will need. For notational convenience we make the following

Definition 2. *Let*

$$s_a = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \ u_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \ w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Remark: These elements generate $SL_2(F)$.

Let $\tilde{G}$ be the two fold metaplectic cover of $G = SL_2(F)$, defined by Kubota's cocycle $\sigma : G \times G \to \{\pm 1\}$. σ is given in terms of the Hilbert symbol by

$$\sigma(g_1, g_2) = \left(\frac{X(g_1)}{X(g_1 g_2)}, \frac{X(g_2)}{X(g_1 g_2)} \right),$$

where $X\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = c$ if $c \neq 0$, d otherwise. Finally, let $s : G \to \tilde{G}$ be the standard section such that $s(g_1)s(g_2) = \sigma(g_1, g_2)s(g_1 g_2)$.

Definition 3. *Let* $\gamma(t) = |t|^{1/2} \lim_{n\to\infty} \int_{\mathfrak{P}^{-n}} \psi(tx^2) dx$.

Theorem 2. *(Weil* [8]*) There exists a representation*

$$\omega : \tilde{G} \to GL\big(S(F)\big),$$

defined by

$$\big(\omega\big(s(u_t)\big)f\big)(x) = \psi(tx^2)f(x),$$

$$\omega\big(s(w)\big)f = \gamma(1)\hat{f},$$

and

$$\omega\big(s(s_a)\big)f = |a|^{1/2}\frac{\gamma(1)}{\gamma(a)}f_a.$$

Remarks: The representation is also known as the oscillator, or metaplectic representation. Using the Stone-von Neumann theorem one can define ω in a more natural way, for instance see chapter 4 in Bump [1]. Note that the usual $L^2(F)$ inner product is $\tilde{G}$-invariant when restricted to $S(F)$.

Let $SO_2(F)$ correspond to the quadratic form x^2+y^2. We are interested in ω restricted to the unique maximal compact subgroup of $SO_2(F)$.

Definition 4. *Let*

$$H = SL_2(\mathfrak{O}) \cap SO_2(F),$$

Remark: If $i \notin F$ (where $i^2 = -1$), then $SO_2(F)$ is contained in the maximal compact subgroup $SL_2(\mathfrak{O}) \subset SL_2(F)$. However, if $i \in F$, then we need to intersect with $SL_2(\mathfrak{O})$ in order for nontrivial H-eigenfunctions to exist. We will call (rightfully so) the first case *anisotropic*, and the second *"split"*.

The restriction of ω to H is a true representation (the cover splits over $SL_2(\mathfrak{O})$ when the residue characteristic is odd, see Kubota [5]), and is given by:

$$(\omega(\begin{bmatrix} a & -b \\ b & a \end{bmatrix})f)(x)$$

$$(1) \qquad = \kappa(a,b)|b|^{-1/2}\gamma(b)^{-1}\int_F \psi(\frac{1}{b}(ax^2+ay^2-2xy))f(y)\,dy,$$

where $\kappa(a,b) = -1$ if we are in the anisotropic case, $ord_{\mathfrak{P}}(b)$ is odd, and $a \equiv -1 \ mod \ \mathfrak{P}$. Otherwise $\kappa(a,b) = 1$.

Since H is a compact abelian group, we know that

$$S(F) = \bigoplus_{\chi \in \hat{H}} V_\chi,$$

where $\hat{H}$ is the unitary dual of H, and

$$V_\chi = \{f \in S(F) \mid \omega(h)f = \chi(h)f \quad \forall h \in H\}.$$

Lemma 3. *All V_χ's are invariant under complex conjugation.*

Proof: Recalling that $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $\gamma(-b) = \gamma(b)^{-1} = \overline{\gamma(b)}$, we see that $\overline{\omega(h)} = \omega(h^{-1})$. (We define $\overline{\omega(h)}$ by $\overline{\omega(h)}f = \overline{\omega(h)\overline{f}}$.) Hence

$$\omega(h)\overline{f} = \overline{\overline{\omega(h)}f} = \overline{\omega(h^{-1})f} = \overline{\chi(h^{-1})f} = \chi(h)\overline{f}. \quad \square$$

Corollary 1. *There exists a $\mathbb{C}$-basis of* real *eigenfunctions for $S(F)$.*

Proof: By lemma 3 we know that both the real and the imaginary part of any eigenfunction f is in the same eigenspace as f. $\square$

2.3. **The Kloosterman decomposition.** In what follows, f will always denote an H-eigenfunction, i.e., $\omega(h)f = \chi(h)f$ for some character $\chi \in \hat{H}$. We first note that any such f will be similar to its Fourier transform since $w \in H$. We will let λ be such that

$$\lambda f(x) = \hat{f}(x) = \int_F \psi(2xy)f(y)dy. \tag{2}$$

The following will show that it is enough to study the action of H on the finite dimensional subspaces $S(\mathfrak{O}, \mathfrak{P}^n)$ for $n \geq 0$.

First note that T_a intertwines the ω_ψ and $\omega_{\psi_{a^2}}$ -actions of H on $S(F)$. Moreover, $\zeta(s, \nu, f_a) = 0 \Leftrightarrow \zeta(s, \nu, f) = 0$, so by replacing f by $f_{\pi^{-k}}$ for k large enough, we can assume that the conductor of ψ is $\mathfrak{P}^n$ for $n \geq 0$, and that the support of f is contained in $\mathfrak{O}$. Recalling that $supp(f) \subset \mathfrak{O}$ implies that $\hat{f}$ is constant on cosets of $\mathfrak{P}^n$, we see that $f \in S(\mathfrak{O}, \mathfrak{P}^n)$ since $\hat{f} = \lambda f$.

It is easily checked that $S(\mathfrak{O}, \mathfrak{P}^n)$ is $SL_2(\mathfrak{O})$-invariant, and hence H-invariant. Moreover, $S(\mathfrak{O}, \mathfrak{P}^n)$ breaks up into an H-direct sum that will enable us to induct on n. The splitting identifies $S(\mathfrak{O}, \mathfrak{P}^{n-2})$ with $S(\mathfrak{P}, \mathfrak{P}^{n-1}) \subset S(\mathfrak{O}, \mathfrak{P}^n)$ via the intertwining map

$$T_{\pi^{-1}} : \big(\omega_{\psi_{\pi^{-2}}}, S(\mathfrak{O}, \mathfrak{P}^{n-2})\big) \to \big(\omega_\psi, S(\mathfrak{O}, \mathfrak{P}^n)\big).$$

This motivates the following:

Definition 5. *An H-eigenfunction f is called a* **lift** *if $f \in L = S(\mathfrak{P}, \mathfrak{P}^{n-1})$. If $f \in L^\perp \subset S(\mathfrak{O}, \mathfrak{P}^n)$, it is said to be* **primitive**.

3. The range of $f \to \zeta(s, v, f)$

In this section we show that the map $f \to \zeta(s, \nu, f)$ has at most one-dimensional range when restricted to $S(\mathfrak{O}, \mathfrak{P}^n) \cap V_\chi$.

3.1. The anisotropic case.

Lemma 4. *$\omega|_H$ is multiplicity free when H is anisotropic.*

Proof: Apply Howe duality to the reductive dual pair $U(1) \times U(1)$. □

Remark: Howe duality is a theorem for any reductive dual pair if the residue characteristic is odd, see Waldspurger [7].

3.2. The "split" case. If $i \in F$, then $\omega|_H$ has multiplicities; but on the other hand we can conjugate H into into a diagonal "torus". This provides enough information about eigenfunctions to prove that the range is at most one-dimensional.

Let $M = \begin{bmatrix} 1/2 & i \\ i/2 & 1 \end{bmatrix} \in SL_2(\mathfrak{O})$. Then $M^{-1}HM = H'$, where

$$H' = \{ \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \mid u \in \mathfrak{O}^\times \}.$$

(M can be thought of as a "Cayley transform".) Moreover, the identity

$$\omega(M^{-1}hM) \circ \omega(M^{-1}) = \omega(M^{-1}) \circ \omega(h)$$

can be interpreted as meaning that

$$(h \to M^{-1}hM, \omega(M^{-1})) : (H, S(\mathfrak{O}, \mathfrak{P}^n)) \to (H', S(\mathfrak{O}, \mathfrak{P}^n))$$

is an intertwining operator. (Note that $S(\mathfrak{O},\mathfrak{P}^n)$ is $SL_2(\mathfrak{O})$-invariant!)

In order to translate the above decomposition back to the H-model, we write

$$M = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & i/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -i/2 & 0 \\ 0 & -2/i \end{bmatrix}.$$

Thus, if $f = \omega(M)g$, then

$$f(x) = \psi_{-i}(x^2) \int_{\mathfrak{O}} \psi(2xy)\psi_{i/2}(y^2) g_{-i/2}(y)dy. \tag{3}$$

The following lemma makes it easy to understand $\omega|_{H'}$:

Lemma 5.

$$S(\mathfrak{O},\mathfrak{P}^n) \cong (\bigoplus_{k=0}^{n-1} S(\pi^k\mathfrak{O}^\times,\mathfrak{P}^n)) \oplus S(\mathfrak{P}^n,\mathfrak{P}^n)$$

as H'-modules. Moreover, each summand is isomorphic to the regular representation of $\mathfrak{O}^\times/1+\mathfrak{P}^{n-k}$. (Abusing notation, we let $1+\mathfrak{P}^0 = \mathfrak{O}^\times$.)

Proof: Clear since

$$\omega(s_a)g = |a|^{1/2}\frac{\gamma(1)}{\gamma(a)} g_a. \quad \square \tag{4}$$

Corollary 2. *If g is an H'-eigenfunction, then so is $g \times 1_{\mathfrak{O}^\times}$ and $g \times 1_{\mathfrak{P}}$.*

Corollary 3. *We can identify H' with $\mathfrak{O}^\times$, and if $\mathfrak{P}^n$ is the conductor of ψ, then ω factors as $H \to H/(1+\mathfrak{P}^n) \to GL\big(S(\mathfrak{O},\mathfrak{P}^n)\big)$. In particular, any character of H' associated with lifts must be trivial on $1+\mathfrak{P}^{n-2}$.*

For notational convenience we define a "zeta operator" Z_ν by

$$(Z_\nu(f))(s) = \zeta(s, \nu, f).$$

Lemma 6. *If* $f = \omega(M)g$ *is primitive,* $0 < m < n$*, and* $supp(g) \subset \mathfrak{P}$*, then* $Z_\nu(f) = 0$.

Proof: Lemma 23 in the appendix. □

Lemma 7. *If* $f \in V_\chi \cap L^\perp$*,* $0 < m < n$*, and the level of* χ *is smaller than* n*, then* $Z_\nu(f) = 0$.

Proof: Lemma 24 in the appendix. □

3.3. **Conclusion.** Putting the previous results together we have:

Theorem 3. *The range of* Z_ν *restricted to* $V_\chi \cap S(\mathfrak{O}, \mathfrak{P}^n)$ *is at most one-dimensional.*

Proof: The anisotropic case follows immediately from lemma 4. In the "split" case we argue as follows: If $m \geq n$, it is easy to see that $\zeta(s, \nu, f)$ is constant; assume that $m < n$. Write f as $f = f_{\text{lift}} + f_{\text{prim}}$, where each term belongs to $L, L^\perp$ respectively. If the level of χ is smaller than n, then $Z_\nu(f_{\text{prim}}) = 0$ by lemma 7. We can thus induct on n; the only thing to check is the case when the level of χ is n. In this case we must have $f_{\text{lift}} = 0$ (corollary 3), and we can assume that g corresponding to f via equation 3 is supported on $\mathfrak{O}^\times$ by corollary 2 and lemma 23. Now, the H'-action on

$S(\mathfrak{O}^\times, \mathfrak{P}^n)$ is just the regular representation (lemma 5), hence each character occurs with multiplicity one, so done. □

Lemma 8. *There exists $\mathbb{C}$-basis for $V_\chi \cap S(\mathfrak{O}, \mathfrak{P}^n)$, consisting of real-valued functions, such that at most one basis element has nonzero image under Z_ν.*

Proof: Use theorem 3 together with corollary 1. □

4. Properties of primitives

Remark: The case when ν is trivial has a different flavor from the non-trivial case; in the former the zeta function will be a rational function of q^{-s}, whereas in the latter it will be a polynomial.

4.1. The case $\nu = 1$.

Lemma 9. *If f is primitive, then $\zeta(s, f)$ satisfies the LRH.*

Proof: See lemma 21 in the appendix.

4.2. The case $\nu \neq 1$. It is easy to see that $m \geq n$ (m being the level of ν) implies that the zeta function is constant, so we will make the assumption that $m < n$.

Lemma 10. *If $f \in L^\perp$, then all the zeroes of $\zeta(s, \nu, f)$ lie on a vertical line.*

Proof: By lemma 22 in the appendix we know that $\zeta(s, \nu, f)$ is of the form $A + Bq^{-(n-m)s}$ for some constants A, B. □

Remark: In view of lemma 22 it is worth mentioning that $|A/B|$ is known, so it possible to prove lemma 10 by a direct calculation as an alternative.

Lemma 11. *If $f \in L^{\perp}$, then $\zeta(s,\nu,f)$ satisfies LRH.*

Proof: By lemma 8 we can assume that $f \sim \overline{f}$, hence

$$\zeta(1-s,\overline{\nu},f) = \overline{\zeta(1-\overline{s},\nu,\overline{f})} \sim \overline{\zeta(1-\overline{s},\nu,f)}.$$

By the functional equation we have

$$\zeta(s,v,f) = c(s)\zeta(1-s,\nu^{-1},\hat{f}) = \lambda c(s)\zeta(1-s,\overline{\nu},f) \sim \lambda c(s)\overline{\zeta(1-\overline{s},\nu,f)}.$$

Since $c(s)$ is nowhere vanishing, we are done by lemma 10. □

4.3. **Conclusion.**

Theorem 4. *The local Riemann hypothesis is true for nonarchimedean local fields of odd residue characteristic.*

Proof: Write $f = f_{\text{lift}} + f_{\text{prim}}$. If f_{lift} is nonzero, then the level of χ is smaller than n, hence $Z_\nu(f) = Z_\nu(f_{\text{lift}})$ by lemma 7, so done by induction. If f is a primitive, then we are done by lemma 11. □

5. The complex case

Let $\psi(z) = e^{2\pi i Re(z)}$ be an additive character on $\mathbb{C}$. As in section 3.2 we will conjugate H into a diagonal "torus" H', where the ω-action is easier to understand. We will show that the LRH does not hold over $\mathbb{C}$ by proving that the dimension of the range of $Z_1|_{V_1}$ is more than one-dimensional. (Linear combinations of two independent functions can be made to have zeroes anywhere.)

Definition 6. *Let*

$$S_e(\mathbb{R}) = \{h \in S(\mathbb{R}) \mid h(x) = h(-x)\},$$

i.e., the space of even Schwartz functions.

Definition 7. *With $q(z,w)$ a quadratic form, let*

$$k_q(r,R) = \int_0^{2\pi}\int_0^{2\pi} \psi\big(q(re^{2\pi i\theta}, Re^{2\pi i\phi})\big)d\theta d\phi.$$

Furthermore, let $K_q : S(\mathbb{R}^+) \to S(\mathbb{R}^+)$ be defined by

$$\big(K_q(h)\big)(r) = \int_0^\infty k_q(r,R)\,R\,h(R)\,dR. \tag{5}$$

Definition 8. *Let $\big(Z_{\mathbb{R}}(h)\big)(s) = \int_0^\infty x^{s-1}h(x)dx$, i.e., the* real *Mellin transform of h.*

Lemma 12.

$$\{g \mid f = \omega(M)g \in V_1 \} = \{g \mid g(z) = h(|z|), \text{ where } h \in S_e(\mathbb{R}) \}.$$

Proof: Use equation 4 and the fact that γ is constant on $\mathbb{C}^\times$. □

Lemma 13.

$$Z_1(V_1) = Z_{\mathbb{R}}\big(K_q(S_e(\mathbb{R}))\big),$$

where $q(z,w) = -iz^2 + 2zw + i/2w^2$.

Proof: Let $f = \omega(M)g$, $g(z) = \sqrt{2}h(|2z|)$, and $h \in S_e(\mathbb{R})$. We have

$$\zeta(s,f) = \int_{\mathbb{C}} f(z)|z|^s d^\times z = \int_0^\infty \Big(\int_0^{2\pi} f(re^{2\pi i\theta})d\theta\Big) r^s\,dr.$$

The last factor in the Bruhat decomposition of M acts by sending $g \to |i/2|^{1/2} g_{i/2}$, hence

$$f(z) = \int_{\mathbb{C}} \psi\big(q(z,w)\big) h(|w|) dw = \int_0^\infty \int_0^{2\pi} \psi\big(q(z, Re^{2\pi i\phi})\big) d\phi\, h(R) R\, dR.$$

Hence it is enough to show that

$$\int_0^{2\pi} \int_0^\infty \int_0^{2\pi} \psi\big(q(re^{2\pi i\theta}, Re^{2\pi i\phi})\big) d\phi\, h(R) R\, dR\, d\theta$$
$$= \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \psi\big(q(re^{2\pi i\theta}, Re^{2\pi i\phi})\big) d\phi\, d\theta\, h(R) R\, dR,$$

but h is of rapid decay, so changing order of integration is justified. □

Corollary 4. *The dimensions of the range of Z_1 and the range of K_q are the same.*

Proof: The real Mellin transform restricted to $S_e(\mathbb{R})$ is invertible.

Lemma 14. *Let $q(z,w) = az^2 + bzw + cw^2$. Then K_q has one-dimensional range only if $b = 0$.*

We can assume that $a = c = 1$ since a diagonal change of variables does not change the dimension of the range. (We might perturb b, but we will not change whether $b = 0$ or not.)

It is an easy consequence of the Riesz representation theorem that an integral operator of the form (5) has one-dimensional range only if $k_q(r,R) = k_{q,1}(r) k_{q,2}(R)$. (We need to be a little bit careful since $k_q(r,R)R$ is unbounded; consider truncations $h \to \int_0^t k_q(r,R)\, R\, h(R)\, dR$ to get the product representation for $r \in [0,t]$. Then let $t \to \infty$, and note that the product representation on any interval is unique if it exists. Also, $S_e(\mathbb{R})$ is L^1-dense in $C([0,t])$ $\forall t$, so we do not have to worry about the "flatness" of h at zero.) Anyhow, by lemma 15 we know that k_q is not a product. □

Lemma 15. *Let $q(z,w) = az^2 + bzw + cw^2$, with $a,b,c \neq 0$. Then $k_q(r,R)$ can not be written as a product of two functions.*

Proof: Again we may assume that $a = c = 1$. If $k_q(r,R) = k_1(r)k_2(R)$, then we must have

(6)

$$
\begin{aligned}
0 &= k_q(r,R)k_q(0,0) - k_q(0,R)k_q(r,0) \\
&= (2\pi)^2 \int_0^{2\pi} \int_0^{2\pi} e^{ir^2\cos(2\theta) + iR^2\cos(2\phi)} \left(e^{i|b|rR\cos(\theta+\phi+arg(b))} - 1 \right) d\phi d\theta.
\end{aligned}
$$

However, when r,R are both small, the first factor is very close to 1, and the second factor has negative real part if $b \neq 0$, integrating it we see that (6) has negative real part for small r,R. (The argument can be made formal by considering Taylor expansions of $e^x, \cos(x)$, and $\sin(x)$.) □

Putting it all together we have:

Theorem 5. *If ν, χ are trivial, then the LRH does not hold for $F = \mathbb{C}$.*

6. Appendix

Here we show that primitives satisfy the local Riemann hypothesis. We will assume that $0 < m < n$, m being the level of ν, and that $\nu(\pi) = 1$. (See lemma 2.)

6.1. **Gauss sums.**

Definition 9. *Let*

$$G(\psi,\nu)=\int_{\mathfrak{O}^\times}\nu(x)\psi(x)d^\times x$$

be the integral form of a Gauss sum.

Lemma 16. $G(\psi_u,\nu)=\nu(u^{-1})G(\psi,\nu)$ *for* $u\in\mathfrak{O}^\times$.

Proof: Change of variables. □

Lemma 17. *Let* $\mathfrak{P}^k$ *be the conductor of an additive character* ρ. *Let* $m>0$ *be the level of* ν, *where* ν *is a character on* $\mathfrak{O}^\times$. *Then* $|G(\rho,\nu)|=\frac{q^{1-k/2}}{q-1}$ *if* $k=m$, *and zero otherwise.*

Proof: If $k>m$,

$$\int_{\mathfrak{O}^\times}\nu(x)\rho(x)d^\times x=\sum_{a\in\mathfrak{O}^\times/(1+\mathfrak{P}^m)}\nu(a)\int_{a(1+\mathfrak{P}^m)}\rho(x)d^\times x=0$$

since $\int_{a+\mathfrak{P}^k}\rho(x)dx=0$, and $d^\times x=Cdx$ on $\mathfrak{O}^\times$. The case $k<m$ is handled similarly. If $m=k$, then

$$|G(\nu,\rho)|^2=\int_{\mathfrak{O}^\times}\int_{\mathfrak{O}^\times}\nu(xy^{-1})\rho(x-y)d^\times x d^\times y$$

$$=\int_{\mathfrak{O}^\times}\int_{\mathfrak{O}^\times}\nu(t)\rho(y(t-1))d^\times y d^\times t$$

$$=C\int_{\mathfrak{O}^\times}\nu(t)\int_{\mathfrak{O}}\rho_{t-1}(y)dy\,d^\times t-C\int_{\mathfrak{O}^\times}\int_{\mathfrak{P}}\nu(t)\rho(y(t-1))dy\,d^\times t.$$

Now, in the first integral, only $t \in 1+\mathfrak{P}^k$ contributes and hence its absolute value equals

$$C\mu^\times(1+\mathfrak{P}^k)\mu(\mathfrak{O}) = \frac{q^{1-n/2}}{q-1}\frac{1}{(q-1)q^{k-1}}q^{n/2}.$$

The second integral can be rewritten as

$$\int_{\mathfrak{P}} \rho(-y) \int_{\mathfrak{O}^\times} v(t)\rho_y(t) d^\times t \, dy,$$

and the inner integral vanishes by the first part of the lemma. Taking square roots we get

$$|G(\rho,\nu)| = \frac{q^{1-k/2}}{q-1}. \quad \square$$

Lemma 18. *If $a \in \mathfrak{O}^\times$ and $0 < k < n$, then $\int_{a+\mathfrak{P}^k} \psi(x^2)dx = 0$.*

Proof:

$$\int_{a+\mathfrak{P}^k} \psi(x^2)dx = \sum_{a_i \in (a+\mathfrak{P}^k)/\mathfrak{P}^{n-1}} \int_{a_i+\mathfrak{P}^{n-1}} \psi(x^2).$$

But

$$\int_{a_i+\mathfrak{P}^{n-1}} \psi(x^2) = \psi(a_i^2)\int_{\mathfrak{P}^{n-1}} \psi(2a_i x + x^2) = \psi(a_i^2)\int_{\mathfrak{P}^{n-1}} \psi(2a_i x) = 0,$$

since ψ_{2a_i} has conductor $\mathfrak{P}^n$, and $x \in \mathfrak{P}^{n-1} \Rightarrow x^2 \in \mathfrak{P}^n$ as $n \geq 2$. $\square$

Remark: This lemma *does not* hold for the even residue characteristic case.

6.2. **Properties of primitives.** *Remarks:* The lemmas of this section holds for both the "split" and the anisotropic case. It should also be noted that the crucial property of the primitives is that they are determined by their values on $\mathfrak{O}^{\times}$ (see next lemma).

Lemma 19. *If f is primitive and $y \in P$, then*

$$\lambda f(y) = \int_{\mathfrak{O}^{\times}} \psi(2xy) f(x) dx.$$

Proof: First we note that

$$f \in L^{\perp} \Leftrightarrow \int_{a+P^{n-1}} f(x) dx = 0 \qquad \forall a \in P.$$

Now,

$$\int_{\mathfrak{O}} \psi(2xy) f(x) dx = \int_{\mathfrak{O}^{\times}} f(x) \psi(2xy) dx + \sum_{a \in P/P^{n-1}} \int_{a+P^{n-1}} \psi(2xy) f(x) dx.$$

Since ψ_{2y} is constant on cosets of P^{n-1}, we can use $f \in L^{\perp}$ to conclude that all but the first terms in the sum vanish. □

Corollary 5. $\lambda f(0) = \int_{\mathfrak{O}^{\times}} f(x) dx$ *for primitive f.*

Lemma 20. *If f is primitive, then*

$$\zeta(s, f) = C f(0)(\lambda - q^{n/2-1-(n-1)s} + C^{-1} \frac{q^{-ns}}{1 - q^{-s}}).$$

Proof:

$$\zeta(s, f) = \int_{\mathfrak{O}} f(x) |x|^s d^{\times} x = \sum_{k=0}^{n-1} q^{-ks} \int_{P^k - P^{k-1}} f(x) d^{\times} x + f(0) \int_{P^n} |x|^s d^{\times} x.$$

Since $f \in L^{\perp}$, all terms except $k = 0, n-1$ and the last one vanishes. The first term is equal to $C\lambda f(0)$. The second term equals $-Cf(0)q^{n/2-1-(n-1)s}$, since $\mu(P^n) = q^{-n/2}$ and

$$\int_{P^{n-1}-P^n} f(x)d^{\times}x = Cq^{n-1}\int_{P^{n-1}-P^n} f(x)dx$$

$$= Cq^{n-1}(\int_{P^{n-1}} - \int_{P^n}) = 0 - Cq^{n-1}\int_{P^n} f(x)dx = -f(0)Cq^{n-1}\mu(P^n).$$

Finally, the third term equals $f(0)\int_{P^n} |x|^s d^{\times}x = f(0)\frac{q^{-ns}}{1-q^{-s}}$. □

Lemma 21. *If f is primitive, then $\zeta(s, f)$ satisfies LRH.*

Proof: Put $x = q^{-s}$ and $y = q^{1/2}x$. Since $C = \frac{q^{1-n/2}}{q-1}$, we see that $\zeta(s, f)$ vanishes only if $f(0) = 0$ or

$$1 = |\lambda| = |C^{-1}\frac{q^{-ns}}{1-q^{-s}} - q^{n/2-1-(n-1)s}|$$

$$= |\frac{(q-1)q^{n/2-1}q^{-ns} - q^{n/2-1-(n-1)s}(1-q^{-s})}{1-q^{-s}}|$$

$$= |\frac{q^{n/2}x^n - q^{n/2-1}x^n - (x^{n-1}q^{n/2-1} - x^n q^{n/2-1})}{1-x}|$$

$$= |\frac{q^{n/2}x^n - x^{n-1}q^{n/2-1}}{1-x}| = |\frac{y^{n-1}(y-q^{-1/2})}{1-yq^{-1/2}}|.$$

Now, both $y \to \frac{y-q^{-1/2}}{1-yq^{-1/2}}$ and $y \to y^{n-1}$ preserve the interior, boundary and exterior of the unit disc, and so does their product. Therefore $|y| = 1$, which implies that $Re(s) = 1/2$. □

Lemma 22. *Let f be primitive, and let $n > m > 0$ be the level of ν. Then*

$$\zeta(s, \nu, f) = \int_{\mathfrak{O}^{\times}} f(x)\nu(x)d^{\times}x + \int_{\mathfrak{O}^{\times}} f(x)\nu(x^{-1})d^{\times}x\frac{G(\psi_{2\pi^{n-m}}, \nu)}{\lambda C}q^{-(n-m)s}.$$

Proof:

$$\zeta(s,\nu,f) = \int_{\mathfrak{O}^\times} \nu(x)f(x)d^\times x + \sum_{k>0} q^{-ks} \int_{\mathfrak{O}^\times} f(\pi^k x)\nu(x)d^\times x.$$

Since f is constant on P^n and $\nu \neq 1$, we see that the terms for which $k \geq n$ vanish. Furthermore,

$$\lambda \int_{\mathfrak{O}^\times} f(\pi^k x)\nu(x)d^\times x = \int_{\mathfrak{O}^\times}\int_{\mathfrak{O}^\times} \nu(x)\psi(2\pi^k xy)f(y)dy\, d^\times x$$

$$= \int_{\mathfrak{O}^\times} G(\psi_{2\pi^k y},\nu)f(y)dy = \int_{\mathfrak{O}^\times} \nu(y^{-1})G(\psi_{2\pi^k},\nu)f(y)dy.$$

Lemma 17 gives that the only non-vanishing term is when $n-k=m$, i.e., $k=n-m$. Thus

$$\zeta(s,\nu,f) = \int_{\mathfrak{O}^\times} \nu(x)f(x)d^\times x \quad + \quad \frac{q^{-(n-m)s}}{\lambda}\int_{\mathfrak{O}^\times} \nu(y^{-1})f(y)G(\psi_{2\pi^{n-m}},\nu)dy$$

$$= \int_{\mathfrak{O}^\times} f(x)\nu(x)d^\times x + \int_{\mathfrak{O}^\times} f(x)\nu(x^{-1})d^\times x \frac{G(\psi_{2\pi^{n-m}},\nu)}{\lambda C} q^{-(n-m)s}. \quad \square$$

Corollary 6. *If f is primitive, $\nu \neq 1$, and*

$$\int_{\mathfrak{O}^\times} f(x)\nu(x)d^\times x = \int_{\mathfrak{O}^\times} f(x)\nu(x^{-1})d^\times x = 0,$$

then $Z_\nu(f) = 0$.

6.3. **Properties of primitives, "split" case.**

Lemma 23. *Let $f = \omega(M)g$ be primitive. If g is supported on P and $0 < m < n$, then $Z_\nu(f) = 0$.*

Proof:

$$f(x) = \psi_{i/2}(x^2) \int_P \psi(2xy)\psi_{-i}(y^2)g_{-i/2}(y)dy$$

and therefore

$$\int_{\mathfrak{O}^\times} f(x)\nu(x)d^\times x = \int_{\mathfrak{O}^\times} \int_P \psi_{i/2}(x^2)\psi(2xy)\psi_{-i}(y^2)g_{-i/2}(y)\nu(x)dy\,dx.$$

Now,

$$\int_{\mathfrak{O}^\times} \nu(x)\psi_{i/2}(x^2)\psi(2xy)dx = \int_{\mathfrak{O}^\times} \nu(x)\psi_{i/2}\big((x+2y/i)^2 - (2y/i)^2\big)dx,$$

and

$$\int_{\mathfrak{O}^\times} \nu(x)\psi_{i/2}\big((x+2y/i)^2\big)dx = \sum_{a\in\mathfrak{O}^\times/(1+P^{n-1})} \nu(a) \int_{a+2y/i+P^{n-1}} \psi_{i/2}(x^2)dx.$$

But $\int_{a+2y/i+P^{n-1}} \psi_{i/2}(x^2)dx = 0$ by lemma 18 ($2y/i \in P$). The same holds for $\overline{\nu}$, so we are done by corollary 6. □

Lemma 24. *If $f \in V_\chi \cap L^\perp$, $0 < m < n$, and the level of χ is smaller than n, then $Z_\nu(f) = 0$.*

Proof: By lemma 23 and corollary 2 we can assume that $supp(g) \subset \mathfrak{O}^\times$. As in the previous lemma, it is enough to show that $\int_{\mathfrak{O}^\times} f(x)\nu(x)d^\times x = 0$. We have $g_{-i/2}(x) = \gamma(x)\chi(x)$ by equation 3. Now, $\gamma(a^2 b) = \gamma(b)$ and hence γ is constant of cosets of squares, i.e., it is constant on $(1+P)$-cosets. Thus $g_{-i/2}$ will be constant on cosets of $1 + P^{n-1}$. Hence

$$f(x) = \psi_{i/2}(x^2) \int_{\mathfrak{O}^\times} \psi(2xy)\psi_{-i}(y^2)g_{-i/2}(y)dy$$

$$= \psi_{i/2}(x^2)\psi_{-i}\big(-(ix)^2\big) \sum_{a\in\mathfrak{O}^\times/1+P^{n-1}} g_{-i/2}(a) \int_{a+P^{n-1}} \psi_{-i}((y+ix)^2)dy.$$

By lemma 18, we see that the only non-vanishing integrals come from $a \in (-ix+P)/(1+P^{n-1})$, and will be of the form

$$\int_{b-ix+P^{n-1}} \psi_{-i}((y+ix)^2)dy = \int_{b+P^{n-1}} \psi_{-i}(z^2)dz,$$

where $b \in P/P^{n-1}$, the point being that those terms do not depend on x. Thus f will be a linear combination of terms of the form

$$\psi_{i/2}(x^2)g_{-i/2}(b-ix)\psi_{-i}(x^2),$$

where $b \in P/P^{n-1}$. Therefore it is enough to show that

$$\int_{\mathfrak{O}^\times} \psi((i/2-i)x^2)g_{-i/2}(b-ix)\nu(x)d^\times x = 0 \;\forall\; b \in P/P^{n-1}.$$

Again, we break up the integral into cosets of $1+P^{n-1}$, since $g_{-i/2}(b-ix)\nu(x)$ is constant on $1+P^{n-1}$-cosets. We can now apply lemma 18 since $i/2-i = -i/2 \in \mathfrak{O}^\times$.

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