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TAUBERIAN THEOREMS FOR THE STIELTJES TRANSFORM

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1. Introduction.

Some years ago I published [1] a remainder theorem for the Laplace transform applicable to remainders of arbitrary order of decrease. The estimates afforded by that theorem are known to be best possible in most interesting cases. I only gave an outline of the method of proof which was a development of the well-known Karamata approximation technique.

In this paper I shall apply Fourier methods to obtain a similar result for the Stieltjes transform. The idea of the proof was given in 1962 in a paper on Wiener's tauberian theorem [2] and the result for the Stieltjes transform (Theorem 2) will in fact be obtained from a general result (Theorem 1). Among the special cases covered I ought to mention the results of Vučković [4, 5]. Theorem 2 is of interest as being applicable to the estimation of spectral functions for certain differential operators, and I have tried to formulate it in a way suitable for these applications.

For Fourier transforms and for convolutions we use the notations

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)\exp(-ixt)\,dx \qquad \text{and} \qquad K*\varphi(x) = \int_{-\infty}^{\infty} K(x-y)\,\varphi(y)\,dy\,.$$

2. The general result.

Theorem 1 may conveniently be stated for a class of kernels defined in the following way.

E_0 is a sub-set of $L(-\infty,\infty)$ consisting of those functions K to which there is an entire function g of exponential type such that

$$g(t) = \hat{K}(t)^{-1}$$

for real t.

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As will be seen in the proof it is not necessary for our purposes that g is entire of exponential type. It is e.g. sufficient that there is a positive b such that g is analytic and

$$|g(t)| \leqq M \exp(m|t|) \quad \text{for} \quad \operatorname{Im} t > -b .$$

THEOREM 1. *Let Q be a positive increasing function to which there is a constant q so that*

$$(2.1) \qquad Q(v) \leqq qQ(x) \quad \textit{for} \quad v \leqq x+1 .$$

Let φ be a bounded measurable function satisfying

$$(2.2) \qquad \varphi(v)-\varphi(x) \geqq -c/Q(x) \textit{ for } x_0 \leqq x \leqq v \leqq x+1/Q(x) ,$$

where x_0 and c are constants. Suppose that $K \in E_0$. Then

$$K*\varphi(x) = O\big(\exp(-Q(x))\big), \quad x \to \infty ,$$

implies

$$(2.3) \qquad \varphi(x) = O\big(1/Q(x)\big), \quad x \to \infty .$$

(Obviously the only interesting cases occur if Q tends to infinity with x.)

As mentioned in the introduction this theorem is proved by the method introduced in [2] and thus the final estimate is obtained by the inequality

$$(2.4) \qquad \sup_x |u(x)| \leqq 30 \left[- \inf_{x \leqq y \leqq x+1/V} \big(u(y)-u(x)\big) + \int_{-V}^{V} |\hat{u}(t)|\, dt \right],$$

which holds for every $u \in L(-\infty,\infty)$ and every positive V.

This formula will be applied with $u = k\varphi$, where k denotes the auxiliary function defined by

$$k(x) = k(x; y, \omega) = \exp\big(-\tfrac{1}{2}(x-y)^2\omega^2\big) ,$$

so that

$$\hat{k}(t) = \omega^{-1}(2\pi)^{\frac{1}{2}} \exp(-iyt - \tfrac{1}{2}t^2\omega^{-2}) .$$

If $\psi = K*\varphi$, then it is easy to see (cf. [2, p. 10]) that

$$(2.5) \qquad \hat{u}(\xi) = (\varphi k)\hat{}\,(\xi) = \int_{-\infty}^{\infty} \psi(x)\, R(x;\xi)\, dx ,$$

where

$$R(x;\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ixt)\, \hat{k}(\xi-t)\, g(t)\, dt .$$

In the following proof $O(1)$ always denotes a constant independent of x, y, ω and ξ. According to the definition of the class E_0, the inequality

$|g(t)| \leqq M \exp(m|t|)$ holds for all complex t. Changing the variable by putting $t=\xi-\tau+i\gamma$ and introducing the expression for k, we get

$$|R(x;\xi)| \leqq O(1)\exp\big(\gamma(x-y)+\tfrac{1}{2}\gamma^2\omega^{-2}+m|\gamma|+m|\xi|\big)\int_{-\infty}^{\infty}\exp(m|\tau|-\tfrac{1}{2}\tau^2\omega^{-2})\,\omega^{-1}\,d\tau$$

and, after evaluation of the integral,

$$|R(x;\xi)| \leqq O(1)\exp\big(\gamma(x-y)+\tfrac{1}{2}\gamma^2\omega^{-2}+m|\gamma|+\tfrac{1}{2}m^2\omega^2+m|\xi|\big)\,.$$

In this estimate γ is at our disposal and will be chosen in suitable ways. We assume that $\omega>1$.

By putting $\gamma=\omega^2(y-m-x)$ we find, if $x<y-m$, that

$$|R(x;\xi)| \leqq O(1)\exp\big(-\tfrac{1}{2}\omega^2(y-m-x)^2+\tfrac{1}{2}m^2\omega^2+m|\xi|\big)\,.$$

Another upper bound is obtained by taking $\gamma=-\gamma_0<0$,

$$|R(x;\xi)| \leqq O(1)\exp\big(-\gamma_0(x-y-m)+\tfrac{1}{2}m^2\omega^2+m|\xi|\big)\,.$$

Introducing these results in (2.5) we find that

$$|(\varphi k)\hat{}\,(\xi)|\exp(-m|\xi|-\tfrac{1}{2}m^2\omega^2)$$

$$\leqq O(1)\left[\int_{-\infty}^{y-2m-\gamma_0}|\psi(x)|\exp\big(-\tfrac{1}{2}\omega^2(y-m-x)^2\big)\,dx+\int_{y-2m-\gamma_0}^{\infty}|\psi(x)|\exp\big(-\gamma_0(x-y-m)\big)\,dx\right]$$

$$\leqq O(1)\left[\int_{m+\gamma_0}^{\infty}|\psi(y-m-u)|\exp(-\tfrac{1}{2}\omega^2u^2)\,du+\exp\big(-Q(y-2m-\gamma_0)+\gamma_0(3m+\gamma_0)\big)\right].$$

To get a bound for the integral on the right we recall that ψ is bounded by our assumptions. Since, for fixed positive a, it holds that

$$\int_a^{\infty}\exp(-\tfrac{1}{2}\omega^2u^2)\,du \leqq a^{-1}\omega^{-2}\exp(-\tfrac{1}{2}a^2\omega^2)\,, \tag{2.6}$$

we get by aid of (2.1) that

$$|(\varphi k)\hat{}\,(\xi)| \leqq O(1)\exp(m|\xi|)\big[\exp(-m\gamma_0\omega^2)+\exp\big(\tfrac{1}{2}m^2\omega^2-Q(y)q^{-\gamma_0-2m-1}\big)\big]\,.$$

Choosing $\omega^2=m^{-2}q^{-\gamma_0-2m-1}Q(y)$ we infer that there is a positive δ depending on m, q and γ_0 such that

$$|(\varphi k)\hat{}\,(\xi)| \leqq O(1)\exp\big(m|\xi|-\delta Q(y)\big)\,. \tag{2.7}$$

We next turn to the first term on the right side of (2.4). We observe that

$$|k(x)| \leqq 1, \quad |k'(x)| < \omega \qquad \text{for all } x\,,$$
$$|k(x)| \leqq \exp(-\tfrac{1}{2}\omega^2), \quad |k'(x)| < 1 \qquad \text{for } |x-y| \geqq 1\,.$$

Obviously

$$\inf\big(\varphi(v)\,k(v)-\varphi(x)\,k(x)\big) \geqq \inf\big(k(x)\big(\varphi(v)-\varphi(x)\big)\big)+\inf\big(\varphi(v)\big(k(v)-k(x)\big)\big)\,.$$

A lower estimate of the first term on the right is obtained by taking the sum of the (non-positive) infima for $|x-y| \leqq 1$ and for $|x-y| \geqq 1$. In the second term we proceed in a similar way after application of the mean-value theorem to the difference $k(v)-k(x)$, but we consider the two cases $|v-y| \leqq 2$ and $|v-y| \geqq 2$. Assuming that $0 \leqq h < 1$, we find that $|v-y| \geqq 2$ and $x \leqq v \leqq x+h$ imply $|x-y| \geqq 1$. Application of the inequalities for k and k' just given, shows that

$$(2.8) \qquad \inf_{x\leqq v\leqq x+h}\big(\varphi(v)\,k(v)-\varphi(x)\,k(x)\big)$$
$$\geqq \inf_{\substack{x\leqq v\leqq x+h\\ |x-y|\leqq 1}}\big(\varphi(v)-\varphi(x)\big)-O(1)\exp(-\tfrac{1}{2}\omega^2)-h\omega\sup_{|v-y|\leqq 2}|\varphi(v)|-O(h)\,.$$

Observing that

$$|\varphi(y)| = |\varphi(y)\,k(y)| \leqq \sup|\varphi(x)\,k(x)|\,,$$

and combining (2.4), (2.7) and (2.8) we obtain

$$(2.9) \quad |\varphi(y)| \leqq O(1)\left\{-\inf_{\substack{x\leqq v\leqq x+V^{-1}\\ |x-y|\leqq 2}}\big(\varphi(v)-\varphi(x)\big)+\omega V^{-1}\sup_{|v-y|\leqq 2}|\varphi(v)|+\right.$$
$$\left.+\exp(-\tfrac{1}{2}\omega^2)+V^{-1}+\exp\big(mV-\delta Q(y)\big)\right\}.$$

Let us now choose $V=\delta(2m)^{-1}Q(y)$ and recall (2.2) and that ω^2 is a multiple of $Q(y)$. Then (2.9) reduces to

$$(2.10) \qquad |\varphi(y)| \leqq O(1)\left\{Q(y)^{-1}+Q(y)^{-\frac{1}{2}}\sup_{|v-y|\leqq 2}|\varphi(v)|\right\}$$

for all sufficiently large y. Remembering that φ is bounded we get

$$|\varphi(y)| \leqq O\big(Q(y)^{-\frac{1}{2}}\big)\,.$$

Introducing this preliminary estimate in (2.10) we get by aid of (2.1) that

$$|\varphi(y)| \leqq O\big(1/Q(y)\big) \quad \text{for } y\to\infty\,,$$

and hence we have obtained (2.3). Our first theorem is proved.

3. A remainder theorem for the Stieltjes transform.

We shall now derive a similar result for a fairly general Stieltjes transform.

THEOREM 2. *Let ϱ and ν be real numbers $\varrho > \nu \geqq 0$, and let r be an increasing function such that Q defined by $Q(x) = r(e^x)$ fulfils* (2.1). *Let σ be of locally bounded variation, $\sigma(0) = 0$ and suppose that*

$$\int_0^\infty (\lambda+\omega)^{-\varrho}\, d\sigma(\lambda) = O(\omega^{\nu-\varrho}) \exp\big(-r(\omega)\big), \quad \omega \to \infty, \tag{3.1}$$

and

$$\sup_{\omega \leqq \Omega \leqq \omega + \omega/r(\omega)} \int_\omega^\Omega d\sigma(\lambda) \leqq O\big(\omega^\nu / r(\omega)\big), \quad \omega \to \infty. \tag{3.2}$$

Then

$$\sigma(\omega) = O\big(\omega^\nu / r(\omega)\big), \quad \omega \to \infty. \tag{3.3}$$

The first part of the proof is the transformation of the problem to a form similar to that treated in section 2.

After an integration by parts in (3.1) we put $\lambda = \exp y$ and $\omega = \exp x$ and obtain

$$\int_{-\infty}^\infty \big(1+\exp(y-x)\big)^{-\varrho-1} \exp\big((\nu+1)(y-x)\big)\, \sigma(\exp y) \exp(-\nu y)\, dy = O\Big(\exp\big(-Q(x)\big)\Big).$$

This formula can be written

$$H * \varphi(x) = O\Big(\exp\big(-Q(x)\big)\Big), \tag{3.4}$$

if

$$H(x) = \big(1+\exp(-x)\big)^{-\varrho-1} \exp\big(-(\nu+1)x\big)$$

and

$$\varphi(x) = \sigma(\exp x) \exp(-\nu x). \tag{3.5}$$

We now investigate $\hat{H}$ in order to see that $H \in E_0$. If B denotes the eulerian function we find

$$\hat{H}(t) = B(\nu+1+it, \varrho-\nu-it) = \Gamma(\nu+1+it)\, \Gamma(\varrho-\nu-it)/\Gamma(\varrho),$$

and since $1/\Gamma$ is entire an application of Stirling's formula reveals that $H \in E_0$.

The other conditions of theorem 1 are not satisfied, since we do not know if φ is bounded. That φ is bounded for positive values of the argument is clear from well-known pure tauberian results, e.g. that

$$\int_0^\infty (\lambda+\omega)^{-\varrho}\, d\sigma(\lambda) = O(\omega^{\nu-\varrho}) \qquad \text{implies} \qquad \sigma(\omega) = O(\omega^\nu)\,,$$

even under weaker tauberian assumptions than (3.2). In fact it is not necessary to invoke these results, since $\sigma(\omega)=O(\omega^\nu)$ may be shown to be a consequence of (3.1) and (3.2) by quite elementary but tedious calculations. I will not insist on this point.

For negative x the immediate estimate is not better than $\varphi(x)= O(\exp(\nu|x|))$ which, however, turns out to be sufficient for our purposes. The derivation of formula (2.5) still holds, since H and R decrease sufficiently rapidly to make the integral

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \varphi(y)\, H(x-y)\, R(x;\, \xi)\, dx\, dy$$

absolutely convergent.

Instead of a bounded ψ we now have to consider a function satisfying

$$|\psi(x)| \leqq O(1)+O(\exp(-\nu x))\,.$$

A glance at the derivation of formula (2.7) reveals that it holds also under this weaker condition. The only change is that (2.6) has to be replaced by

$$\int_a^\infty \exp(\nu u - \tfrac{1}{2}\omega^2 u^2)\, du \leqq (a\omega^2-\nu)^{-1} \exp(\nu a - \tfrac{1}{2}a^2\omega^2)\,,$$

true for $\nu < a\omega^2$.

There remains to check the estimates connected with the tauberian condition, and we reconsider (2.8). According to (3.5) we have

$$(3.6) \qquad \varphi(v)-\varphi(x) = \big(1-\exp\nu(v-x)\big)\exp(-\nu v)\,\sigma(\exp v) + \\ + \exp(-\nu x)\big(\sigma(\exp v)-\sigma(\exp x)\big)\,.$$

If $x_0 \leqq x \leqq v \leqq x+1/Q(x)$ we get by (3.2) that

$$\varphi(v)-\varphi(x) \geqq -\big(\exp(\nu/Q(x))-1\big) - O\big(1/Q(x)\big) \geqq O\big(1/Q(x)\big)\,.$$

If $x<x_0$ and $x\leqq v\leqq x+c$, then (3.6) shows that

$$\varphi(v)-\varphi(x) \geqq O\big(\exp(-\nu v)\big)\,,$$

since σ is bounded for arguments less than some fixed number. Returning to (2.8) we have to consider the terms $\inf[k(x)(\varphi(v)-\varphi(x))]$ for $|x-y|\geqq 1$ and $\inf[\varphi(v)(k(v)-k(x))]$ for $|v-y|\geqq 2$.

Since

$$\sup_{|x-y|\geqq 1} |k(x)\exp(-\nu x)| \leqq \exp\big(-\tfrac{1}{2}\omega^2-\nu(y-1)\big)\,,$$

we get exactly the same inequality as before, that is

$$|\varphi(y)| \leqq O(1/Q(y)) .$$

Introducing the form of φ given in (3.5) we find

$$\sigma(\omega) = O(\omega^\nu/r(\omega)) ,$$

and hence formula (3.3) is proved.

We add two remarks concerning more complicated results which can be obtained by the same method.

REMARK 1. Under the assumptions of theorem 2

$$\int_0^\omega (1-\lambda/\omega)^{m-1}\, d\sigma(\lambda) = O(\omega^\nu r(\omega)^{-m}) \tag{3.7}$$

for any natural m. This follows if we apply the formula

$$\sup_x |u(x)| \leqq C\left(-V^{-m} \inf_{x\leqq v\leqq x+1/V} \left(u^{(m)}(v)-u^{(m)}(x)\right) + \int_{-V}^{V} |\hat{u}(t)|\, dt\right)$$

instead of (2.4). For this formula see Ganelius [3].

REMARK 2. Standard arguments may be invoked to prove that theorem 2 holds also if $\omega^\nu L(\omega)$ is substituted for ω^ν on the right side of (3.1), (3.2) and (3.7), L being a slowly oscillating function.

ADDED IN PROOF. I have observed that results overlapping with my previous results but also with those of Section 3 have been obtained by M. A. Subhankulov, Trudy Mat. Inst. Steklov 64 (1961), 239–266. (Review no. 3305 in Math. Rev. 25 (1963)).

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