



Article

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## TAUBERIAN THEOREMS FOR THE STIELTJES TRANSFORM

TORD GANELIUS

### 1. Introduction.

Some years ago I published [1] a remainder theorem for the Laplace transform applicable to remainders of arbitrary order of decrease. The estimates afforded by that theorem are known to be best possible in most interesting cases. I only gave an outline of the method of proof which was a development of the well-known Karamata approximation technique.

In this paper I shall apply Fourier methods to obtain a similar result for the Stieltjes transform. The idea of the proof was given in 1962 in a paper on Wiener's tauberian theorem [2] and the result for the Stieltjes transform (Theorem 2) will in fact be obtained from a general result (Theorem 1). Among the special cases covered I ought to mention the results of Vučković [4, 5]. Theorem 2 is of interest as being applicable to the estimation of spectral functions for certain differential operators, and I have tried to formulate it in a way suitable for these applications.

For Fourier transforms and for convolutions we use the notations

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) \exp(-ixt) dx \quad \text{and} \quad K * \varphi(x) = \int_{-\infty}^{\infty} K(x-y) \varphi(y) dy.$$

### 2. The general result.

Theorem 1 may conveniently be stated for a class of kernels defined in the following way.

$E_0$  is a sub-set of  $L(-\infty, \infty)$  consisting of those functions  $K$  to which there is an entire function  $g$  of exponential type such that

$$g(t) = \hat{K}(t)^{-1}$$

for real  $t$ .

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As will be seen in the proof it is not necessary for our purposes that  $g$  is entire of exponential type. It is e.g. sufficient that there is a positive  $b$  such that  $g$  is analytic and

$$|g(t)| \leq M \exp(m|t|) \quad \text{for } \operatorname{Im} t > -b.$$

**THEOREM 1.** *Let  $Q$  be a positive increasing function to which there is a constant  $q$  so that*

$$(2.1) \quad Q(v) \leq qQ(x) \quad \text{for } v \leq x+1.$$

*Let  $\varphi$  be a bounded measurable function satisfying*

$$(2.2) \quad \varphi(v) - \varphi(x) \geq -c/Q(x) \quad \text{for } x_0 \leq x \leq v \leq x+1/Q(x),$$

*where  $x_0$  and  $c$  are constants. Suppose that  $K \in E_0$ . Then*

$$K * \varphi(x) = O(\exp(-Q(x))), \quad x \rightarrow \infty,$$

*implies*

$$(2.3) \quad \varphi(x) = O(1/Q(x)), \quad x \rightarrow \infty.$$

(Obviously the only interesting cases occur if  $Q$  tends to infinity with  $x$ .)

As mentioned in the introduction this theorem is proved by the method introduced in [2] and thus the final estimate is obtained by the inequality

$$(2.4) \quad \sup_x |u(x)| \leq 30 \left[ - \inf_{x \leq y \leq x+1/V} (u(y) - u(x)) + \int_{-V}^V |\hat{u}(t)| dt \right],$$

which holds for every  $u \in L(-\infty, \infty)$  and every positive  $V$ .

This formula will be applied with  $u = k\varphi$ , where  $k$  denotes the auxiliary function defined by

$$k(x) = k(x; y, \omega) = \exp(-\frac{1}{2}(x-y)^2 \omega^2),$$

so that

$$\hat{k}(t) = \omega^{-1} (2\pi)^{\frac{1}{2}} \exp(-iyt - \frac{1}{2}t^2 \omega^{-2}).$$

If  $\psi = K * \varphi$ , then it is easy to see (cf. [2, p. 10]) that

$$(2.5) \quad \hat{u}(\xi) = (\varphi k)^\wedge(\xi) = \int_{-\infty}^{\infty} \varphi(x) R(x; \xi) dx,$$

where

$$R(x; \xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ixt) \hat{k}(\xi-t) g(t) dt.$$

In the following proof  $O(1)$  always denotes a constant independent of  $x$ ,  $y$ ,  $\omega$  and  $\xi$ . According to the definition of the class  $E_0$ , the inequality

$|g(t)| \leq M \exp(m|t|)$  holds for all complex  $t$ . Changing the variable by putting  $t = \xi - \tau + i\gamma$  and introducing the expression for  $k$ , we get

$$|R(x; \xi)| \leq O(1) \exp(\gamma(x-y) + \frac{1}{2}\gamma^2\omega^{-2} + m|\gamma| + m|\xi|) \int_{-\infty}^{\infty} \exp(m|\tau| - \frac{1}{2}\tau^2\omega^{-2}) \omega^{-1} d\tau$$

and, after evaluation of the integral,

$$|R(x; \xi)| \leq O(1) \exp(\gamma(x-y) + \frac{1}{2}\gamma^2\omega^{-2} + m|\gamma| + \frac{1}{2}m^2\omega^2 + m|\xi|).$$

In this estimate  $\gamma$  is at our disposal and will be chosen in suitable ways. We assume that  $\omega > 1$ .

By putting  $\gamma = \omega^2(y - m - x)$  we find, if  $x < y - m$ , that

$$|R(x; \xi)| \leq O(1) \exp(-\frac{1}{2}\omega^2(y - m - x)^2 + \frac{1}{2}m^2\omega^2 + m|\xi|).$$

Another upper bound is obtained by taking  $\gamma = -\gamma_0 < 0$ ,

$$|R(x; \xi)| \leq O(1) \exp(-\gamma_0(x - y - m) + \frac{1}{2}m^2\omega^2 + m|\xi|).$$

Introducing these results in (2.5) we find that

$$\begin{aligned} & |(\varphi k)^\wedge(\xi)| \exp(-m|\xi| - \frac{1}{2}m^2\omega^2) \\ \leq & O(1) \left[ \int_{-\infty}^{y-2m-\gamma_0} |\psi(x)| \exp(-\frac{1}{2}\omega^2(y - m - x)^2) dx + \int_{y-2m-\gamma_0}^{\infty} |\psi(x)| \exp(-\gamma_0(x - y - m)) dx \right] \\ \leq & O(1) \left[ \int_{m+\gamma_0}^{\infty} |\psi(y - m - u)| \exp(-\frac{1}{2}\omega^2u^2) du + \exp(-Q(y - 2m - \gamma_0) + \gamma_0(3m + \gamma_0)) \right]. \end{aligned}$$

To get a bound for the integral on the right we recall that  $\psi$  is bounded by our assumptions. Since, for fixed positive  $a$ , it holds that

$$(2.6) \quad \int_a^{\infty} \exp(-\frac{1}{2}\omega^2u^2) du \leq a^{-1}\omega^{-2} \exp(-\frac{1}{2}a^2\omega^2),$$

we get by aid of (2.1) that

$$|(\varphi k)^\wedge(\xi)| \leq O(1) \exp(m|\xi|) [\exp(-m\gamma_0\omega^2) + \exp(\frac{1}{2}m^2\omega^2 - Q(y)q^{-\gamma_0-2m-1})].$$

Choosing  $\omega^2 = m^{-2}q^{-\gamma_0-2m-1}Q(y)$  we infer that there is a positive  $\delta$  depending on  $m, q$  and  $\gamma_0$  such that

$$(2.7) \quad |(\varphi k)^\wedge(\xi)| \leq O(1) \exp(m|\xi| - \delta Q(y)).$$

We next turn to the first term on the right side of (2.4). We observe that

$$\begin{aligned} |k(x)| &\leq 1, & |k'(x)| &< \omega & \text{for all } x, \\ |k(x)| &\leq \exp(-\tfrac{1}{2}\omega^2), & |k'(x)| &< 1 & \text{for } |x-y| \geq 1. \end{aligned}$$

Obviously

$$\inf(\varphi(v)k(v) - \varphi(x)k(x)) \geq \inf(k(x)(\varphi(v) - \varphi(x))) + \inf(\varphi(v)(k(v) - k(x))).$$

A lower estimate of the first term on the right is obtained by taking the sum of the (non-positive) infima for  $|x-y| \leq 1$  and for  $|x-y| \geq 1$ . In the second term we proceed in a similar way after application of the mean-value theorem to the difference  $k(v) - k(x)$ , but we consider the two cases  $|v-y| \leq 2$  and  $|v-y| \geq 2$ . Assuming that  $0 \leq h < 1$ , we find that  $|v-y| \geq 2$  and  $x \leq v \leq x+h$  imply  $|x-y| \geq 1$ . Application of the inequalities for  $k$  and  $k'$  just given, shows that

$$\begin{aligned} (2.8) \quad &\inf_{x \leq v \leq x+h} (\varphi(v)k(v) - \varphi(x)k(x)) \\ &\geq \inf_{\substack{x \leq v \leq x+h \\ |x-y| \leq 1}} (\varphi(v) - \varphi(x)) - O(1) \exp(-\tfrac{1}{2}\omega^2) - h\omega \sup_{|v-y| \leq 2} |\varphi(v)| - O(h). \end{aligned}$$

Observing that

$$|\varphi(y)| = |\varphi(y)k(y)| \leq \sup |\varphi(x)k(x)|,$$

and combining (2.4), (2.7) and (2.8) we obtain

$$(2.9) \quad |\varphi(y)| \leq O(1) \left\{ - \inf_{\substack{x \leq v \leq x+V^{-1} \\ |x-y| \leq 2}} (\varphi(v) - \varphi(x)) + \omega V^{-1} \sup_{|v-y| \leq 2} |\varphi(v)| + \exp(-\tfrac{1}{2}\omega^2) + V^{-1} + \exp(mV - \delta Q(y)) \right\}.$$

Let us now choose  $V = \delta(2m)^{-1}Q(y)$  and recall (2.2) and that  $\omega^2$  is a multiple of  $Q(y)$ . Then (2.9) reduces to

$$(2.10) \quad |\varphi(y)| \leq O(1) \left\{ Q(y)^{-1} + Q(y)^{-\frac{1}{2}} \sup_{|v-y| \leq 2} |\varphi(v)| \right\}$$

for all sufficiently large  $y$ . Remembering that  $\varphi$  is bounded we get

$$|\varphi(y)| \leq O(Q(y)^{-\frac{1}{2}}).$$

Introducing this preliminary estimate in (2.10) we get by aid of (2.1) that

$$|\varphi(y)| \leq O(1/Q(y)) \quad \text{for } y \rightarrow \infty,$$

and hence we have obtained (2.3). Our first theorem is proved.

**3. A remainder theorem for the Stieltjes transform.**

We shall now derive a similar result for a fairly general Stieltjes transform.

**THEOREM 2.** *Let  $\rho$  and  $\nu$  be real numbers  $\rho > \nu \geq 0$ , and let  $r$  be an increasing function such that  $Q$  defined by  $Q(x) = r(e^x)$  fulfils (2.1). Let  $\sigma$  be of locally bounded variation,  $\sigma(0) = 0$  and suppose that*

$$(3.1) \quad \int_0^\infty (\lambda + \omega)^{-\rho} d\sigma(\lambda) = O(\omega^{\nu-\rho}) \exp(-r(\omega)), \quad \omega \rightarrow \infty,$$

and

$$(3.2) \quad \sup_{\omega \leq \Omega \leq \omega + \omega/r(\omega)} \int_\omega^\Omega d\sigma(\lambda) \leq O(\omega^\nu/r(\omega)), \quad \omega \rightarrow \infty.$$

Then

$$(3.3) \quad \sigma(\omega) = O(\omega^\nu/r(\omega)), \quad \omega \rightarrow \infty.$$

The first part of the proof is the transformation of the problem to a form similar to that treated in section 2.

After an integration by parts in (3.1) we put  $\lambda = \exp y$  and  $\omega = \exp x$  and obtain

$$\int_{-\infty}^\infty (1 + \exp(y-x))^{-\rho-1} \exp((\nu+1)(y-x)) \sigma(\exp y) \exp(-\nu y) dy = O(\exp(-Q(x))).$$

This formula can be written

$$(3.4) \quad H * \varphi(x) = O(\exp(-Q(x))),$$

if

$$H(x) = (1 + \exp(-x))^{-\rho-1} \exp(-(\nu+1)x)$$

and

$$(3.5) \quad \varphi(x) = \sigma(\exp x) \exp(-\nu x).$$

We now investigate  $\hat{H}$  in order to see that  $H \in E_0$ . If  $B$  denotes the eulerian function we find

$$\hat{H}(t) = B(\nu+1+it, \rho-\nu-it) = \Gamma(\nu+1+it) \Gamma(\rho-\nu-it) / \Gamma(\rho),$$

and since  $1/\Gamma$  is entire an application of Stirling's formula reveals that  $H \in E_0$ .

The other conditions of theorem 1 are not satisfied, since we do not know if  $\varphi$  is bounded. That  $\varphi$  is bounded for positive values of the argument is clear from well-known pure tauberian results, e.g. that

$$\int_0^{\infty} (\lambda + \omega)^{-\epsilon} d\sigma(\lambda) = O(\omega^{\nu-\epsilon}) \quad \text{implies} \quad \sigma(\omega) = O(\omega^{\nu}),$$

even under weaker tauberian assumptions than (3.2). In fact it is not necessary to invoke these results, since  $\sigma(\omega) = O(\omega^{\nu})$  may be shown to be a consequence of (3.1) and (3.2) by quite elementary but tedious calculations. I will not insist on this point.

For negative  $x$  the immediate estimate is not better than  $\varphi(x) = O(\exp(\nu|x|))$  which, however, turns out to be sufficient for our purposes. The derivation of formula (2.5) still holds, since  $H$  and  $R$  decrease sufficiently rapidly to make the integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(y) H(x-y) R(x; \xi) dx dy$$

absolutely convergent.

Instead of a bounded  $\psi$  we now have to consider a function satisfying

$$|\psi(x)| \leq O(1) + O(\exp(-\nu x)).$$

A glance at the derivation of formula (2.7) reveals that it holds also under this weaker condition. The only change is that (2.6) has to be replaced by

$$\int_a^{\infty} \exp(\nu u - \frac{1}{2}\omega^2 u^2) du \leq (a\omega^2 - \nu)^{-1} \exp(\nu a - \frac{1}{2}a^2\omega^2),$$

true for  $\nu < a\omega^2$ .

There remains to check the estimates connected with the tauberian condition, and we reconsider (2.8). According to (3.5) we have

$$(3.6) \quad \varphi(v) - \varphi(x) = (1 - \exp \nu(v-x)) \exp(-\nu v) \sigma(\exp v) + \exp(-\nu x) (\sigma(\exp v) - \sigma(\exp x)).$$

If  $x_0 \leq x \leq v \leq x + 1/Q(x)$  we get by (3.2) that

$$\varphi(v) - \varphi(x) \geq -(\exp(\nu/Q(x)) - 1) - O(1/Q(x)) \geq O(1/Q(x)).$$

If  $x < x_0$  and  $x \leq v \leq x + c$ , then (3.6) shows that

$$\varphi(v) - \varphi(x) \geq O(\exp(-\nu v)),$$

since  $\sigma$  is bounded for arguments less than some fixed number. Returning to (2.8) we have to consider the terms  $\inf[k(x)(\varphi(v) - \varphi(x))]$  for  $|x - y| \geq 1$  and  $\inf[\varphi(v)(k(v) - k(x))]$  for  $|v - y| \geq 2$ .

Since

$$\sup_{|x-y| \geq 1} |k(x) \exp(-\nu x)| \leq \exp(-\frac{1}{2}\omega^2 - \nu(y-1)),$$

we get exactly the same inequality as before, that is

$$|\varphi(y)| \leq O(1/Q(y)).$$

Introducing the form of  $\varphi$  given in (3.5) we find

$$\sigma(\omega) = O(\omega^r/r(\omega)),$$

and hence formula (3.3) is proved.

We add two remarks concerning more complicated results which can be obtained by the same method.

REMARK 1. Under the assumptions of theorem 2

$$(3.7) \quad \int_0^{\omega} (1 - \lambda/\omega)^{m-1} d\sigma(\lambda) = O(\omega^r r(\omega)^{-m})$$

for any natural  $m$ . This follows if we apply the formula

$$\sup_x |u(x)| \leq C \left( -V^{-m} \inf_{x \leq v \leq x+1/V} (u^{(m)}(v) - u^{(m)}(x)) + \int_{-V}^V |\hat{u}(t)| dt \right)$$

instead of (2.4). For this formula see Ganelius [3].

REMARK 2. Standard arguments may be invoked to prove that theorem 2 holds also if  $\omega^r L(\omega)$  is substituted for  $\omega^r$  on the right side of (3.1), (3.2) and (3.7),  $L$  being a slowly oscillating function.

ADDED IN PROOF. I have observed that results overlapping with my previous results but also with those of Section 3 have been obtained by M. A. Subhankulov, Trudy Mat. Inst. Steklov 64 (1961), 239–266. (Review no. 3305 in Math. Rev. 25 (1963)).

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