An alternative Schrödinger (-Caldeón) momentum operator enabling a quantum gravity theory

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Abstract

Based on the Hilbert resp. Riesz transform operators we provide an alternative Schrödinger operator P^* with related appropriate domain given by

$$D(P^*) = H_{1/2} = H_1 \otimes H_1^\perp \cdot$$

The corresponding physical states of quantum mechanics are correspondently described by vectors of the Hilbert space

$$H_{-1/2} = H_0 \otimes H_o^\perp \cdot$$

The related location-momentum commutator vanishes for all quantum states $\in H_0$. The complementary closed space $H_o^\perp$ can be interpreted as vacuum quantum state space. As a consequence the Fourier wave concept keeps valid for quantum states $\in H_0$, while it is replaced by the Calderón (admissible) wavelet concept for the vacuum energy space $H_1^\perp$ governed by the wavelet admissibility ($H_{1/2}$ – compatible) condition

$$\int \frac{\left|\hat{\psi}^2(\xi)\right|}{|\xi|} d\xi < \infty \quad \cdot$$

The alternative framework is valid for all energy-momentum related differential equations (in its weak corresponding variational representation forms), i.e. all 4 Nature "forces" can be modelled in the same Hilbert space framework with underlying identical space-time continuum and space-time dimension $n = m+1 = 4$. With respect to the quantum states $\in H_0$ we recall the conjecture "*that distortion-free families of progressive, spherical wave of higher levels only exist if and only if the Huygens' principle is valid and that families of spherical, progressive families only can exist for space-time dimensions $n = m+1 = 2$ and $n = m+1 = 4$*" ([CoR], VI, §10) . At the same time the Riesz operators are rotation invariant ([StE], appendix).

As the Hilbert resp. Riesz operators are skew-symmetric the corresponding alternative Lorentz-invariant Schrödinger (Klein-Gordon) equation corresponds to the weak representation the wave equation, newly with respect to the inner product of the $H_{-1/2}$ – Hilbert space. For space dimension $m = 1$ the newly proposed Schödinger operator is equal to the Calderón operator

$$\Lambda := DH$$

whereby H denotes the Hilbert transform operator and $D := -i \cdot d/dx$ ([MeY], 7.1).

Our proposed concept replaces a dynamical matter-fields (laws of interaction) world (Yang-Mills field) with a truly continuum world with its own laws, being compatible also with the Einstein field equations. This means, that "*the obscure problem of laws of interaction between matter and field does* no longer *arise. This conception of the world can hardly be described as dynamical any more, since the fields neither generated by nor acting upon an agent spate from the field, but following its own laws is in a quiet continuous flow. It is of the essence of the continuum*". The proposed new framework enables "*universal field laws of atomic nuclei and electrons spreading out continuously and being subject to fine fluent changes, where e.g. the mass of an electron derives completely from the accompanying electromagnetic field. Even the atomic nuclei and the electrons are not ultimate unchangeable elements that are pushed back and forth by natural forces acting upon them, but they are themselves spread out continuously and are subject to fine fluent changes*". In other words, "*all physical phenomena can be reduced to a simple universal field law (in the form of the Hamiltonian principle)*", ([WeH1] p. 171, appendix).

In quantum theory the Schrödinger momentum operator P is defined by

$$\overline{P} := \frac{1}{i\eta} P := -\nabla = -\partial \ .$$

The corresponding commutator property

$$[x, \overline{P}] := x\overline{P} - \overline{P}x = Id$$

leads to the Heisenberg uncertainty inequality, being valid in the corresponding Hilbert space framework L_2. We propose the following alternative momentum operator in an extended quantum state Hilbert state framework $H_{-1/2}$ building on some (rotation) properties of the Hilbert transformation operator H (resp. on the corresponding properties of the Riesz operators for space dimensions m>1) and the related Symm`s log kernel operator A (see appendix):

$$\overline{P}^* := \frac{1}{i\eta} H[P] := -H[\nabla] = -H[\partial] \ .$$

Let $u_H := H[u]$ denote the Hilbert transform of $u \in H_\beta$. It holds

Lemma: The alternative Schrödinger quantized momentum operator fulfills the commutator properties

$$([x, \overline{P}^*]u, u)_{-1/2} = (u_H, u)_{-1/2} = \begin{cases} = 0 & u \in H_0 \\ \neq 0 & u \in H_0^\perp = H_{-1/2} - H_0 \end{cases}$$

Proof: From

$$[x, \overline{P}^*]u(x) = u_H(x) - (xH - Hx)u'(x)$$

it follows

$$([x, \overline{P}^*]u, u)_{-1/2} = (u_H, u)_{-1/2} - ([x, H]u', u)_{-1/2}$$

$$= (u_H, u)_{-1/2} - ([x, H][u'], Au)_0$$

$$= (u_H, u)_{-1/2} - ([x, H]A[u'], u)_0 = (u_H, u)_{-1/2} + ([x, H]u_H, u)_0$$

Applying the commutator property of the Hilbert transform for functions/distributions with vanishing constant Fourier term then leads to

$$([x, \overline{P}^*]u, u)_{-1/2} = (u_H, u)_{-1/2} \ .$$

The second property related to the orthogonality between a L_2-function and its corresponding Hilbert transform then proves the proposition above.

Note 1 ([MeY], 7.1): For space dimension $m=1$ the newly proposed Schödinger operator is equal to the Calderón operator

$$\Lambda := DH \quad \text{with} \quad D := -i \cdot d/dx$$

and H denotes the Hilbert transform operator. In [EsG], (3.17), (3.35), its corresponding generalization, the Caldeón-Zygmund integrodifferential operator, for space dimensions $m > 1$ is provided in the form

$$(\Lambda u)(x) = (\sum_{k=1}^{n} R_k D_k u)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{n} p.v. \int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta \Lambda^{-1}) u(x)$$

with ([EsG] (3.15'))

$$\Lambda^{-1} u = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}} \int_{-\infty}^{\infty} \frac{u(y) dy}{|x-y|^{n-1}} \quad , \quad n \geq 2$$

whereby R_k denotes the Riesz operators

$$R_k u = -i \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy \quad .$$

Let B denote the operator of pointwise multiplication by a function $b(x)$, then Calderón showed that the commutator $[B, \Lambda]$ is bounded on $L_2(R)$ if and only if the function $b(x)$ is Lipschitz. This characterization opened the way to the study of the operators on Hardy spaces.

Note 2: The $H_{-1/2}$ − Hilbert space is proposed to model quantum states. For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1}^{\infty} e^{-\sqrt{\lambda_i} t} (x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 \quad .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^{α}. Putting $\sigma := t := \delta$ it holds that for any bounded $x \in H_0$ (appendix)

$$\|x\|_{-1/2}^2 \leq \delta \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2 = \sigma \|x\|_0^2 + e \|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i} \sigma} x_i^2 \quad .$$

For

$$\psi = \psi_0 + \psi_0^{\perp} \in H_0 \otimes H_0^{\perp}$$

with

$$\|\psi_0\|_0 = 1 \quad , \quad \sigma := \|\psi_0^{\perp}\|_{-1/2}^2$$

one therefore gets

$$\|x\|_{-1/2}^2 \leq \sigma \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i} \sigma} x_i^2 \quad .$$

Note 3: The one-dimension Dirac "function" δ is not an element of the Hilbert space $\delta \notin H_{-1/2}$, but $\delta \in H_{-1/2-\varepsilon}$.

Note 4: For the harmonic quantum oscillator model the saw-tooth function

$$\rho(x) := x - [x] = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n} =: \frac{1}{2} - \psi(x)$$

plays a key role ([BrK1]). Its related Hilbert transform is given by

$$\rho_H(x) = \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{\pi n} = -\frac{1}{\pi}\log(2\sin(\pi x)) \in L_2^{\#}(0,1)$$

fulfilling the wavelet condition $\hat{\rho}_H(0) = 0$. The Fourier series to its related Clausen integral is given by ([AbM] 27.8)

$$f(\theta) := \int_0^{\theta} \log(2\sin\frac{t}{2})dt = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} \quad , \quad 0 \le \theta \le \pi$$

fulfilling the following identity

$$f(\pi - \theta) = f(\theta) - \frac{1}{2}f(2\theta) \quad , \quad 0 \le \theta \le \pi/2 \quad .$$

The relationship to the Riemann function is given by the following (Davenport-Chowla) identity

$$R(x) := -\sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{\pi n^2} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n} \psi(nx)$$

with (formally)

$$R'(x) := -2\sum_{n=1}^{\infty} \cos(2\pi n^2 x) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n} \psi'(nx)$$

whereby

$$\lambda(n) := (-1)^{\Omega(n)}$$

being the Liouville function, whereby $\Omega(n)$ denotes the total number of distinct prime factors of n. A corresponding function, which does not satisfy a Lipschitz condition, is given by ([ChK])

$$\phi(x) := \begin{cases} x & 0 \le x \le 1/2 \\ 1-x & 1/2 \le x \le 1 \end{cases}$$

resp.

$$\phi(x) = \frac{1}{4} - 2\sum \frac{\cos(2\pi nx)}{\pi^2 n^2} \quad , \quad n = 1,3,5,.....$$

with its formal derivative

$$\phi'(x) = 4\sum \frac{\sin(2\pi nx)}{\pi n} =: 4\tilde{\psi}(x) \quad , \quad n = 1,3,5,..... \quad .$$

Note 5: This note puts the notes above into relationship to the Planck black body radiation law. The latter one is basically about an appropriate energy density model in the electromagnetic field in the range of frequencies ω to $\omega + d\omega$ in the form ([FeR] 10-4)

$$\frac{8\pi}{(2\pi c)^3}\frac{\eta\omega^3}{e^{\eta\omega/kT}-1}d\omega .$$

The common denominator to the notes above is the Hilbert transform of the saw-tooth function

$$\rho_H(x) = H[\rho](x) = A[\rho'](x) = \sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{\pi n} = -\frac{1}{\pi}\log(2\sin(\pi x))$$

and the (exponential degree) norm estimate (note 3)

$$\|x\|_{-1/2}^2 \le \delta\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2 = \sigma\|x\|_0^2 + e\|x\|_{(\sigma)}^2 = \sigma\|x\|_0^2 + \sum_{i=1}^{\infty}e^{1-\sqrt{\lambda_i}\sigma}x_i^2 .$$

[FeR] 10-4: *The Planck black body radiation law is derived from the partition function for any system of interacting oscillators. Such a system is equivalent to a set of independent oscillators of frequencies ω_n. However, the value of the free energy F for independent systems is the sum of the values of F for each of the separate systems. This gives the free energy of a linear system in the form*

$$F = kT \cdot \sum_{n=1}^{\infty}\log(2\sinh(\frac{\eta\omega_n}{2kT}))$$

It is split into the two infinite sums

$$F = kT \cdot \sum_{n=1}^{\infty}\log(1-e^{-\eta\omega_n/kT}) + \sum_{n=1}^{\infty}\frac{1}{2}\eta\omega_n$$

whereby the last term is the ground state energy of the system. As $\omega_n \approx n$ this infinite series is diverge. The first term (while the second term is completely ignored) is used to derive the free energy of the electromagnetic field per unit volume of the blackbody radiation, i.e.

$$\frac{F}{V} = kT\int 2\log(1-e^{-\eta cK/kT})\frac{d\vec{K}}{(2\pi)^3}$$

The internal energy U is the partial derivative of βF with respect to β which becomes (putting $\omega := Kc$)

$$\frac{U}{V} = 2\int\frac{\eta\omega}{e^{-\eta\omega/kT}-1}\frac{d\vec{K}}{(2\pi)^3} .$$

The zero-point energy (which is anyway a divergent infinite series) is completely left out of this model.

Note 6: The alternative momentum operator enables a mathematical model, where "*the obscure problem of laws of interaction between matter and field does not arise. This conception of the world can hardly be described as dynamical any more, since the fields neither generated by nor acting upon an agent spate from the field, but following its own laws is in a quiet continuous flow. It is of the essence of the continuum*". The text in *italic* is quoted from ([WeH1], p 170 ff).

Note 7: For $m > 1$ the Hilbert transform is replaced by the Riesz transforms (([StE]). We especially note the "rotation invariance" property of the Riesz transform. "*It can be shown by the means of the wave equation of light (which can be immediately extended to n dimension) that only the space of an odd number of dimensions is the extinction of a candle followed by complete darkness about the candle (within a radius that increases as rapidly as light travels). This, at least, shows up an important inner difference regarding the propagation of effects between even and odd number of dimensions. Those particularly simple and harmonious laws which Maxwell had developed for the electromagnetic field in empty space are invariant with respect to an arbitrary change of the standard unit length at every point, provided the world is four-dimension. This principle of "gauge invariance" holds for no other number of dimensions*". The text in *italic* is quoted from ([WeH1], p. 136 ff). In [WeH2] a theory is provided, which links organically the electric field with the mass field. In this theory the simple and harmonic laws, which can be derived from the Maxwell electric field equations, can only be derived in a four-dimensional world. From Einstein (foundation of relativity theory) we quote: "*the Maxwell equations determine the electromagnetic field based on known distribution of charges and currents. The laws, by which those charges and currents behave, are unknown.*"

Note 8: In the context of the note above we recall from [CoR], VI, §10, the conjecture "*that distortion-free families of progressive, spherical wave of higher levels only exist if and only if the Huygens' principle is valid and that families of spherical, progressive families only can exist for space-time dimensions n=2 and n=4*".

Note 9: ([ScE] p. 157):
"*Atoms or quanta – the counter-spell of old standing, to escape the intricacy of the continuum*

Our helplessness vis-à-vis the continuum, reflected a late arrival, it stood godmother to the birth of science – an evil godmother, if you please, like the thirteenth fairy in the tale of the Sleeping Beauty. Her evil spell had for a long time been stemmed by the genial invention of atomism. This explains why atomism has proved so successful and durable and indispensable. It was not happy guess by thinkers who "really did not know anything about it" – it was the powerful counter-spell which naturally cannot be dispensed with as long as the difficulty it is to exorcise survives. By this I will not say that atomism will ever go by the board. Its valuable findings – especially the statistical theory of heat – certainly never will. Nobody can tell the future. Atomism finds itself facing a serious crisis. Atoms – our modern atoms, the ultimate particles – must no longer be regarded as identifiable individuals. This is a stronger deviation from the original idea of an atom than anybody had ever contemplated. We must be prepared for anything."

The closed space $H_0^\perp = H_{-1/2} - H_0$ can be interpreted as wave package space. It is proposed as an alternative wave package model for the Pauli-Weißkopf multi particle theory overcoming the challenge of negative energy values of the Klein-Gordon Equation (KGE).

The proposed alternative framework overcomes challenges like "for only a sufficiently high number of quanta …." or vice versa challenges like "for small number of quanta, where the term "particle" gets useless", building on a distributional wave(let) "function" in the form

$$\psi = \sum_1^\infty a_n \varphi_n + \int a_\lambda \varphi_\lambda d\lambda \cdot$$

The sub-(Hilbert) space $H_0 \subset H_{-1/2}$ "supports" a discrete spectrum while its orthogonal complementary space "supports" the corresponding continuous (ground state energy) spectrum.

The conservation laws for energy and momentum are also valid for quantum theory. Taking Planck's formula

$$E = \eta\omega = \eta k^0$$

in the single particle theory as a baseline, then the Lorentz-invariant KGE

$$\psi \in H_2: \quad KG[\psi](x) := \left((\frac{\partial^2}{\partial t^2} - \Delta) + (\frac{mc}{\eta})^2 \right)\psi(x) = 0$$

is derived from the Schrödinger equation by the canonical energy and momentum operators mapping into quantum world by

$$E \to i\frac{\partial}{\partial t} \quad, \quad \vec{p} \to -i\vec{\nabla}$$

with its related (average) energy equation

$$E^2 = p^2 + m^2 \to i^2 \frac{\partial^2}{\partial t^2}\psi = i^2\nabla^2\psi + m^2\psi \,.$$

The energy term E^2 allows negative energy values

$$E = \pm c\sqrt{\vec{p}^2 + (mc)^2}$$

of the (plane) wave solution

$$\psi(x^\mu) = Ne^{i\vec{p}\vec{x} - Et}$$

which conflicts to the current definition of the ground state energy. This creates a physical interpretation problem for those values. Pauli-Weißkopf overcame this issue by developing the multi particle theory.

An operator in a Hilbert scale framework is only completely defined in combination with the definition of its corresponding domain. In quantum mechanics the domain for the location operator Q is $D(Q) = L_2 = H_0$, i.e. the corresponding domains for the Schrödinger resp. the Laplace operator are given by $D(P) = H_1, \; D(\Delta) = H_2 \,.$

The weak representations of the location and the momentum operators are defined by

$$\psi \in H_0: \; (\psi, \chi) \to (Q\psi, \chi) \quad \forall \chi \in H_0 \quad, \quad \psi \in H_1: \; (\psi, \chi) \to (P\psi, \chi) \quad \forall \chi \in H_0 \;.$$

Applying the alternative Schrödinger operator definition above in combination with a less regular domain (i.e. interpreting those operators as Pseudo Differential Operators) leads to

$$\psi \in H_{-1/2} \quad (\psi, \chi)_{-1/2} \to (Q^*\psi, \chi)_{-1/2} \quad \forall \chi \in H_{-1/2}$$

$$\psi \in H_{1/2} \quad (\psi, \chi)_{-1/2} \to (P^*\psi, \chi)_{-1/2} \quad \forall \chi \in H_{-1/2} \,.$$

The Hilbert and the Riesz transforms are skew-symmetric. Therefore, putting as alternative covariant derivatives

$$\partial_\mu^* := (\frac{1}{c}H\left[\frac{\partial}{\partial t}\right], R[\nabla]) \quad, \quad \partial_*^\mu := (\frac{1}{c}H\left[\frac{\partial}{\partial t}\right], -R[\nabla])$$

in combination with a reduced domains $H_{1/2}$ leads to

$$\psi \in H_{3/2}: \quad KG^*[\psi](x) := \left((\frac{\partial^2}{\partial t^2} - \Delta^*) \right)\psi(x) = 0 \quad \text{resp.} \quad p_\mu^*(p^*)^\mu = 0$$

The corresponding weak variational equation is given by

$$\psi = \psi_1 + \psi_1^\perp \in H_{1/2} = H_1 \otimes H_1^\perp :$$

$$(\dot{\psi}, \chi)_{-1/2} + (\nabla\psi, \nabla\chi)_{-1/2} = (\dot{\psi}, \chi)_{-1/2} + (\psi, \chi)_{1/2} = 0 \quad \forall \chi \in H_{1/2}.$$

Now the modified "Laplace" operator (as there is a different domain defined) respectively its related (Dirichlet integral) inner product with respect to the restricted to the subset $H_1^\perp = H_{1/2} - H_1$ anticipates also the ground state matter/energy value m^2 of the classical energy formula above. In other words, both summands $\dot{p}^2 + m^2$ are governed by the (extended) Laplace operator with domain $H_{3/2}$.

Putting $\chi := \dot{\psi}$ gives for the "energy equation"

$$\psi = \psi_1 + \psi_1^\perp \in H_{1/2} = H_1 \otimes H_1^\perp :$$

$$\frac{1}{2}\frac{d}{dt}\left\{\|\dot{\psi}\|_{-1/2}^2 + \|\psi\|_{1/2}^2\right\} = 0.$$

Note 10: The closed (wave package) space $H_0^\perp = H_{-1/2} - H_0$ also provides an alternative framework to model superconductivity, super-fluids and condensates ([AnJ]). Its related $H_1^\perp = H_{1/2} - H_1$ wavelet package (momentum) space is also proposed in [BrK] to enable a finite energy balance inequality for the 3D non-linear, non-stationary Navier-Stokes Equations (NSE). From a mathematical point of view the central change is about

$$D(P) = H_1 \qquad\qquad \rightarrow\ D(P^*) = H_{1/2} = H_1 \otimes H_1^\perp$$

$$\textit{Fourier } (H_1)\ \textit{waves} \qquad \rightarrow\ \textit{Fourier } (H_1)\ \textit{waves}$$
$$\rightarrow \textit{Calderon (admissible } H_1^\perp)\ \textit{wavelets}$$

with the wavelet admissibility ($H_{1/2}$ – compatible) condition

$$\int \frac{|\hat{\psi}^2(\xi)|}{|\xi|}d\xi < \infty$$

In [BrK] a global unique weak $H_{-1/2}$ – solution of the generalized 3D Navier-Stokes initial value problem (for all $v \in H_{-1/2}$)

$$(\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} = 0$$

$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} \qquad\qquad\qquad .$$

is provided, whereby the global boundedness of the energy inequality for the space dimension 3 is a consequence of the crucial Sobolevskii estimate (see also Note 4 above) of the non-linear NSE term ([SoP])

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le |(Bu, u)_{-1/2}| \le c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

It enables the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u_0\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\left\{\|u_0\|_{-1/2} + \|u_0\|_0^2\right\} \qquad .$$

In other words, the standard 3D non-stationary, non-linear NSE is not well posed, which is the negative answer to the corresponding Millennium problem.

The H_0-based 3D NSE model provides only potential flows to model appropriate boundary layer circulation. The lift of an airfoil in inviscid flow requires circulation in the flow around the airfoil, but a single potential function that is continuous throughout the domain around the airfoil cannot represent a flow with non-zero circulation.

Note 11 (Hardy spaces H^p): From [StE1] IV 6.3, we note that a periodic function on R in the form

$$u(x) = \sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \quad \text{with} \quad |u_\nu| \le \frac{1}{\nu}$$

is an element of the function space of bounded mean oscillation, i.e. $u \in BMO(R)$.

Suppose

$$\sum_{-\infty}^{\infty} v_\nu e^{i\nu x} \in BMO(R) \;,\quad v_\nu \ge 0$$

then

$$\sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \in BMO(R) \;,\quad |u_\nu| \le v_\nu \;.$$

The Hardy spaces H^p are equivalent to L_p for $p > 1$.

For $p = 1$ we note $H^1 \cong BMO^*$, which can be seen as proper substitution of L_1 and L_∞ ([AbM] 4.7). Our concept above is about the alternative duality $H_{-1} \cong H_1^*$ and $H_{-1/2} \cong H_{1/2}^*$ of Hilbert spaces, embedded in a Hilbert scale framework instead of Banach spaces, only.

The Hilbert transform operator is skew symmetric in the space $L_2^{\#}(0,2\pi)$.

The Hilbert transform also defines a bijective mapping from the Hölder space $C^{k,\lambda}$ onto itself ([MuN]). The proof is essentially building on the Plemelj formulae. In [NiJ] for the Poisson equation, plate equation and the two-dimensional Stokes equations the regularity of the solution in dependence of a reduced regularity of the right hand sides based on standard estimates of the Newtonian potential are given.

Hölder (Banach) spaces equipped with the appropriate norm are usually applied to analyze especially non-linear PDE problems. Then the Banach Fixed Point theorem is applied to prove unique (fixed point) solutions [Ni1].

Note 12: A Galerkin analysis for Schrödinger equation by wavelets is provided in [DaD].

Appendix

Some key properties of the Hilbert transform

$$(Hu)(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{u(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy$$

are given in

Lemma: i)   The constant Fourier term vanishes, i.e. $(Hu)_0 = 0$

ii) $$H(xu(x)) = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy$$

iii)   For odd functions it hold $H(xu(x)) = x(Hu)(x)$

iv)   If $u, Hu \in L_2$ then u and Hu are orthogonal, i.e.

$$\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = 0$$

v) $\|H\| = 1$, $H^* = -H$, $H^2 = -I$, $H^{-1} = H^3$

vi) $H(f * g) = f * Hg = Hf * g$ $f * g = -Hf * Hg$

vii)   If $(\varphi_n)_{n \in N}$ is an orthogonal system, so it is for the system $(H(\varphi_n))_{n \in N}$, i.e.
$$(H\varphi_n, H\varphi_n) = -(\varphi_n, H^2\varphi_n) = (\varphi_n, \varphi_n)$$

viii) $\|Hu\|^2 = \|u\|^2$, i.e. if $u \in L_2$,   then $Hu \in L_2$.

Proof:

ii)   Consider the Hilbert transform of $xu(x)$
$$H(xu(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yu(y)}{x-y} dy \quad .$$

The insertion of a new variable $z = x - y$ yields

$$H(xu(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-z)u(x-z)}{z} dz = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(xu(x-z)}{z} dz - \frac{1}{\pi} \int_{-\infty}^{\infty} u(x-z) dz = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy$$

iv) $$\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = \frac{i}{2\pi} \int_{-\infty}^{\infty} sign((\omega)|\hat{u}(\omega)|^2 d\omega \quad \text{whereby } |\hat{u}(\omega)|^2 \text{ is even .}$$

The convolution operator

$$(Au)(x) := -\oint \log 2 \sin \frac{x-y}{2} u(y) dy =: \oint k(x-y)u(y) dy$$

has the following Fourier coefficients

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|} u_\nu \quad .$$

The eigenfunctions of the harmonic quantum oscillator

The nth Hermite polynomial is given by ([SzG])

$$H_n(z) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (z-it)^n e^{-t^2} dt \quad .$$

Hermite polynomials have only real zeros all of which are simple. The weighted Hermite polynomials

$$\varphi_n(x) := \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \quad \text{with} \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad , \quad H_0(x) = 1 \, , \quad H_1(x) = x$$

form a set of orthonormal functions. The relation to the Gauss function is given by

$$f(x) = \pi^{1/4} \varphi_0(\sqrt{2\pi} x) \quad .$$

It holds

$$\varphi_n \in L_2 \, , \quad H\varphi_n \in L_2 \, , \quad (\varphi_n, H\varphi_n) = 0$$

and

$$L_2 := H := span[\varphi_n(x)] = span[H(\varphi_n(x))] \quad .$$

All zeros of the Mellin transforms of the weighted Hermite polynomials lie on the critical line [BuD]. The proof is based on the recursion formula of the Hermite polynomials in combination with an argument from Polya ([PoG]).

The Hermite polynomials $H_n(x)$ fulfill the recursion formula

$$H_n(\sqrt{2\pi} x) = 2x H_{n-1}(\sqrt{2\pi} x) - (n-1) b_n \varphi_{n-2}(x) - 2(n-1) H_{n-2}(\sqrt{2\pi} x) \quad .$$

Using the abbreviation

$$a_n := \sqrt{\frac{2(n-1)!}{n!}} \qquad b_n := \sqrt{\frac{(n-2)!}{n!}}$$

this gives the recursion formula

$$\varphi_n(x) := a_n x \varphi_{n-1}(x) - (n-1) b_n \varphi_{n-2}(x) \, , \quad \varphi_0(x) := \pi^{-1/4} e^{-\frac{x^2}{2}} \, , \quad \varphi_1(x) := 2^{-1/2} \pi^{-1/4} x e^{-\frac{x^2}{2}}$$

The corresponding recursion formula for the Hilbert transformed Hermite polynomials are given by

$$\hat{\varphi}_n(x) := a_n \left[x\hat{\varphi}_{n-1}(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi_{n-1}(y) dy \right] - (n-1) b_n \hat{\varphi}_{n-2}(x)$$

$$\hat{\varphi}_0(x) = \pi^{1/4} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} \sin(\omega x) d\omega$$

The eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product $(.,.)$ and the norm $\|...\|$. We will consider mappings $K : H \to H$. Unless otherwise noticed the standard assumptions on K are:

    i) K is symmetric, i.e. for all $x, y \in H$ it holds $(x, Ky) = (x, Ky)$

    ii) K is compact, i.e. for any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent,

    iii) K is injective, i.e. $Kx = 0$ implies $x = 0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} \quad .$$

Lemma: Let K be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|} \quad .$$

Theorem: There exists a countable sequence $\{\lambda_i, \varphi_i\}$ of eigenelements and eigenvalues $K\varphi_i = \lambda_i \varphi_i$ with the properties

    i) the eigenelements are pair-wise orthogonal, i.e. $(\varphi_i, \varphi_k) = \delta_{i,k}$

    ii) the eigenvalues tend to zero, i.e. $\lim_{i \to \infty} \lambda_i$

    iii) the generalized Fourier sums

$$S_n := \sum_{i=1}^{n} (x, \varphi_i) \varphi_i \to x \quad \text{with } n \to \infty \text{ for all } x \in H$$

    iv) the Parseval equation

$$\|x\|^2 = \sum_{i}^{\infty} (x, \varphi_i)^2$$

holds for all $x \in H$.

Hilbert scales

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm $\|...\|$ and A be a linear operator with the properties

    i) A is self-adjoint, positive definite
    ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A)$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

    i)   there is a countable sequence $\{\lambda_i, \varphi_i\}$ with

$$A\varphi_i = \lambda_i \varphi_i \quad , \quad (\varphi_i, \varphi_k) = \delta_{i,k} \text{ and } \lim_{i \to \infty} \lambda_i$$

    ii)   any $x \in H$ is represented by

$$(*) \qquad x = \sum_{i=1}^{\infty} (x, \varphi_i)\varphi_i \quad \text{and} \quad \|x\|^2 = \sum_{1}^{\infty} (x, \varphi_i)^2 .$$

Lemma: Let $x \in D(A)$, then

$$(**) \quad Ax = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i)\varphi_i \quad , \quad \|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)^2 ,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)(y, \varphi_i) .$$

Because of (*) there is a one-to-one mapping I of H to the space $\hat{H}$ of infinite sequences of real numbers

$$\hat{H} := \{\hat{x} | \hat{x} = (x_1, x_2, ...)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \varphi_i) .$$

If we equip $\hat{H}$ with the norm

$$\|\hat{x}\|^2 = \sum_{1}^{\infty} (x, \varphi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x}, \hat{y})_\alpha = \sum_{i} \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_{i} \lambda_i^\alpha x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x}, \hat{x})_\alpha$$

Let $\hat{H}_\alpha$ denote the set of all sequences with finite α − norm. then $\hat{H}_\alpha$ is a Hilbert space. The proof is the same as the standard one for the space l_2.

Similarly one can define the spaces H_α: they consist of those elements $x \in H$ such that $Ix \in \hat{H}_\alpha$ with scalar product

$$(x, y)_\alpha = \sum_i^\infty \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x, x)_\alpha.$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (**) it holds

$$\|x\|_2^2 = (Ax, Ax)_0, \quad H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $\lim \lambda_i^\alpha \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, ...)$ with

$$\|\hat{x}\|_2^2 < \infty, \quad \|\hat{x}\|_0^2 = \infty.$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then
$$\|x\|_\alpha \leq \|x\|_\beta$$
and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then
$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$
with

$$\mu = \frac{\gamma - \beta}{\gamma - \alpha} \text{ and } \nu = \frac{\beta - \alpha}{\gamma - \alpha}.$$

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\alpha \le t^{\beta - \alpha} \|x\|_\beta$

ii) $\|x - y\|_\beta \le \|x\|_\beta \;,\; \|y\|_\beta \le \|x\|_\beta$

iii) $\|y\|_\gamma \le t^{-(\gamma - \beta)} \|x\|_\beta \;.$

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\rho \le t^{\beta - \rho} \|x\|_\beta$ for $\alpha \le \rho \le \beta$

ii) $\|y\|_\sigma \le t^{-(\sigma - \beta)} \|x\|_\beta$ for $\beta \le \sigma \le \gamma \;.$

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \;\; R(A) = H_0 \;\text{and}\; \|x\|_2 = \|Ax\| \;.$$

Extensions and generalizations

[NiJ2]: For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i} t} (x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^α.

Obviously we have

$$\|x\|_{(t)} \le c(\alpha, t) \|x\|_\alpha \text{ for } x \in H_\alpha$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the $(t) - norm$ is weaker than any $\alpha - norm$. On the other hand any negative norm, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the $0 - norm$ and the newly introduced $(t) - norm$. As it holds for any $t, \delta, \alpha > 0$ and $\lambda \ge 1$ the inequality

$$\lambda^{-\alpha} \le \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$$

one gets

Lemma: Let $\alpha > 0$ be fixed. The $\alpha - norm$ of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha} \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the $\alpha - norm$, which can be reformulated in the form ($\mu, \nu > 0, \mu + \nu > 1$)

$$\|x\|_\beta^2 \le \nu \varepsilon \|x\|_\gamma^2 + \mu e^{-\nu/\mu} \|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2 \le (\|x\|_\alpha^2)^\mu (\|x\|_\gamma^2)^\nu .$$

The counterpart of the fourth lemma above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

i) $\qquad \|x - y\| \le \|x\|$

ii) $\qquad \|y\|_1 \le \delta^{-1} \|x\|$

iii) $\qquad \|x - y\|_{(t)} \le e^{-t/\delta} \|x\| .$

In this paper we are especially concerned with the $H_{-1/2}$ – Hilbert space, as the proposed alternative framework to model quantum states. With respect to the above lemma this means that for any bounded $x \in H_0$ it holds with $\sigma := t := \delta$

$$\|x\|_{-1/2}^2 \le \delta \|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2 = \sigma \|x\|_0^2 + e\|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1}^\infty e^{1-\sqrt{\lambda_i}\sigma} x_i^2 \ .$$

For

$$\psi = \psi_0 + \psi_0^\perp \in H_0 \otimes H_0^\perp$$

with

$$\|\psi_0\|_0 = 1 \ , \quad \left\|\psi_0^\perp\right\|_{-1/2}^2 =: \sigma$$

one therefore gets

$$\|x\|_{-1/2}^2 \le \sigma \|x\|_0^2 + \sum_{i=1}^\infty e^{1-\sqrt{\lambda_i}\sigma} x_i^2 \ .$$

Eigenfunctions and eigendifferentials

Let H *be* a (infinite dimensional) Hilbert space with inner product $(.,.)$, the norm $\|...\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1} compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \to L(H,H)$$

$$\lambda \to E_\lambda := \int_{\lambda_0}^{\lambda} \varphi_\mu (\varphi_\mu,*) d\mu \quad , \quad \mu \in [\lambda_0, \infty) \;,$$

i.e.
$$dE_\lambda = \varphi_\lambda (\varphi_\lambda,*) d\lambda \;.$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ. The set E_λ fulfills the following properties:

    i) E_λ is a projection operator for all $\lambda \in R$

    ii) for $\lambda \leq \mu$ it follows $E_\lambda \leq E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

    iii) $\lim\limits_{\lambda \to -\infty} E_\lambda = 0$ and $\lim\limits_{\lambda \to \infty} E_\lambda = Id$

    iv) $\lim\limits_{\substack{\mu \to \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having the compact support $[a,b]$. Let $f : [a,b] \to R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \to H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) \;.$$

Symbolically one writes

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda \;.$$

Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \quad , \quad d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax, y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \quad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2 x, y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \quad , \quad \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \;.$$

The function

$$\sigma(\lambda) := \left\| E_\lambda x \right\|^2$$

is called the spectral function of A for the vector x. It has the properties of a distribution function. It holds the following eigenpair relations

$$A\varphi_i = \lambda_i \varphi_i \qquad A\varphi_\lambda = \lambda \varphi_\lambda \qquad \left\| \varphi_\lambda \right\|^2 = \infty \; , \; (\varphi_\lambda, \varphi_\mu) = \delta(\varphi_\lambda - \varphi_\mu) \; .$$

The φ_λ are not elements of the Hilbert space. The so-called eigen-differentials, which play a key role in quantum mechanics, are built as superposition of such eigen-functions.

Let I be the interval covering the continuous spectrum of A. We note the following representations:

$$x = \sum_1^\infty (x, \varphi_i) \varphi_i + \int_I \varphi_\mu (\varphi_\mu, x) d\mu \; , \quad Ax = \sum_1^\infty \lambda_i (x, \varphi_i) \varphi_i + \int_I \lambda \varphi_\mu (\varphi_\mu, x) d\mu$$

$$\left\| x \right\|^2 = \sum_1^\infty \left| (x, \varphi_i) \right|^2 + \int_I \left| (\varphi_\mu, x) \right|^2 d\mu \; ,$$

$$\left\| x \right\|_1^2 = \sum_1^\infty \lambda_i \left| (x, \varphi_i) \right|^2 + \int_I \lambda \left| (\varphi_\mu, x) \right|^2 d\mu$$

$$\left\| x \right\|_2^2 = \left\| Ax \right\|^2 = \sum_1^\infty \lambda_i^2 \left| (x, \varphi_i) \right|^2 + \int_I \lambda^2 \left| (\varphi_\mu, x) \right|^2 d\mu \; .$$

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\varphi_\lambda(x) = \lambda \varphi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\varphi_\lambda(x)$ it results into elements of H. The practical way to find eigen-differentials is looking for solutions of a distribution equation.

Rotation-invariance property of the Riesz operators

For this section we refer to [StE] III, 1.2, IV 4.7, 4.8:

For the Riesz operators

$$R_k u = -i c_n \, p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy \quad \text{with} \quad c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$$

let

$$m := m(x) := (m_1(x),...m_n(x))$$

denote the vector of the corresponding Mikhlin multipliers. Then for

$$\rho = \rho_{ik} \in SO(n)$$

it holds

$$m(\rho(x)) = \rho(m(x)) \quad ,$$

i.e.

$$m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$$

whereby

$$m(\rho(x)) = c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(x\rho^{-1}(y)) + \log\left|\frac{1}{x\rho^{-1}(y)}\right|) \frac{y}{|y|} d\sigma(y) = c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(xy) + \log\left|\frac{1}{xy}\right|) \frac{y}{|y|} d\sigma(y) \quad .$$

Current Standard MEP vs. proposed Non-Standard MEP

Comparison table

	Standard Model of Elementary Particles (SMEP)	Grand Unified Theory (GUT) *Proposed mathematical model*
Physical model interpretation	all "forces", which interact between different particles, are due to the interaction of related different quantum "types", resp. fields; Standard Model of Elementary Particles by the Yang-Mills field $SU(3)xSU(2)xU(1) = SU(3)xU(2)$,	no single force only "at" big bang and generation of 4 forces during inflation phase; 4 pseudo forces, only, which are measurable actions as a consequence of vacuum state energy as defined per considered variational PDE or PDO
Mathematical framework	additive Gauge fields concept per "force type" in combination with variational principles $G = U(1)$: group of rotations in the plane $G = SU(2)$: the unit of quaternions, Grassman-valued fermion field	single vacuum energy concept for all "pseudo-force" types in combination with variational principles per considered PDE/PDO
Hankel & Hilbert transforms	n/a	$H_\nu^2 = -H^2 = Id$ H_ν, H self-adjoint, L_2-isometric
Wave and wave package concept	"Dirac" function; its regularity depending from space dimension	distributional Hilbert space element of $H_{-1/2}$, the Hilbert space regularity is independent from the space dimension and better than "Dirac function" regularity for any space dimension
quantum state framework	$H_1 \subset H_0 \subset H_{-N/2-\varepsilon}, \varepsilon > 0$	$H_{1/2} \subset H_0 \subset H_{-1/2}, N > 0$
H_0 polynomial system	Hermite polynomials φ_n whereby φ_1 is the Gaussian function	Hilbert transformed Hermite polynomials φ_n^H : $(\varphi_n, \varphi_n^H) = 0$, $\hat{\varphi}_n^H(0) = 0$, i.e. the commutator $[x, P] = 0$ for any convolution (singular integral) Pseudo-Differential Operator
$H_{-1/2} = H_0 \oplus H_0^\perp$	n/a	orthogonal projection operator $P_0 : H_{-1/2} \to H_0$ $(P_0 u, v)_{-1/2} = (u, v)_{-1/2} \ \forall v \in H_0$ whereby for $u' \in H_{-1/2}$ $(u', v)_{-1/2} = (u, v)_0 \ \forall v \in H_{-1/2}$; note: the least action principle is equivalent to operator norm minimization problem
Hamiltonian operator(s)	sum of Hamiltonian operators per considered gauge field combinations $H = H_{Dirac} + H_{Yang-Mills} + H_{Higgs} + H_{Einstein}$	single Hamiltonian operator per considered variational PDE/PDO

Harmonic oscillator, Hamiltonian operator $H = -\dfrac{d^2}{dx^2} + x^2$ creation & annihilation operators with different domains H Hilbert transform operator	$\underline{A} = -\dfrac{d}{dx} + x : H_1 \to H_0{}'$ $\underline{A}^* = \dfrac{d}{dx} + x : H_1 \to H_0$ $(\underline{A}u, v)_0{}' \quad (u, \underline{A}^* v)_0 \quad \forall v \in H_0$ $\underline{H}$ self adjoint, eigenvalues real, $\underline{A}^* = -\underline{A}$ $\underline{H} = \underline{A}^*\underline{A} + 1 = \underline{A}\,\underline{A}^* - 1$	$\underline{a} = -\dfrac{d}{dx} + x : H_{1/2} \to H_{-1/2}{}'$ $\underline{a}^* = \dfrac{d}{dx} + x : H_{1/2} \to H_{-1/2}$ $(\underline{a}u, v)_{-1/2}{}' \quad (u, \underline{a}^* v)_{-1/2} \quad \forall v \in H_{-1/2}$ $\underline{a}, \underline{a}^*$ are self adjoint, because of $(u', v)_{-1/2} = (A[u'], v)_0 = (H[u], v)_0$ $= -(u, H[v])_0 = (u, A[-v'])_0 = (u, [-v'])_{-1/2}$
dependency of space dimension	yes	no
space-time dimensions	$N > 10$	4-dimensional scalar field $n = m + 1 = 4$
Poincare "conjecture"	n/a	applicable
Huygens' principle; spherical waves for temporal lines of space-time dimension	n/a	the Cauchy & radiation problem for the wave equation
particles vs. photons ratio	$6 \cdot 10^{-10}$	particles $\in H_0$, photons $\in H_{-1/2} - H_0$
Planck's action quantum	smallest action quantum, which can be measured in the test space H_0	smallest action quantum, which can be measured in the test space H_0
"time"	eigen-time of a photon is zero, i.e. a photon dies not realize that times goes by; how a photon can act? de Broglie time: $t = \dfrac{h}{E}$	R. Penrose, The Emperor's New Mind: "*the time of our perception does not "really" flow in quite the linear forward-moving way that we perceive it to do. The temporal ordering that we "appear" to perceive is, …, something that we impose upon our perceptions in order to make sense of them in relation to the uniform forward time-progression of an external physical reality*"
Schrödinger momentum operator $P := -i\eta\nabla = \dfrac{\eta}{i}\nabla$ $H := -\dfrac{\eta^2}{2m}\Delta = \dfrac{1}{2m}\left(\dfrac{\eta}{i}\nabla\right)^2$	$D(P) = H_0$: "eigen-functions" are plane-waves $f(x) = e^{ikx}$ which are not defined in H_0, i.e. $\notin H_0$	$D(P^*) = H_{-1/2}$: eigenfunctions are plane-waves $f(x) = e^{ikx}$ which are defined in H_0, i.e. $\in H_0$
Harmonic quantum oscillator (generation & annihilation operators) $H_{osc} = \dfrac{p^2}{2m} + \dfrac{1}{2}m\omega^2 q^2 = \dfrac{1}{2}\eta\omega(a^+ a + \dfrac{1}{2})$	$H_{osc} = \dfrac{1}{2}\eta\omega(a^+ a + \dfrac{1}{2})$ $(H_{osc}\psi, \psi)_0 = \infty \quad \forall \psi \in H_0$ $E_n := \eta\omega(n + \dfrac{1}{2}), \sum E_n = \infty$	$\underline{H} = c(\underline{a}^{*2} + \underline{a}^2)$ $(\underline{H}\psi, \psi)_{-1/2} < \infty \quad \forall \psi \in H_0$
"empty" vacuum	"a closed vacuum space cannot be diluted" (causing inflation/mass generation challenges in the early state of the universe)	no particles in a vacuum, "just" $H_{-1/2} - H_0 -$photons/waves
Heisenberg uncertainty principle	per affected gauge field/Hamiltonian operator	valid in $H_{-1/2}$ not valid in H_0

Schrödinger equation	classical mechanics relationship: continuity equation, Fourier waves $$\frac{\partial}{\partial t}\|\psi\|_0^2 = \frac{\eta}{2im}\left[(\psi^*, grad\psi)_0 - (\psi, grad\psi^*)_0\right]$$	classical mechanics relationship continuity equation, Calderón wavelets $$\frac{\partial}{\partial t}\|\psi\|_{-1/2}^2 = \frac{\eta}{2im}\left[(\psi^*, grad\psi)_{-1/2} - (\psi, grad\psi^*)_{-1/2}\right]$$ $$\frac{\partial}{\partial t}\|\psi\|_{-1/2}^2 \cong \frac{\eta}{2im}\left[(\psi^*,\psi)_0 - (\psi,\psi^*)_0\right] = 0' \quad \forall \psi \in H_0$$
Generation of mass of sub-atomic particles and its mass value	energy breaking, phase transition; Lagrange (Klein-Gordon) density and potential of the Higgs field Φ, which is a kind of ether existing throughout the universe $$L_{Higgs} = (D_\mu\Phi)(D^\mu\Phi)^+ - V(\Phi)$$ $$V(\Phi) := \mu^2\Phi\Phi^+ + \lambda(\Phi\Phi^+)^2'$$ D_μ covariant derivative	phase transition between $H_0 \subset H_{-1/2}$ and the closed sub-space $H_{-1/2} - H_0$
Casimir, Lamb effect	verification in test space H_0	verification in test space H_0
Elementary particles and the options $$H := H_0$$ or $$H := H_{-1/2} = H_0 \oplus H_0^\perp$$	Einstein-Podolsky-Rosen paradoxon $$H = H_0' \quad H_0^\perp = \{ \}$$ gauge/Higgs/(hypothetical) graviton bosons with only symmetric Schrödinger wave function solutions $$H = H_0' \quad H_0^\perp = \{ \}$$ "matter" particles fermions with only anti-symmetric Schrödinger wave function solutions $$\psi = \psi_0 \in H_0$$ with $\|\psi_0\|_0 = 1$ one therefore gets $$\|\psi\|_0 = \|\psi_0\|_0 = 1$$	$$H = H_{-1/2}$$ "wavelet" function/field space $$H_0$$ fermions (sub-) space = (matter) test space $$\psi = \psi_0 + \psi_0^\perp \in H_{-1/2} = H_0 \otimes H_0^\perp$$ with $$\|\psi_0\|_0 = 1 \quad \|\psi_0^\perp\|_{-1/2} =: \sigma < \infty$$ one therefore gets $$\|\psi\|_{-1/2}^2 \leq \sigma + \sum_{i=1}^\infty e^{1-\sqrt{\lambda_i}\sigma} x_i^2$$
Single and double layer potential	Lebesgue integrals, mass density $$\mu'(s) = \frac{d\mu}{ds}$$ and existing normal derivative with corresponding (mathematically required) regularity assumptions $$\frac{dU}{dn}$$	Stieltjes integrals, Plemelj's concepts of an mass element $$d\mu_s$$ and related "current of a force" **on** the boundary through the boundary element ds $$\bar{U}(\sigma) = -\int_{\sigma_0}^{\sigma}\frac{dU}{dn_s}ds$$
Quantum theory of radiation, intensity = surface force density $$[\text{int}\, ensity]$$ $$= [surface\_force\_density]$$ $$= [energy\_density]$$ $$= \frac{N}{cm^2} == \frac{Ncm}{cm^3} = Pascal$$	regularity requirement/assumptions of an existing normal derivative with exterior/interior domain w/o physical meaning	very poor regularity assumptions in the form $$\int_{\sigma_0}^{\sigma}|d\mu| < \infty$$ to ensure continuous single layer potential; concept of an apparent size of the element ds as "seen" from the interior point p and double layer potential

Maxwell-Lorentz equations, (retarded) potentials and corresponding field equations	Lorentz gauge $$A \to A - \nabla\chi$$ $$\phi \to \phi + \frac{\dot{\chi}}{c}$$ because of the assumption $$H = curlA, \quad curl(E + \frac{\dot{A}}{c}) = 0$$ and therefore $$E + \frac{\dot{A}}{c} = -\nabla\phi$$ Vector field A is not completely determined by the magnetic field H	Vector field A is completely determined by the magnetic field H; no differentiability assumption required; no calibration required due to this purely mathematical assumptions (w/o physical meaning; same situation as for the "differential manifolds" assumptions for the Einstein field equations)

Statistical Thermodynamics,
E. Schrödinger's view [ScE]

There is, essentially, only one problem in statistical thermodynamics: the distribution of a given amount of energy E over N identical systems. Or perhaps better: to determine the distribution of an assembly of N identical systems over the possible states in which this assembly can find itself, given that the energy of the assembly is a constant E. The idea is that there is weak interaction between them, so weak that the energy of interaction can be disregarded, that one can speak of the "private" energy of every one of them and that the sum of their "private" energies has to equal E. ….

"To determine the distribution" .. mean in principle to make oneself familiar with any possible distribution-of-the-energy (or state-of-the-assembly) …. is (always the same) the mathematical problem; we shall (soon) present its general solution, from which in the case of every particular kind of system enery particular classification that may be desirable can be found as a special case:

But there are two different attitudes as regards the physical application of the mathematical result. …

The older and more naïve application is to N actually exsiting physical systems in actual physical interaction with each other, e.g. gas molecules or electrons or Planck oscillators or degrees of freedom ("ether oscillators") of a "hohlraum". The N of them together represent the actual physical system under consideration. This original point of view is associated with the names of Maxwell, Boltzmann and others.

But it suffices only dealing with a very restricted class of physical systems – virtually only with gases. It is not applicable to a system which does not consist of a great number of identical constituents with "private" energies. …

Hence a second point of view … has been developed. It has a particular beauty of its own, is applicable quite generally to every physical system, and has some advantages to be mentioned forthwith. Here the N identical systems are mental copies of the one system under consideration – of the one macroscopic device that is actually erected on our laboratory table. Now what on earth could it mean, physically, to distribute a given amount of energy E over these N mental copies? The idea is, in my view, that you can, of course, imagine that you really had N copies of your system, that they really were in "weak interaction" with each other, but isolated from the rest of the world. Fixing your attention on one of them, you find it in a peculiar kind of "heat-bath" which consists of the $N-1$ others.

Now you have, on the one hand, the experience that in thermodynamical equilibrium the behavior of a physical which you place in a heat-bath is always the same whatever be the nature of the heat-bath that keeps it at constant temperature, provided, of course, that the bath is chemically neutral towards your system, i.e. that there is nothing else but heat exchange between them. On the other hand, the statistical calculations do not refer to the mechanism of interaction: they only assume that it is "purely mechanical", that it does not affect the nature of the single systems (e.g. that it never blows them to pieces), but merely transfers energy from one to the other.

These considerations suggest that we may regard the behavior of any one of those N systems as describing the one actually existing system when placed in a heat-bath of given temperature. Moreover, since N systems are alikely and number similar conditions, we can then obviously, from their simultaneous statistics, judge of the probability of finding our

system, when placed in a heat-bath of given temperature, in one or other of its private states. Hence all questions concerning the system in a heat-bath can be answered. …

The advantage consists not only in the general applicability, but also in the following two points:

    i) N can be made arbitrarily large. In fact, in case of doubt, we always mean $\lim N = \infty$ (… Stirling's formula for $N!$)

    ii)   No question about the individuality of the members of the assembly can ever arise – as it does, according to the "new statistics", with particles. Our systems are macroscopic systems, which we could, in principle, furnish with labels. Thus two states of the assembly differing by system No. 6 and system No. 13 having exchanged their roles are, of course, to be counted as different states, while the same may not be true when two similar atoms within system No. 6 have exchanged their roles; …

Remark: The approach in [BrK5] with the proposed quantum state Hilbert space given by

$$x = x_0 + x_0^\perp \in H_{-1/2} = H_0 \otimes H_0^\perp$$

fulfills the same "advantages" points i), ii) above, while the (compactly embedded, (!)) subspace $H_0 \subset H_{-1/2}$ "covers" the statistically relevant (measurable) macroscopic world and its related orthogonal space $H_0^\perp$ "covers" the "particle-interaction-world. The "heat-bath-room" of given temperature is in line with the corresponding domain of the alternatively proposed Schrödinger operator, given by $H_1^\perp$ of the corresponding energy space

$$H_1 \underset{(c)}{\subset} H_{1/2} = H_1 \otimes H_1^\perp .$$

The second section of [ScE] is concerned with the method of the most probable distribution, allowing infinite numbers of identical systems over their energy levels. We briefly sketch the central mathematical idea of chapter II:

for an assembly of N identical systems the nature of any of them is described by its possible state by enumerating them with labels $1,2,3,...l,..$ In a quantum-mechanical system those states are to be described by the eigenvalues of a complete set of commuting variables. The eigenvalues of the energy in these states are called $\varepsilon_1, \varepsilon_2, ..., \varepsilon_l,...$ so that $\varepsilon_{l+1} \geq \varepsilon_l$. In a "classical system" the schema can also be applied, when the states will have to be described as cells in phase-space (p_k, q_k) of equal volume – whether infinitesimal in all directions or not – at any rate such that the energy does not vary appreciable within the cell. The considered model parameters are

State No.	$1,2,3...l,...$
Energy	$\varepsilon_1, \varepsilon_2, ...\varepsilon_l,...$
Occupation No.	$\alpha_1, \alpha_2, ...\alpha_l,...$

The number of single states belonging to this class is

$$P = \frac{N!}{\alpha_1! \alpha_2! ... \alpha_l!,...} .$$

The set of numbers must comply with the conditions

$$N = \sum \alpha_s \quad , \quad E = \sum \varepsilon_l \alpha_s .$$

"In this form the result is wholly unsurveyable. … For N large, but finite, the assumption is only approximately true. Indeed, in the application to the Boltzmann case, the distributions with occupation numbers deviating from the "maximum set" must not be entirely disregarded. They give information on the thermodynamic fluctuations of the Boltzmann system, when kept at constant energy E, i.e. in perfect heat isolation. …

Now the fluctuations of a system in a heat bath at constant temperature are much more easily obtained directly from the Gibbs point of view."
We choose the logarithm of

$$P = \frac{N!}{\alpha_1! \alpha_2! ... \alpha_l!, ...}$$

as the function whose maximum we are determine, taking care of the accessory conditions in the usual way by Lagrange multipliers, λ and μ, seeking the unconditional maximum of

$$\log P - \lambda \sum_l \alpha_l - \mu \sum_l \varepsilon_l \alpha_l$$

leading to

$$N = \sum_l e^{-\lambda - \mu \varepsilon_l} \quad , \quad E = \sum_l \varepsilon_l e^{-\lambda - \mu \varepsilon_l} .$$

Calling $U := E/N$ the average share of energy of one system, the whole result is given by

$$U = \frac{E}{N} = \frac{\sum_l \varepsilon_l e^{-\mu \varepsilon_l}}{\sum_l e^{-\mu \varepsilon_l}} = -\frac{\partial}{\partial \mu} \log(\sum_l e^{-\mu \varepsilon_l})$$

whereby

$$\alpha_l = \frac{e^{-\mu \varepsilon_l}}{\sum_l e^{-\mu \varepsilon_l}} = -\frac{N}{\mu} \frac{\partial}{\partial \varepsilon_l} \log(\sum_l e^{-\mu \varepsilon_l})$$

indicates the distribution of the N systems over their energy levels.

Chapter IV provides three examples, which are

    i)   Free mass-point (ideal monatomic gas)

$$\Psi = k \log Z = kL \log V + \frac{3}{2} kL \log T + cons \tan t \quad \text{for } L \text{ atoms}$$

$$U = T^2 \frac{\partial \Psi}{\partial T} = \frac{3}{2} kLT \quad , \quad p = T \frac{kL}{V}$$

    ii)   Planck oscillator

$$\Psi = k \log Z = k \log(\sum e^{-\mu h \nu (l+1/2)}) = -k \log(\sinh(\frac{x}{2}) - k \log 2 ,$$

        with $x := (h \nu)/(kT) = \mu h \nu =: \mu \varepsilon$

$$U = \frac{\varepsilon}{2} + \frac{\varepsilon}{e^{\varepsilon/(kT)} - 1}$$

    iii)   Fermi oscillator

$$\Psi = k \log Z = k \log(1 + e^{-\varepsilon/kT})$$

$$U = \frac{\varepsilon}{e^{\varepsilon/(kT)} + 1} .$$

We also briefly sketch the corresponding mathematical approach for the n-particle problem ([ScE] chapter VII):

The sum-over-state of the considered n-particle problem is given by

$$Z = \sum_{n_s} e^{\mu \sum n_s \alpha_s}$$

where $n_s, s = 1,2,...$ denote the numbers of particles on level $\alpha_s, s = 1,2,...$, $\mu := 1/kT$ and the levels ε_l is given by

$$\varepsilon_j = \sum n_s \alpha_s .$$

For the Bose-Einstein gas and Fermi-Dirac gas the values admitted for every n_s are

    i) $n_s = 0,1,2,3,4,....$ Bose-Einstein gas

    ii) $n_s = 0,1$ Fermi-Dirac gas

Putting

$$z_s = e^{-\mu \alpha_s}$$

thus

$$Z = \sum_{n_s} z_1^{n_1} \cdot z_2^{n_2} z_3^{n_3} ... z_s^{n_s} ....$$

This results into

    i) $Z = \prod_s (1 - z_s)^{-1}$ Bose-Einstein gas

    ii) $Z = \prod_s (1 + z_s)$ Fermi-Dirac gas

which is combined into the following formula

$$Z = \prod_s (1 \, \mu \, z_s)^{\mu 1} .$$

In case the condition

$$n = \sum n_s$$

is imposed, this formula is not yet the final result. For a glance at the original form

$$Z = \sum_{n_s} z_1^{n_1} \cdot z_2^{n_2} z_3^{n_3} ... z_s^{n_s} ....$$

indicates that one have to select from

$$Z = \sum_{n_s} z_1^{n_1} \cdot z_2^{n_2} z_3^{n_3} ... z_s^{n_s} ....$$

only the terms homogeneous of order n in all the z_s. That is most conveniently done by the method of the residue integral.

Putting

$$f(\zeta) := \prod_s (1 \mp \zeta \cdot z_s)^{\mp 1}$$

the correct is rigorously represented by the following integral:

$$Z = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta^n} \frac{d\zeta}{\zeta} \quad.$$

The corresponding analysis leads to

i) $$N = \sum (\frac{1}{\zeta} e^{\mu\alpha_s} \mp 1)^{-1}$$

ii) $$\log Z = -n \log \zeta + \log f(\zeta) = -n \log \zeta \mp \sum_s \log(1 \mp \zeta e^{\mu\alpha_s})$$

iii) $$n_s = -\frac{1}{\mu} \frac{\partial \log Z}{\partial \alpha_s} = (\frac{1}{\zeta} e^{\mu\alpha_s} \mp 1)^{-1} \quad.$$

The related thermodynamic parameters are given by

$$n = \frac{4\pi V}{h^3} (2mkT)^{3/2} \int_0^\infty (\frac{1}{\zeta} e^{x^2} \mp 1)^{-1} x^2 dx$$

$$\Psi = k \log Z = -nk \log(\zeta) \mp \frac{4\pi V}{h^3} (2mkT)^{3/2} \int_0^\infty \log(1 \mp \zeta e^{x^2}) x^2 dx$$

$$U = kT \frac{4\pi V}{h^3} (2mkT)^{3/2} \int_0^\infty (\frac{1}{\zeta} e^{x^2} \mp 1)^{-1} x^4 dx \quad.$$

At the end this leads to a "*thermodynamic potential*" in the form

$$nkT \log(\zeta) = U - TS + pV = T^2 \frac{\partial \Psi}{\partial T} - TS + VT \frac{\partial \Psi}{\partial V} = T^2 \frac{\partial \Psi}{\partial T} - TS + \frac{2}{3} U$$

[WeH1] p. 171:

With Newton, gravitation still appears as an instantaneous action into distance. When only nearby action is considered admissible, ether theories of gravitation arise, which at first however are still under the pressure of the purely mechanical interpretation of nature. Of course Newton too was aware of the difficulty, but declined to "frame hypotheses" about the cause of gravitation (apparently he thought of a non-material transmission by virtue of a "spiritual substance" or of the all-penetrating space filled with the omnipresence of God.) The difficulty was overcome by physical means after Faraday had developed the idea of a field for the electric phenomena. Maxwell found that the field propagates from the centers of excitation not instantaneously but with the velocity of light. Nearby action laws, in the form of differential equations, connect the physical quantities characteristic of matter and field, namely charge and current densities and electrical and magnetic field strengths. The force, which with Newton is not an activity determined by and emanating from a single body k but a bond between two bodies k and k' which join hands across an abyss, is a split up into an activity of k (excitation of the filed determined by k alone) and suffering of k (temporal change of its momentum caused by that field). Between them the expanse of the field is spread out according to laws of its own of the utmost simplicity and harmony. The field transmits momentum as well as energy from one body to another; a radiating body not only loses energy, but as it radiates light in one direction it recoils in the opposite direction. In the field we therefore have spatially localized energy and momentum. The scalar densities and the components of the vectorial current densities of energy and momentum can be computed by means of simple laws from the two field strengths. The ponderomotoric effect of bodies upon one another is due to an exchange of field energy and momentum against kinetic energy and momentum of matter and vice versa; the increase or decrease in time of total energy or total momentum of any part V of the field is compensated by the current of energy or momentum going through the surface of V. If we determine the center of energy of a portion of space containing both matter and radiation, in the same way as we determine the center of gravity (mass center) of a "ponderable" body, it turns out that the total momentum $\overset{p}{I}$ contained in this portion has the same direction as the velocity $\overset{}{v}$ of the center of energy. If we set $\overset{p}{I} = m\overset{p}{v}$ the proportionality factor m may well again be called the inert mass. It is connected with the energy E by the universal relation $m = E/c^2$, where c is the velocity of light. A portion of a field such as the radiation in empty space enclosed by massless shell (Hohlraumstrahlung) possesses inert mass like an ordinary body. Thus the strength with which a body, in the face of diverting forces, persists on its natural course as prescribed by the field of inertia depends on the energy compressed in the body. The mass of the electron certainly derives in part from accompanying electromagnetic field. *Or even completely?* Since all physically important properties of an elementary material particle, as we have seen, belong to the surrounding field rather than the substantial nucleus at the field center, the question becomes inevitable whether the existence of such a nucleus is not a presumption that may be completely dispensed with.

This question is answered in the affirmative by the *field theory of matter*. According to the latter a material particle such as an electron is merely a small domain of electrical field within which the field strength assumes enormously high values, indicating that comparatively huge field energy is concentrated in a very small space. Such an energy knot, which by no means is clearly delineated against the remaining field, propagates through empty space like a water wave across the surface of a lake; there is no such thing as one and the same substance of which the electron consists at all times. Just as the velocity of a water wave is not a substantial but a phase velocity, so that the velocity with which an electron moves is only the velocity of an ideal "center of energy," constructed out of the field distribution. According to this view, there exists but one kind of natural law, namely, field laws of the same transparent nature as Maxwell had established for the electromagnetic field. The obscure problem of laws of interaction between matter and field does not arise. This conception of the world can hardly be described as dynamical any more, since the fields neither generated by nor acting upon an agent spate from the field, but following its own laws is in a quiet continuous flow. It is of the essence of the continuum. Even the atomic nuclei and the electrons are not ultimate unchangeable elements that are pushed back and forth by natural forces acting upon them, but they are themselves spread out continuously and are subject to fine fluent changes. …

Beside the electromagnetic field we have the metric or gravitation field. The task of merging both into one unit arises.

Auszüge aus

Was ist Materie?

Hermann Weyl, [WeH]

Seite 28: Aufgrund der angegebenen Tatsachen kommt man notgedrungen zu der Auffassung, daß die Definition „*Kraft = Ableitung des Impulses*" das Wesen der Kraft nicht richtig wiedergibt. Der wirkliche Sachverhalt ist vielmehr umgekehrt: Die Kraft ist der Ausdruck für eine selbständige, die Körper zufolge ihren inneren Natur und ihrer gegenseitigen Lage und Bewegungsbeziehung verknüpfende Potenz, welche die zeitliche Änderung des Impulses verursacht

Seite 31: Die Definition des Feldes mit Hilfe seiner ponderomotorischen Wirkung auf einen Probekörper ist nur ein Provisorium. Durch hereinbringen des geladenen Probekörpers stört man immer in etwas das Feld, das es eigentlich zu beobachten galt; befindet es sich im Felde, so gehört er so gut wie die übrigen Konduktoren mit zu den das Feld erzeugenden Ladungen. Das wahre Naturgesetz, das an die Stelle von „Kraft = Probekörperladung * Feldstärke" tritt, wird also anzugeben haben, was für Kräfte das von irgendwie verteilten Ladungen erregte elektrische Feld *auf diese Ladungen selber* ausübt.

Seite 35: Wenn so alle physikalisch wesentlichen Eigenschaften des Elektrons an dem umgebenden Felde und nicht an dem im Feldzentrum steckenden substantiellen Kerne hängen, so muß man sich doch fragen, *ob denn überhaupt die Annahme eines derartigen Kernes nötig ist oder ob wir ihn nicht ganz entbehren können.* Die letzte Frage beantwortet die Feldtheorie der Materie mit Ja; ein Materieteilchen wie das Elektron ist für sie lediglich ein kleines Gebiet des elektrischen Feldes, in welchem die Feldstärke enorm hohe Werte annimmt und wo demnach auf kleinstem Raum eine gewaltige Feldenergie konzentriert ist. Ein solcher Energieknoten, der gegen das übrige Feld keineswegs scharf abgegrenzt ist _ der geometrische Begriff des Elektronenradius verliert also seinen präzisen Sinn – pflanzt sich durch den leeren Raum nicht anders fort, wie eine Wasserwelle über die Seefläche fortschreitet; es gibt da nicht ein und dieselbe Substanz, aus der das Elektron zu allen Seiten besteht. Wie die Geschwindigkeit einer Wasserwelle nicht substanzielle, sondern Phasengeschwindigkeit ist, so handelt es sich bei der Geschwindigkeit, mit der sich das Elektron bewegt, auch nur um die Geschwindigkeit eines ideellen, aus dem Feldverlauf konstruierten „Energiemittelpunktes". Läßt sich diese Auffassung durchführen, durch welche der die Physik seit FARADAY und MAXWELL beherrschende Dualismus vom Materie und Feld zugunsten des Feldes überwunden wird, so ergäbe sich ein außerordentlich einheitliches Weltbild. Statt der drei Arten von Gesetzen, nach denen das Feld 1. durch die Materie erregt, emittiert wird, 2. sich ausbreitet und 3. auf die Materie wirkt, behalten wir nur die Feldgesetze 2 übrig vom Typus der Maxwellschen Gleichungen, deren Struktur uns völlig durchsichtig ist, während die Gesetze 1 und 3, in deren Dunkel die Physik auch heute noch kaum eingedrungen ist, überflüssig werden. Insbesondere die Gültigkeit der mechanischen Gleichungen gewährleisten durch den aus den Feldgesetzen folgenden differentiellen Energie-Impulssatz, dessen Energiekomponente für das Maxwellsche Feld in Formel (15) angegeben wurde. Man kann dieses Weltbild kaum mehr als ein dynamisches mehr bezeichnen, weil hier das Feld weder von einem dem Felde gegenüberstehenden materiellen Agens erzeugt wird noch auf ein solches wirkt, sondern lediglich, seiner Eigengesetzlichkeit folgend, in einem still kontinuierlichem Fließen begriffen ist. Es ruht ganz und gar im *Kontinuum*, auch die Atomkerne und Elektronen sind keine letzten unveränderlichen, von den angreifenden Naturkräften hin und her geschobenen Elemente, sondern selber stetig ausgebreitet und feinen fließenden Veränderungen unterworfen.

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