

On A Rapidly Converging Series For The Riemann Zeta Function

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Abstract

To evaluate Riemann's zeta function is important for many investigations related to the area of number theory, and to have quickly converging series at hand in particular. We investigate a class of summation formulae and find, as a special case, a new proof of a rapidly converging series for the Riemann zeta function. The series converges in the entire complex plane, its rate of convergence being significantly faster than comparable representations, and so is a useful basis for evaluation algorithms. The evaluation of corresponding coefficients is not problematic, and precise convergence rates are elaborated in detail. The globally converging series obtained allow to reduce Riemann's hypothesis to similar properties on polynomials. And interestingly, Laguerre's polynomials form a kind of leitmotif through all sections.

Keywords: Riemann Zeta function, Kummer function, Laguerre polynomials, Fourier transform, Riemann hypothesis.

1. Introduction and Definitions

1.1. Confluent Hypergeometric Functions

Confluent hypergeometric functions are special hypergeometric functions, sometimes called also Kummer's function of first and second kind. They are linear independent solutions of Kummer's differential equation

$$z y''(z) + (b - z) y'(z) - a y(z) = 0.$$

The first solution, Kummer's function of the first kind, is usually given as a globally converging power series

$$M(a; b; z) := \sum_{i=0}^{\infty} \frac{\binom{a+i-1}{i} z^i}{\binom{b+i-1}{i} i!} \quad (1.1)$$

(Kummer's function of the first kind), and Kummer's function of the second kind is often given by

$$U(a; b; z) := \frac{\Gamma(1-b)}{\Gamma(1-b+a)} M(a; b; z) - \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1; 2-b; z) \quad (1.2)$$

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Kummer's transformation states that $M(a; b; z) = e^z M(b-a; b; -z)$, which subsequently leads to the identity $U(a; b; z) = z^{1-b} U(1+a-b; 2-b; z)$.

Another solution of this differential equation – which turns out to be identical to U and thus is simply another representation of U – is obtained as an integral representation by the Laplace transform

$$\begin{aligned} U(a; b; z) &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt \\ &= \frac{z^{1-b}}{\Gamma(1+a-b)} \int_0^\infty e^{-zt} \frac{t^{a-b}}{(1+t)^a} dt. \end{aligned} \quad (1.3)$$

Some particular and frequently used confluent hypergeometric functions are the upper and lower incomplete Gamma function, which have the representations

$$\begin{aligned} \Gamma(s, z) &:= \int_z^\infty t^{s-1} e^{-t} dt \\ &= e^{-z} U(1-s, 1-s, z) = e^{-z} z^s U(1, 1+s, z) \end{aligned} \quad (1.4)$$

and

$$\begin{aligned} \gamma(s, z) &:= \int_0^z t^{s-1} e^{-t} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{z^{s+k}}{s+k} \\ &= \frac{z^s}{s} M(s, s+1, -z) = \frac{z^s}{s} e^{-z} M(1, s+1, z). \end{aligned} \quad (1.5)$$

1.2. Laguerre's Polynomials

For a a negative integer the defining series for M reduces to a polynomial, which turns out to be closely related to Laguerre's polynomials: The explicit representation is

$$L_i^{(\alpha)}(z) = \binom{i+\alpha}{i} M(-i, \alpha+1, z) = \sum_{j=0}^i (-1)^j \binom{i+\alpha}{i-j} \frac{z^j}{j!}. \quad (1.6)$$

In view of (1.2) Laguerre's polynomial may be given by Kummer's second function as well, that is $L_i^{(\alpha)}(z) = \frac{(-1)^i}{i!} U(-i, \alpha+1, z)$. This somehow suggests that Laguerre's polynomials are somewhat in between of both solutions M and U of Kummer's differential equation. This is central in our investigations and reflected in the results of the next sections.

In addition to that it is well-known that Laguerre's polynomials are orthogonal with respect to the weight-function $z^\alpha e^{-z}$; more precisely we find that

$$\int_0^\infty z^\alpha e^{-z} L_i^{(\alpha)}(z) L_j^{(\alpha)}(z) dz = \begin{cases} 0 & i \neq j \\ \binom{i+\alpha}{i} \Gamma(\alpha+1) & i = j, \end{cases} \quad (1.7)$$

Laguerre's polynomials thus are orthogonal with respect to the inner product

$$\langle g|f \rangle := \int_0^\infty \frac{z^\alpha e^{-z}}{\Gamma(\alpha+1)} g(z) f(z) dz.$$

As a result of the classical theory on Hilbert spaces we may expand a function f in a series with respect to this orthogonal basis, provided that $\|f\|_2 := \sqrt{\langle f|f \rangle} < \infty$. The function has the

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expansion $f(z) = \sum_{i=0}^{\infty} f_i^{(\alpha)} L_i^{(\alpha)}(z)$ (and is therefore in the closed hull of all $L_i^{(\alpha)}$), its coefficients taking the explicit form

$$f_i^\alpha = \int_0^\infty \frac{L_i^{(\alpha)}(z)}{\binom{i+\alpha}{i}} \frac{z^\alpha e^{-z}}{\Gamma(\alpha+1)} f(z) dz \quad (1.8)$$

(cf. (1.7)). Conversely, the norm can be recovered from the function's coefficients, as $\|f\|_2^2 = \sum_{i=0}^{\infty} \binom{i+\alpha}{i} |f_i^\alpha|^2$.

An elementary example of such a representation in explicit terms is

$$e^{-tz} = \frac{1}{(1+t)^{\alpha+1}} \sum_{i=0}^{\infty} L_i^{(\alpha)}(z) \left(\frac{t}{1+t}\right)^i, \quad (1.9)$$

as can be verified straight forward by evaluating the respective integrals for the coefficients (1.8). Notably, this series converges point-wise if $t > -\frac{1}{2}$ ($\|f\|_2 < \infty$), even if $\alpha \leq -1$.

To give a reference of these classical ingredients aggregated in this section above we would like to refer to the standard work [AS64].

2. Fourier Series

Kummer's functions allow some explicit representation as series of Laguerre polynomials.

Theorem 1 (Expansion of Kummer's functions in terms of Laguerre polynomials). *Suppose that $\operatorname{Re}(b-a) > 0$ and $\operatorname{Re}(\alpha - 2b) > -\frac{5}{2}$, then*

$$M(a; b; z) = \frac{\binom{b-1}{a}}{\binom{b-\beta-1}{a}} \sum_{i=0}^{\infty} (-1)^i L_i^{(\beta-i)}(z) \frac{\binom{-a}{i}}{\binom{\beta-b}{i}}$$

and

$$U(a; b; z) = \frac{(1+\alpha-b)!}{(1+\alpha-b+a)!} \sum_{i=0}^{\infty} L_i^{(\alpha)}(z) \frac{\binom{-a}{i}}{\binom{b-a-\alpha-2}{i}}; \quad (2.1)$$

moreover,

$$\Gamma(s; z) = z^s e^{-z} \sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}(z)}{(k+1) \binom{k+1+\alpha-s}{k+1}} \quad (2.2)$$

for $\operatorname{Re}\left(s - \frac{\alpha}{2}\right) < \frac{1}{4}$.

Proof. As for the proof notice first that $\frac{z^i}{i!} = (-1)^i \sum_{j=0}^i \binom{-\beta}{i-j} L_j^{(\beta-j)}(z)$, which is a kind of converse of (1.6) and verified straight forward by comparing the coefficients of respective powers of z . Substituting this into (1.1), interchanging the order of summation and employing Gauss'

hypergeometric theorem (cf. [Koe98]) gives

$$\begin{aligned}
 M(a; b; z) &= \sum_{i=0}^{\infty} \frac{\binom{a+i-1}{i}}{\binom{b+i-1}{i}} (-1)^i \sum_{j=0}^i \binom{-\beta}{i-j} L_j^{(\beta-j)}(z) \\
 &= \sum_{j=0}^{\infty} L_j^{(\beta-j)}(z) \sum_{i=j}^{\infty} (-1)^i \frac{\binom{a+i-1}{i}}{\binom{b+i-1}{i}} \binom{-\beta}{i-j} \\
 &= \sum_{j=0}^{\infty} L_j^{(\beta-j)}(z) (-1)^j \frac{\Gamma(b) \Gamma(b-a-\beta) \Gamma(a+j)}{\Gamma(a) \Gamma(b-a) \Gamma(b-\beta+j)} \\
 &= \sum_{j=0}^{\infty} L_j^{(\beta-j)}(z) (-1)^j \frac{\binom{b-1}{a}}{\binom{b-\beta-1}{a}} \frac{\binom{a+j-1}{j}}{\binom{b-\beta-1+j}{j}} \\
 &= \frac{\binom{b-1}{a}}{\binom{b-\beta-1}{a}} \sum_{j=0}^{\infty} (-1)^j L_j^{(\beta-j)}(z) \frac{\binom{-a}{j}}{\binom{\beta-b}{j}},
 \end{aligned}$$

which is the desired assertion. As regards convergence it follows from the elementary recursion

$$L_j^{(\beta-j)}(z) = -\left(1 + \frac{z-\beta-1}{j}\right) L_{j-1}^{(\beta-j+1)}(z) - \frac{z}{j} L_{j-2}^{(\beta-j+2)}(z) \quad (2.3)$$

that $(-1)^j L_j^{(\beta-j)}(z) = \mathcal{O}(j^{-\beta-1})$, a more thorough analysis even discloses that $\frac{L_j^{(\beta-j)}(z)}{\binom{\beta}{j}} = e^z \left(1 + \mathcal{O}\left(\frac{1}{j}\right)\right)$.

Hence, $(-1)^j L_j^{(\beta-j)}(z) \frac{\binom{-a}{j}}{\binom{\beta-b}{j}} = \mathcal{O}\left(j^{-\beta-1} \frac{j^{a-1}}{j^{b-\beta-1}}\right) = \mathcal{O}\left(\frac{1}{j^{b-a+1}}\right)$, and the series converges – irrespective of β – if $\operatorname{Re}(b-a) > 0$.

To verify the second statement notice that

$$\begin{aligned}
 U_i^\alpha &= \int_0^\infty \frac{L_i^{(\alpha)}(z)}{\binom{i+\alpha}{i}} \frac{z^\alpha e^{-z}}{\Gamma(\alpha+1)} U(a; b; z) dz \\
 &= \int_0^\infty \frac{L_i^{(\alpha)}(z)}{\binom{i+\alpha}{i}} \frac{z^\alpha e^{-z}}{\Gamma(\alpha+1) \Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt dz \\
 &= \int_0^\infty \frac{t^{a-1}}{\Gamma(a)} (1+t)^{b-a-1} \int_0^\infty \frac{L_i^{(\alpha)}(z)}{\binom{i+\alpha}{i}} \frac{z^\alpha e^{-z}}{\Gamma(\alpha+1)} e^{-zt} dz dt \\
 &= \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} (1+t)^{b-a-1} \frac{t^i}{(1+t)^{\alpha+i+1}} dt,
 \end{aligned}$$

which is a consequence of (1.3) and (1.9). The latter integral is a beta function, we thus continue and find the coefficient

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Algorithm 1 to evaluate $L_i^{(\alpha)}(z)$, based on $L_i^{(\alpha)}(z) = \left(2 + \frac{\alpha-1-z}{i}\right)L_{i-1}^{(\alpha)}(z) - \left(1 + \frac{\alpha-1}{i}\right)L_{i-2}^{(\alpha)}(z)$.

```

L1:=0; Laguerre:= 1
For j:= 1 to i
    L0:= L1; L1:= Laguerre;
    Laguerre:= ((2* j+ alpha - 1- z)* L1- (j+ alpha - 1)* L0)/ j;
Next j
Return Laguerre

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Algorithm 2 to evaluate $L_i^{(\beta-i)}(z)$, based on (2.3)

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L1:=0; Laguerre:= 1
For j:= 1 to i
    L0:= L1; L1:= Laguerre;
    Laguerre:= ((beta+ 1- j- z)* L1- z* L0)/ j;
Next j
Return Laguerre

```

$$\begin{aligned}
 U_i^\alpha &= \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1+i} (1+t)^{b-a-\alpha-i-2} dt \\
 &= \frac{1}{\Gamma(a)} \frac{\Gamma(a+i)\Gamma(\alpha+2-b)}{\Gamma(i+a+\alpha+2-b)} \\
 &= \frac{(\alpha+1-b)!}{(i+a+\alpha+1-b)!} \frac{\binom{a+i-1}{i}}{\binom{i+a+\alpha+2-b}{i}}
 \end{aligned}$$

which is the respective coefficient for U .

Convergence is more difficult compared to Kummer's function of the first kind. However, we will show below (Theorem 9) that $\limsup_i \frac{L_i^{(\alpha)}(z)}{i^{\frac{\alpha}{2}-\frac{1}{4}}} < \infty$ and the series thus converges if $\operatorname{Re}(\alpha - 2b) > -\frac{5}{2}$. \square

Remark 2. Identity (2.1) can be found for the special case $\alpha = b - 1$ in [EMOT53].

To evaluate the series above using Laguerre polynomials it is necessary to have good evaluations of Laguerre polynomials at hand for given parameters α and z . Moreover, numerical approximations should be sufficiently good and the computation stable. Although there are explicit expressions or Horner's scheme available to evaluate Laguerre polynomials, the resulting algorithms usually behave unstable very soon.

We have found the algorithms described much better, adaptations even allow to evaluate and then successively store intermediary results. However, they should not be interchanged, as this will cause numerical instability again.

3. Continuous Fourier Transform and Poisson Summation Formula

3.1. Continuous Fourier Transform.

It is well-known that Poisson's summation formula provides an efficient tool to evaluate sums, some authors dedicate entire chapters to these summation techniques, see for instance [IK04]. To apply these effective summation identities we need to have a good expression for the functions involved at hand, which involve the continuous Fourier transform.

In literature there occur a few variants for the continuous Fourier transform of a function f , which are interchanged frequently; for our purposes it is most convenient to state

$$\hat{f}(k) := \mathcal{F}(f)(k) := \int_{-\infty}^{\infty} f(z) e^{-2\pi i k z} dz$$

as a definition.

We will sometimes abuse the notation just introduced and improperly write $\mathcal{F} f(z)$ shortly for $\mathcal{F}(f(z))(z)$, that is to say we use the argument z for both functions – f and $\mathcal{F}(f)$ – synonymously.

The following result establishes that Laguerre's polynomials, as well as Kummer's functions are each others Fourier transform in the following sense:

Theorem 3 (Fourier transform of Kummer's functions).

1. $\mathcal{F} e^{-z^2\pi} L_i^{(\alpha)}(z^2\pi) = (-1)^i e^{-z^2\pi} L_i^{(-i-\alpha-\frac{1}{2})}(z^2\pi)$,
2. $\mathcal{F} e^{-z^2\pi} U(a, b, z^2\pi) = \frac{\Gamma(\frac{3}{2}-b)}{\Gamma(a+\frac{3}{2}-b)} e^{-z^2\pi} M(a, a+\frac{3}{2}-b, z^2\pi)$,
3. $\mathcal{F} e^{-z^2\pi} M(a, b, z^2\pi) = \frac{\Gamma(b)}{\Gamma(b-a)} e^{-z^2\pi} U(a, a+\frac{3}{2}-b, z^2\pi)$.

Proof. Notice first, that $e^{-z^2\pi}$ is an eigenfunction of the Fourier transform \mathcal{F} , which already covers the desired statement for $i = 0$. Now recall that – due to integration by parts – $\mathcal{F}\left(\frac{d^j}{dz^j} f(z)\right) = (2\pi iz)^j \mathcal{F}(f(z))$ and observe that differentiating this first eigenfunction involves Laguerre's polynomials again, as $\frac{d^{2j}}{dz^{2j}} e^{-z^2\pi} = (-4\pi)^j j! e^{-z^2\pi} L_j^{(-\frac{1}{2})}(z^2\pi)$. Hence,

$$\begin{aligned} \frac{(z^2\pi)^j}{j!} e^{-z^2\pi} &= \frac{(-1)^j}{(4\pi)^j j!} (2\pi iz)^{2j} \mathcal{F} = e^{-z^2\pi} \frac{(-1)^j}{(4\pi)^j j!} \mathcal{F}\left(\frac{d^{2j}}{dz^{2j}} e^{-z^2\pi}\right) \\ &= \frac{(-1)^j}{(4\pi)^j j!} \mathcal{F}\left((-4\pi)^j j! e^{-z^2\pi} L_j^{(-\frac{1}{2})}(z^2\pi)\right) = \mathcal{F} e^{-z^2\pi} L_j^{(-\frac{1}{2})}(z^2\pi). \end{aligned}$$

Next, by (1.6) and the identity just derived,

$$\begin{aligned} (-1)^i e^{-z^2\pi} L_i^{(-i-\alpha-\frac{1}{2})}(z^2\pi) &= e^{-z^2\pi} \sum_{j=0}^i (-1)^{i-j} \binom{-\alpha-\frac{1}{2}}{i-j} \frac{(z^2\pi)^j}{j!} \\ &= \mathcal{F} e^{-z^2\pi} \sum_{j=0}^i \binom{\alpha-\frac{1}{2}+i-j}{i-j} L_j^{(-\frac{1}{2})}(z^2\pi) = \mathcal{F} e^{-z^2\pi} L_i^{(\alpha)}(z^2\pi), \end{aligned}$$

the latter identity being a special case ($x = 0, \beta = -\frac{1}{2}$) of the more general identity

$$\sum_{j=0}^i L_{i-j}^{(\alpha)}(x) L_j^{(\beta)}(y) = L_i^{(\alpha+\beta+1)}(x+y).$$

This proves the first statement.

The other statements are an immediate consequence of this first one and Theorem 1. \square

3.2. Poisson Summation Formula.

Given the notation introduced Poisson's summation formula reads $\sum_{z \in \mathbb{Z}} f(z) = \sum_{k \in \mathbb{Z}} \hat{f}(k)$ (cf. [Zyg68, Pin02]), or a bit more generally

$$\sum_{z \in \mathbb{Z}} f((z - z_0)x) e^{-2\pi i(z - z_0)k_0} = \frac{1}{x} \sum_{k \in \mathbb{Z}} \hat{f}\left(\frac{k + k_0}{x}\right) e^{-2\pi i k z_0} \quad (3.1)$$

when applied to the same function f but translated (z_0), contracted (x) and shifted in phase (k_0).

As already mentioned, the Poisson summation formula often transforms slowly converging series into very rapidly converging series. We will exploit this fact to obtain the following results, which are an application of Poisson's summation formula to the Fourier transforms elaborated in the latter section.

4. Application to Riemann Zeta Function

It turns out that a particular application of summation techniques outlined above is a very rapidly converging series for the Riemann zeta function. We are even able to prove these following variants, and Riemann's function equation follows as a by-product:

Theorem 4. *For any $s \in \mathbb{C}$ we have*

$$s(s-1) \frac{\zeta(s) \Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = 1 - s(1-s) \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{s}{2}, k^2 \pi\right)}{(k^2 \pi)^{\frac{s}{2}}} + \frac{\Gamma\left(\frac{1-s}{2}, k^2 \pi\right)}{(k^2 \pi)^{\frac{1-s}{2}}}. \quad (4.1)$$

More generally, for every arbitrarily chosen x satisfying $\operatorname{Re}(x) > |\operatorname{Im}(x)|$ we find

$$s(s-1) \frac{\zeta(s) \Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = (1-s)x^s + s x^{s-1} + s(s-1) \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{s}{2}, k^2 x^2 \pi\right)}{(k^2 \pi)^{\frac{s}{2}}} + \frac{\Gamma\left(\frac{1-s}{2}, \frac{k^2 \pi}{x^2}\right)}{(k^2 \pi)^{\frac{1-s}{2}}};$$

moreover

$$(2^s - 1)(1 - 2^{1-s}) \frac{\zeta(s) \Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = \Upsilon_x(s) + \Upsilon_{\frac{1}{x}}(1-s),$$

where $\Upsilon_x(s)$ is the 2^{nd} forward difference

$$\Upsilon_x(s) = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{s}{2}, \frac{(4k+1)^2 \pi x^2}{4}\right)}{\left(\frac{(4k+1)^2 \pi}{4}\right)^{\frac{s}{2}}} - 2 \frac{\Gamma\left(\frac{s}{2}, \frac{(4k+2)^2 \pi x^2}{4}\right)}{\left(\frac{(4k+2)^2 \pi}{4}\right)^{\frac{s}{2}}} + \frac{\Gamma\left(\frac{s}{2}, \frac{(4k+3)^2 \pi x^2}{4}\right)}{\left(\frac{(4k+3)^2 \pi}{4}\right)^{\frac{s}{2}}}.$$

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Remark 5. It should be stressed that these are globally convergent series for a function analytic in the *entire* plane, converging in particular for $s = 1$. Moreover, by de l'Hôpital's rule, $\lim_{z \rightarrow 1} \frac{\Gamma(s; z)}{z^{s-1} e^{-z}} = \lim_{z \rightarrow 1} \frac{z^{s-1} e^{-z}}{z^{s-1} e^{-z} - (s-1)z^{s-2} e^{-z}} = 1$, the rate of convergence thus is of order

$$\begin{aligned} O\left(\sum_{k'=k}^{\infty} e^{-k'^2 \pi}\right) &= O(e^{-k^2 \pi}) \\ &= O(0.04321 \dots^{k^2}), \end{aligned}$$

which is pretty quick.

As an additional result, Riemann's function equation follows immediately from equation (4.1).

It seems that an identity close to (4.1) already was known to Riemann himself, although the proof being based on integrals involving the Jacobi function $\theta_3(z) := \sum_{k \in \mathbb{Z}} e^{-k^2 \pi z}$, or rather the function $\tilde{\theta}(z) := \theta'_3(z) + \frac{3}{2} \theta''_3(z)$: Both satisfy $\theta_3(z) = \frac{1}{\sqrt{z}} \theta_3\left(\frac{1}{z}\right)$, but $\theta_3^{(k)}(z) \rightarrow 0$ as $z \rightarrow 0$ and $z \rightarrow \infty$ for any $k \in \{0, 1, 2, \dots\}$.¹ By exploiting this fact (4.1) can be recovered as well – this is for example demonstrated in the third method (out of 7) of proving the function equation in [Tit86].

Remark 6. The famous, fast algorithm for multiple evaluations of the Riemann Zeta function in [OS88] is based on an a representation for the zeta function, which is not converging on the entire complex plane, and the rate of convergence being slower, but allowing precise bounds and estimates.

The representation given by [Son94], $\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \sum_{k=0}^n \frac{(-1)^{k-1} \binom{n}{k}}{(k+1)^s}$, converges globally as well, however, the rate of convergence is significantly slower.

A similar algorithm is proposed in [Bor91] with exponential convergence rate.

So to summarize and compare the representations described in Theorem 4 are faster, the only downside is that $\Gamma\left(\frac{s}{2}\right)$ is very small for huge imaginary parts, and incomplete Gamma functions have to be evaluated. However, this can be done very quickly, as will be further outlined below.

Proof. As for the proof notice first that

$$\sum_{z=-\infty}^{\infty} e^{-z^2 \pi x^2} M(a, b, z^2 \pi x^2) = \frac{1}{x} \frac{\Gamma(b)}{\Gamma(b-a)} \sum_{z=-\infty}^{\infty} e^{-\frac{z^2 \pi}{x^2}} U\left(a, a + \frac{3}{2} - b, \frac{z^2 \pi}{x^2}\right),$$

or, using (1.2),

$$\begin{aligned} &1 + 2 \sum_{z=1}^{\infty} e^{-z^2 \pi x^2} M(a, b, z^2 \pi x^2) \\ &= \frac{1}{x} \frac{\Gamma(b)}{\Gamma(b-a)} \left(\frac{\Gamma(b - \frac{1}{2} - a)}{\Gamma(b - \frac{1}{2})} + 2 \sum_{z=1}^{\infty} e^{-\frac{z^2 \pi}{x^2}} U\left(a, a + \frac{3}{2} - b, \frac{z^2 \pi}{x^2}\right) \right). \end{aligned}$$

We choose $a = 1$ and $b = 1 + \frac{s}{2}$. Thus, using (1.5) and (1.4),

$$1 + 2 \frac{s}{2} \sum_{z=1}^{\infty} \frac{\gamma\left(\frac{s}{2}, z^2 \pi x^2\right)}{(z^2 \pi x^2)^{\frac{s}{2}}} = \frac{s}{(s-1)x} + \frac{2}{x} \frac{\Gamma\left(1 + \frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \sum_{z=1}^{\infty} \frac{\Gamma\left(\frac{1-s}{2}, \frac{z^2 \pi}{x^2}\right)}{\left(\frac{z^2 \pi}{x^2}\right)^{\frac{1-s}{2}}},$$

¹By the way: Suppose that $\theta(z) = z^{-\alpha-1} \cdot \theta\left(\frac{1}{z}\right)$, then $(-1)^i \frac{z^{1+\alpha+i}}{i!} \theta(z) = \sum_{j=0}^i \binom{i+\alpha}{i-j} \frac{1}{z^j} \theta^{(j)}\left(\frac{1}{z}\right)$, strikingly reminding to the explicit representation of Laguerre polynomials (1.6).

or

$$\frac{x^s}{s} + \frac{x^{s-1}}{1-s} + \sum_{z=1}^{\infty} \frac{\gamma\left(\frac{s}{2}, z^2\pi x^2\right)}{(z^2\pi)^{\frac{s}{2}}} = \sum_{z=1}^{\infty} \frac{\Gamma\left(\frac{1-s}{2}, \frac{z^2\pi}{x^2}\right)}{(z^2\pi)^{\frac{1-s}{2}}}. \quad (4.2)$$

This nice identity at hand we find that

$$\begin{aligned} s(s-1) \frac{\zeta(s)\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} &= s(s-1) \sum_{k=1}^{\infty} \frac{\gamma\left(\frac{s}{2}; k^2\pi x^2\right) + \Gamma\left(\frac{s}{2}; k^2\pi x^2\right)}{(k^2\pi)^{\frac{s}{2}}} \\ &= (1-s)x^s + s x^{s-1} \\ &\quad - s(1-s) \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{1-s}{2}; \frac{k^2\pi}{x^2}\right)}{(k^2\pi)^{\frac{1-s}{2}}} + \frac{\Gamma\left(\frac{s}{2}; k^2\pi x^2\right)}{(k^2\pi)^{\frac{s}{2}}}, \end{aligned}$$

which is the second statement. Notice, that the condition $\operatorname{Re}(x) > |\operatorname{Im}(x)|$ insures both, $\operatorname{Re}(x^2) > 0$ and $\operatorname{Re}\left(\frac{1}{x^2}\right) > 0$, which is necessary for convergence.

The first statement is obvious by the choice $x = 1$.

As for the next statement recall that

$$(1-2^{1-s}) \frac{\zeta(s)\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\left(\gamma\left(\frac{s}{2}, k^2\pi x^2\right) + \Gamma\left(\frac{s}{2}, k^2\pi x^2\right)\right)}{(k^2\pi)^{\frac{s}{2}}}.$$

In order to get rid of slowly converging γ (and replace it by rapidly converging Fourier transform Γ) we apply Poisson summation formula (3.1) again, now with $z_0 = \frac{1}{2}$ and $k_0 = 0$. Hence,

$$\begin{aligned} (1-2^{1-s}) \frac{\zeta(s)\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} &= \frac{x^s}{s} - \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{1-s}{2}, \frac{(k-\frac{1}{2})^2\pi}{x^2}\right)}{\left(\left(k-\frac{1}{2}\right)^2\pi\right)^{\frac{1-s}{2}}} + (-1)^k \frac{\Gamma\left(\frac{s}{2}, k^2\pi x^2\right)}{(k^2\pi)^{\frac{s}{2}}} \\ &= \frac{x^s}{s2^s} - \frac{1}{2^{s-1}} \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{1-s}{2}, \frac{(2k-1)^2\pi}{x^2}\right)}{\left((2k-1)^2\pi\right)^{\frac{1-s}{2}}} + (-1)^k \frac{\Gamma\left(\frac{s}{2}, \frac{k^2\pi x^2}{4}\right)}{(k^2\pi)^{\frac{s}{2}}}, \end{aligned}$$

as the statement holds for $\frac{s}{2}$ as well. Combining the latter two identities and rearranging the terms is cumbersome, but finally gives the assertion. \square

Remark 7. $x := \frac{s}{s-1}$ is notably a possible choice if $\operatorname{Im}(s) > \frac{1+\sqrt{2}}{2} \approx 1.21$, as in this case $\operatorname{Re}(x) > |\operatorname{Im}(x)|$. The advantage of this this particular choice is that the term $(1-s)x^s + s x^{s-1}$ vanishes.

Remark 8. It should be noticed that the identity (4.2) is central here. It allows to replace the slowly converging series (which involves γ) by its Fourier transform, which is the very fast converging series (which involves Γ). This is the key strategy in all improvements of convergence above.

5. Asymptotics of the Laguerre Polynomials

Theorem 9. (Asymptotics of the Laguerre Polynomial) Let $\operatorname{Re}(z) > 0$. Then, as $i \rightarrow \infty$,

$$L_i^{(\alpha)}(z) \approx \frac{i^{\frac{\alpha}{2}-\frac{1}{4}}}{\sqrt{\pi}} \frac{e^{-\frac{z}{2}}}{z^{\frac{\alpha}{2}+\frac{1}{4}}} \cos\left(2\sqrt{z\left(i+\frac{\alpha+1}{2}\right)} - \frac{\pi}{2}\left(\alpha+\frac{1}{2}\right)\right) \text{ and}$$

$$L_i^{(\alpha)}(-z) \approx \frac{i^{\frac{\alpha}{2}-\frac{1}{4}}}{\sqrt{\pi}} \frac{e^{-\frac{z}{2}}}{z^{\frac{\alpha}{2}+\frac{1}{4}}} \exp\left(2\sqrt{z\left(i+\frac{\alpha+1}{2}\right)}\right).$$

Remark 10. We write $f_i \approx g_i$ as $i \rightarrow \infty$ to indicate that $\lim_{i \rightarrow \infty} \frac{f_i}{g_i} = 1$.

Remark 11. To prove the statements of the theorem we will involve Bessel functions. Bessel functions, of first and second kind, are

$$J_\alpha(x) := \sum \frac{(-1)^m}{m! \Gamma(\alpha+m+1)} \left(\frac{x}{2}\right)^{\alpha+2m} \text{ and } I_\alpha(x) := \sum \frac{1}{m! \Gamma(\alpha+m+1)} \left(\frac{x}{2}\right)^{\alpha+2m}.$$

Notice, that $\frac{J_\alpha(ix)}{(ix)^\alpha} = \frac{I_\alpha(x)}{x^\alpha}$ are entire, even functions. Moreover, Kummer's second formula links Bessel functions to confluent hypergeometric functions, as $\frac{J_\alpha(z)}{\left(\frac{z}{2}\right)^\alpha} = e^{-z} \frac{M\left(\alpha+\frac{1}{2}, 2\alpha+1, 2z\right)}{\Gamma(\alpha+1)}$.

Proof. We start with identity 13.3.7 from [AS64] which states that

$$\frac{M(a, b, z)}{\Gamma(b)} = e^{\frac{z}{2}} \cdot \sum_{n=0}^{\infty} A_n \cdot \left(\frac{z}{2}\right)^n \frac{J_{b-1+n}\left(\sqrt{2z(b-2a)}\right)}{\left(\frac{1}{2}\sqrt{2z(b-2a)}\right)^{b-1+n}}, \quad (5.1)$$

where J is the Bessel function of the first kind and A satisfies the recursion $A_0 := 1, A_1 := 0, A_2 := \frac{b}{2}$ and $A_n := \left(1 + \frac{b-2}{n}\right)A_{n-2} - \frac{b-2a}{n}A_{n-3}$. Although we refer to this Identity (5.1) without proof we mention that it follows straight forward by successively comparing the respective coefficients of z in (5.1). The coefficients satisfy $A_n = \mathcal{O}\left(\binom{a-b}{n}\right)$ for $b > 2a$, as follows directly from the recursive definition.

Following [AW05],

$$J_\alpha(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\left(\alpha + \frac{1}{2}\right)\right), \quad I_\alpha(x) \approx \frac{e^x}{\sqrt{2\pi x}} \quad (5.2)$$

as $x \rightarrow \infty$.

Combining all those ingredients we see that the initial term ($n = 0$) dominates all other summands in (5.1) when $a \rightarrow -\infty$, so we may neglect them for $n \geq 1$ to identify the rate of convergence. But for $n = 0$ the assertion states that

$$\frac{M(a, b, z)}{\Gamma(b)} \approx e^{\frac{z}{2}} \cdot \frac{J_{b-1}\left(\sqrt{2z(b-2a)}\right)}{\left(\frac{1}{2}\sqrt{2z(b-2a)}\right)^{b-1}},$$

which is a useful approximation itself.

There is another, similar approach, which is quite useful, which we want to give here as well for the sake of completeness and further reference: It starts with identity 13.3.8 from [AS64] with parameter $h = \frac{1}{2}$. This reads

$$\frac{M(a, b, z)}{\Gamma(b)} = e^{\frac{z}{2}} \cdot \sum_{n=0}^{\infty} C_n z^n \frac{J_{b-1+n}\left(2\sqrt{-az}\right)}{\left(\sqrt{-az}\right)^{b-1+n}},$$

where $C_0 = 1$, $C_1 = -b$, $C_2 = \frac{b(b+1)}{2}$ and $C_n = -\frac{b}{2n}C_{n-1} + \left(\frac{1}{4} + \frac{b-2}{4n}\right)C_{n-2} + \frac{a}{4n}C_{n-3}$. Here, $C_n = O\left(\frac{1}{2^n} \binom{a-b}{n}\right)$ and similar to the reasoning above we obtain

$$\frac{M(a, b, z)}{\Gamma(b)} \approx e^{\frac{z}{2}} \cdot \frac{J_{b-1}(2\sqrt{-az})}{(\sqrt{-az})^{b-1}}.$$

From both asymptotic identities the theorem follows in view of (1.6), that is

$$L_i^{(\alpha)}(z) := \binom{i+\alpha}{i} M(-i, \alpha+1, z),$$

and (5.2) as $i \rightarrow \infty$. □

Remark 12. We want to stress that the method used in the proof above allows to compute terms of higher order of the expressions given in Theorem 9. The higher order correction terms give successive improvements of order $\frac{1}{\sqrt{i}}$.

For an interesting treatment to evaluate Laguerre polynomials for large i we refer the reader to [BBC08].

6. Applications to the incomplete Gamma Function

In order to make use of the rapidly converging series (4.1) it is necessary to have a good implementation for the upper incomplete gamma function at hand. To this end we further elaborate on a procedure based on continued fractions (cf. [AT99]), which is always a good candidate for rapid convergence. The formula is a special case of Gauss' continued fraction method using confluent hypergeometric functions (see [JT80, Wal48]):

$$\Gamma(s, z) = \frac{z^s e^{-z}}{1 - s} \cfrac{z + \cfrac{1}{1 + \cfrac{2-s}{z + \cfrac{2}{1 + \cfrac{3-s}{z + \cfrac{3}{1 + \cfrac{\ddots}{z + \ddots}}}}}}}}{1} \tag{6.1}$$

(as a formal power series this is equivalent to $\Gamma(s, z) = z^s e^{-z} \cdot {}_2F_0\left(1, 1-s, -\frac{1}{z}\right)$).

Stopping a general continued fraction gives its convergents, the k^{th} -convergent is

$$\frac{p_k}{q_k} := \frac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{\ddots}{\ddots + \cfrac{a_{k-1}}{b_{k-1} + \cfrac{a_k}{b_k}}}}}$$

To simplify the respective values there is – from standard theory [Wal48] – the recursion

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = a_k \begin{pmatrix} p_{k-2} \\ q_{k-2} \end{pmatrix} + b_k \begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix}$$

with initial conditions $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$, and additionally the somewhat more explicit formula

$$\frac{p_k}{q_k} = \sum_{i=1}^k (-1)^{i-1} \frac{a_1 \cdot a_2 \cdot \dots \cdot a_i}{q_i \cdot q_{i-1}}. \quad (6.2)$$

When applied to the continuous fraction of the incomplete gamma function (6.1) the recursions for the numerator and denominator $\frac{p_k(s,z)}{q_k(s,z)}$ read

$$p_k(s, z) = \begin{cases} \left(\frac{k}{2} - s\right) p_{k-2}(s, z) + p_{k-1}(s, z) & k \text{ even} \\ \frac{k-1}{2} p_{k-2}(s, z) + z p_{k-1}(s, z) & k \text{ odd} \end{cases}$$

and

$$q_k(s, z) = \begin{cases} \left(\frac{k}{2} - s\right) q_{k-2}(s, z) + q_{k-1}(s, z) & k \text{ even} \\ \frac{k-1}{2} q_{k-2}(s, z) + z q_{k-1}(s, z) & k \text{ odd}, \end{cases} \quad (6.3)$$

but different initial values: $p_0 = 0$, $p_1 = 1$, $q_{-1} = 0$ and $q_0 = 1$. It comes without surprise that Laguerre polynomials appear again, as these recursions have the closed expression

$$p_{2k}(s, z) = k! L_k^{(-s)}(-z) \text{ and } q_{2k+1}(s, z) = k! z L_k^{(1-s)}(-z).$$

In view of (6.2) the knowledge of the denominator is already sufficient to get the convergents, in explicit terms we get the simple expressions

$$\frac{p_{2k}(s, z)}{q_{2k}(s, z)} = \sum_{i=0}^{k-1} \frac{\frac{1}{i+1} \binom{i-s}{i}}{L_i^{(-s)}(-z) L_{i+1}^{(-s)}(-z)} \rightarrow \frac{\Gamma(s, z)}{z^s e^{-z}}$$

and

$$\frac{p_{2k+1}(s, z)}{q_{2k+1}(s, z)} = \frac{1}{z} + \frac{1}{z} \sum_{i=1}^k \frac{\binom{i-s}{i}}{L_{i-1}^{(1-s)}(-z) L_i^{(1-s)}(-z)} \rightarrow \frac{\Gamma(s, z)}{z^s e^{-z}} \quad (6.4)$$

which have been observed in [BBC08]. To find a handy expression for the numerator as well is surprisingly much more difficult – but yes, the somewhat curious result involves Laguerres again:

Theorem 13. *For any s and $z \neq 0$ we have*

$$\begin{aligned} \frac{\Gamma(s, z)}{z^s e^{-z}} &= \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} L_{k-2i+1}^{(2i-s)}(-z) \frac{\binom{s-1}{i-1}}{\binom{k-1}{i-1}}}{k L_k^{(-s)}(-z)} \\ &= \frac{1}{z} + \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} L_{k-2i+1}^{(2i+1-s)}(-z) \frac{\binom{s-1}{i}}{\binom{k}{i}}}{z L_k^{(1-s)}(-z)}, \end{aligned} \quad (6.5)$$

the rate of approximation for $\operatorname{Re}(z) > 0$ being of order $\mathcal{O}\left(\frac{e^{-4\sqrt{kz}}}{\sqrt{kz}}\right)$.

7 APPLICATION TO RIEMANN'S ZETA FUNCTION

Remark. The integer valued floor function $\lfloor \cdot \rfloor$ satisfies $x - 1 < \lfloor x \rfloor \leq x$.

Proof. The fractions in the limit represent explicit expressions for $\frac{p_{2k}(s,z)}{q_{2k}(s,z)}$ ($\frac{p_{2k+1}(s,z)}{q_{2k+1}(s,z)}$, resp.). Again, the proof links the recursions (6.3) introduced above to well known recursions of Laguerre polynomials.

The rate of approximation is an interesting consequence of (6.2) (same for (6.4)):

$$\begin{aligned} \mathcal{O}\left(\sum_{i=k}^{\infty} \frac{\frac{1}{i+1} \binom{i-s}{i}}{L_i^{(-s)}(-z) L_i^{(1-s)}(-z)}\right) &= \mathcal{O}\left(\sum_{i=k}^{\infty} \frac{i^{-s-1}}{i^{-\frac{s}{2}-\frac{1}{4}} i^{\frac{1-s}{2}-\frac{1}{4}} e^{4\sqrt{iz}}}\right) \\ &= \mathcal{O}\left(\int_k^{\infty} \frac{i^{-1}}{e^{4\sqrt{iz}}} di\right) \\ &= \mathcal{O}\left(\Gamma(0, 4\sqrt{kz})\right) \\ &= \mathcal{O}\left(\frac{e^{-4\sqrt{kz}}}{\sqrt{kz}}\right), \end{aligned}$$

which is finally the desired rate. □

7. Application to Riemann's Zeta Function

We have given a few approximations for the upper incomplete Gamma function which involve polynomials, for example (2.2), (6.2) ((6.4), respectively) and (6.5). Those expressions converge sufficiently quick and do not impose any difficulties as their argument z tends to infinity. So they can be substituted in (4.1) to give recent approximations and a variety of possible investigations on the geometry of the Riemann zeta function.

As an initial example combine (2.2) and (4.1) to get the representation

$$\zeta_n^{(\Delta)}(s) := 1 - s(1-s) \sum_{k=0}^{n-1} a_k^{(\Delta)} \left(\frac{1}{\binom{\frac{s}{2}+k+\Delta}{k+1}} + \frac{1}{\binom{\frac{1-s}{2}+k+\Delta}{k+1}} \right) \xrightarrow{n \rightarrow \infty} s(s-1) \frac{\zeta(s) \Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}},$$

where $a_k^{(\Delta)} := \frac{\sum_{z=1} e^{-z^2 \pi} L_k^{(\frac{\Delta-1}{2})}(z^2 \pi)}{k+1}$. Notice, that this representation converges, as $n \rightarrow \infty$, uniformly on $-2\Delta - 1 < \operatorname{Re}(s) \leq 2\Delta$.

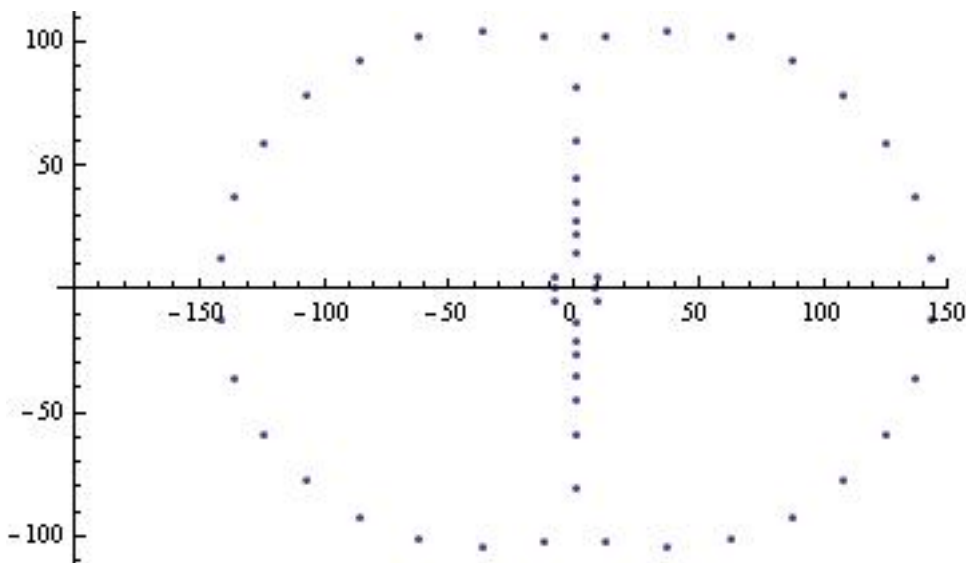
Moreover – and this is a key observation – the numerator of the function

$$\zeta_n^{(\Delta)}(s) \binom{\frac{s}{2} + n - 1 + \Delta}{n} \binom{\frac{1-s}{2} + n - 1 + \Delta}{n},$$

as a function in variable s , is a *polynomial* of degree $2n$. Thus, by Hurwitz' Theorem in complex analysis (cf. [Con78]), the zeros of ζ are just the accumulation points of $\zeta_n^{(\Delta)}$'s zeros.

Interestingly, the zeros of the polynomials above have a very nice, symmetric pattern in common, a typical result is plotted in figure 7.1²: The total of zeros is 50, 18 on the left (right)

²Figures have been computed using Mathematica.

Figure 7.1: Zeros of $\zeta_{25}^{(5)}$.

of $\text{Re}(s) = \frac{1}{2}$, 7 have a positive (negative) imaginary part on the critical line, a few exemptions occur close to 2Δ ($1 - 2\Delta$, respectively).

However: all zeros in the area of convergence, which is $-2\Delta - 1 < \text{Re}(s) \leq 2\Delta$, lie on the critical line.

Another pattern is found when employing (6.2) or (6.5), in our next example for $k = 6$ and involving $z^2\pi$ in $a_k^{(\Delta)}$ for z from 1 to 5: The corresponding polynomial has degree 48, its zeros are depicted in figure 7.2 (6 do not fit to the scale chosen).

Interestingly, the most convenient pattern is observed when substituting (2.2) into the equation $\Upsilon_x(s)$. Stopping the infinite sum including $\frac{L_k^\Delta}{(k+1)\binom{k+1+a-\frac{s}{2}}{i+1}}$ as a final term and solving $\Upsilon_1(s) + \Upsilon_1(1-s) = 0$ again leads to finding the roots of polynomials in s , which are of degree $2k$ here. Obviously, the zeros corresponding to the factor $(2^s - 1)(1 - 2^{1-s})$ have to become visible as well, but, as numerical experiments show, those zeros are being added very slowly, as k increases. We have depicted some zeros for $k = 50$ in figure 7.3, where only 4 zeros corresponding to $s = \frac{1}{2} \pm \frac{1}{2} \pm \frac{2\pi i}{\ln 2}$ ($\frac{2\pi}{\ln 2} \approx 9.06$) appear. However, and this is potentially a big advantage in comparison to the other figures, all other figures are located on the critical line and two symmetric bubbles except a few others usually located close to the real line. The complete pattern of zeros is depicted in figure 7.4. Do these associated polynomials always have their zeros in the region of convergence on the critical strip? The answer “Yes” obviously implies RH. So this question seems being worth an attempt and has to be investigated in much more detail in promising, further research.

8. Acknowledgment

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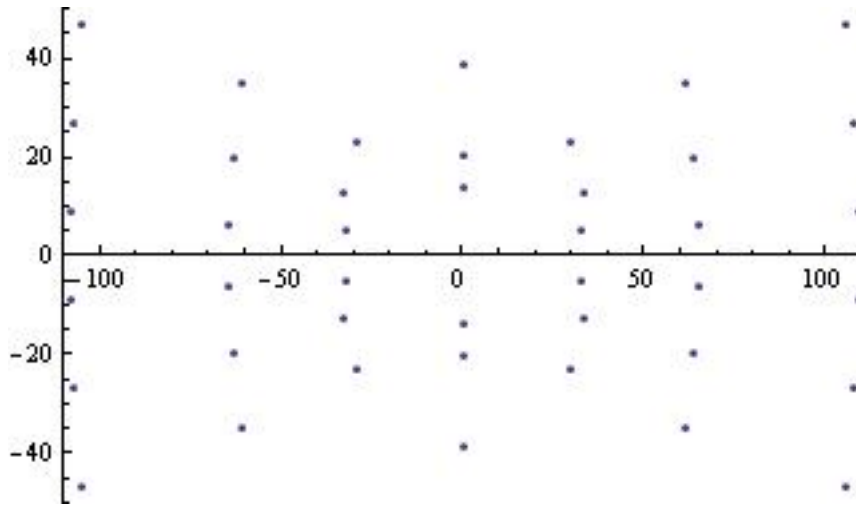


Figure 7.2: Zeros of $\zeta_{548}^{(5,6)}$.

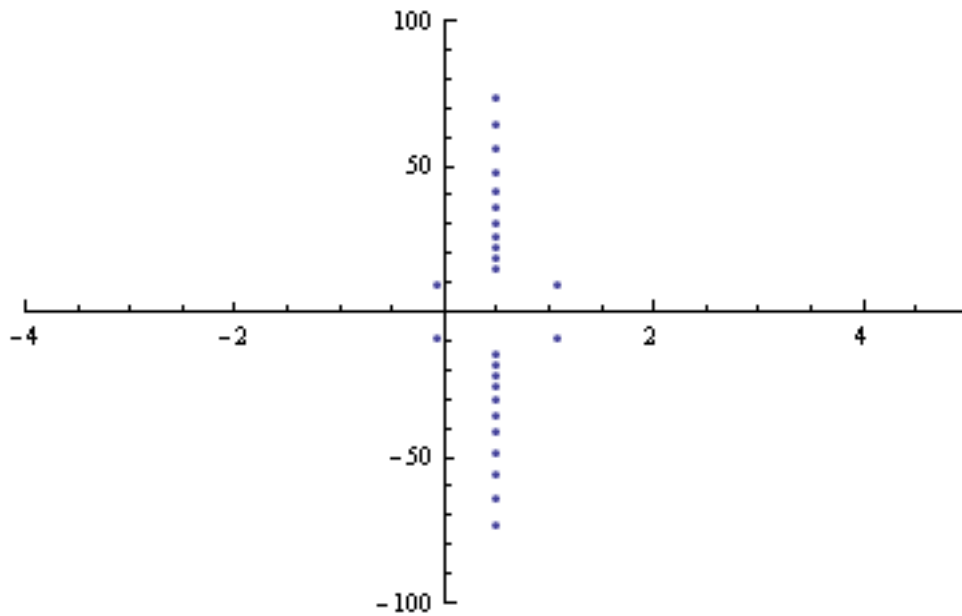
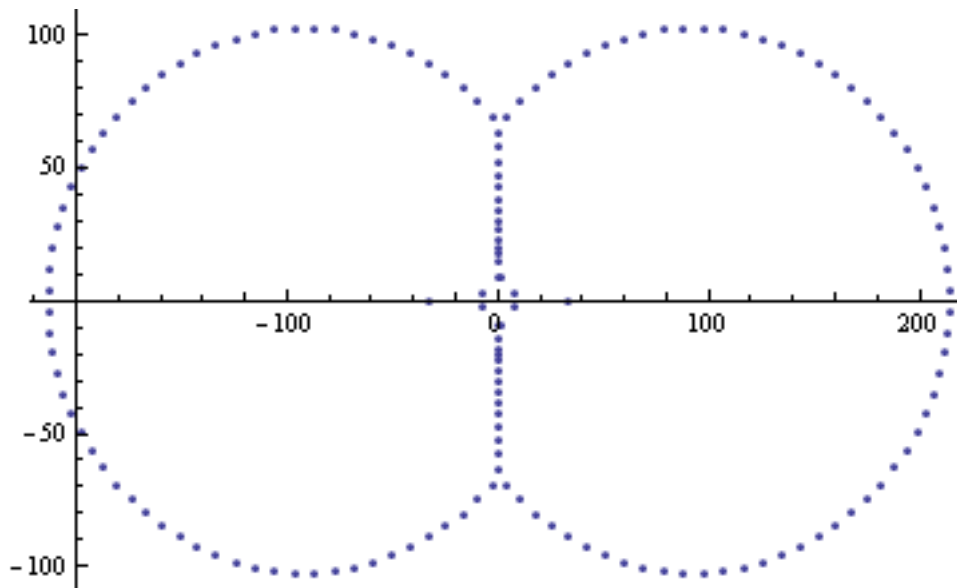


Figure 7.3: Zeros of ζ_{50} , 4 zeros corresponding to the factor $(2^s - 1)(1 - 2^{1-s})$.

Figure 7.4: Zeros of ζ_{50} , $\alpha = 5$

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