§17. *Initial Value Problems for Non-Space-Like Initial Manifolds*

The mean value theorem of §16 illuminates the situation regarding initial value problems for ultrahyperbolic differential equations and for hyperbolic differential equations with non-space-like initial manifolds. In particular, we shall see why initial value problems of this kind are not meaningful or "well posed" in the sense of Ch. III, §6.

1. Functions Determined by Mean Values over Spheres with Centers in a Plane. The integral of a function $f(x, t) = f(x_1, \cdots, x_n, t)$ over the sphere of radius r and center $(x, 0)$ in x, t-space is given by

$$(1) \qquad g(x, r) = \int_{\xi^2+\tau^2=r^2} f(x + \xi, \tau)\, dS = Q[f].$$

Obviously $Q[f]$ only depends on the even part $f(x, t) + f(x, -t)$ of f. We wish to determine $f(x, t) + f(x, -t)$ for prescribed $g(x, r)$. For this purpose we form the integral of f over the solid sphere of radius r and center $(x, 0)$:

$$(2) \qquad G(x, r) = \int_0^r g(x, \rho)\, d\rho = \int_{\xi^2+\tau^2<r^2} f(x + \xi, \tau)\, d\xi\, d\tau.$$

Differentiating G with respect to one of the x variables, x_i, we find

$$(3) \qquad G_{x_i} = \int_{\xi^2+\tau^2<r^2} f_{\xi_i}(x + \xi, \tau)\, d\xi\, d\tau = \frac{1}{r}\int_{\xi^2+\tau^2=r^2} f(x + \xi, \tau)\xi_i\, dS.$$

Hence

$$Q[f(x, t)x_i] = \int_{\xi^2+x^2=r^2} f(x + \xi, \tau)(x_i + \xi_i)\, dS = x_i g(x, r) + rG_{x_i}(x, r) = x_i g(x, r) + r\frac{\partial}{\partial x_i}\int_0^r g(x, \rho)\, d\rho = D_i g,$$

where D_i is a linear operator acting on functions $g(x, r)$. Applying the same argument to the function $x_i f$ in place of f and repeating this process indefinitely we see that for any polynomial $P(x_1, \cdots, x_n)$ the integral of the function Pf over the sphere of radius r and center $(x, 0)$ is given by

$$Q[Pf] = P[D_1, \cdots, D_n)g$$

and hence is known when g is known. We have also

$$(4) \qquad Q[Pf] = \int_{(\eta-x)^2<r^2} P(\eta)(f(\eta, \tau) + f(\eta, -\tau))\frac{r\, d\eta}{\tau},$$

where

$$\tau = \sqrt{r^2 - (\eta - x)^2}.$$

In virtue of the completeness of the polynomials P in the sphere the function

$$(5) \qquad \frac{f(\eta, \tau) + f(\eta, -\tau)}{\tau} \qquad (\tau = \sqrt{r^2 - (\eta - x)^2})$$

and hence f itself is determined uniquely by the known expressions $P(D_1, \cdots, D_n)g$.[1]

We now observe the following important fact: To calculate the operator $D_i g$ for any system of values $x_1^0, \cdots, x_n^0, r^0$ it is sufficient to know the mean value $g(x, r)$ of $f(x, t)$ for

$$(6) \qquad 0 \le r \le r^0, \qquad \sum_{i=1}^n (x_i - x_i^0)^2 \le \epsilon^2,$$

where ϵ may be arbitrarily small. The same is true for the calculation of all polynomials $P(D_1, \cdots, D_n)g$.

From this follows: The values of g in the region characterized by (6) uniquely determine the even part of the function f in the entire solid sphere

$$\sum_{i=1}^n (x_i - x_i^0)^2 + t^2 \le r_0^2.$$

This, in turn, uniquely determines the integral $g(x, t, r)$ over any

[1] The fact that the function (5) is singular for $\tau = 0$ does not affect this conclusion. We only have to smooth out the function (5) in a neighborhood of the sphere $|\eta - x| = r$ and choose for the polynomials P a sequence approximating uniformly this smoothed function.

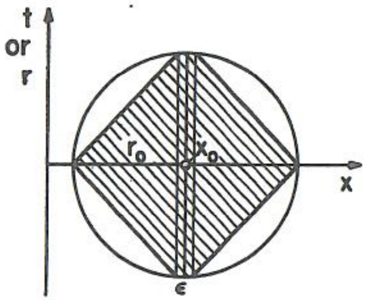

FIGURE 59

sphere with center at $t = 0$ in the interior of the sphere in which f is known, provided that

$$(7) \qquad r + \sqrt{\sum_{i=1}^{n} (x_i - x_i^0)^2} \leq r_0 .$$

Thus we have shown:

g is uniquely determined in the entire double cone (7) *by its values in the cylindrical region* (6) *of arbitrarily small thickness* ϵ. See Figure 59.

2. Applications to the Initial Value Problem. We consider the ultrahyperbolic equation

$$(8) \qquad \sum_{i=1}^{l} u_{y_i y_i} = \sum_{i=1}^{n} u_{x_i x_i} + u_{tt} ,$$

in which we single out $x_{n+1} = t$ and suppose $n \geq 2$, but do not necessarily assume $l = n + 1$. We try to determine a solution by prescribing its values on the plane $t = 0$. Thus for $t = 0$, let

$$u(x, y, 0) = \psi(x, y) \qquad \text{and} \qquad u_t(x, y, 0) = \phi(x, y)$$

be given. We consider the initial values in a domain of the x,y-space, where y is assumed to lie in some domain G of the y-space, while x varies inside the small sphere

$$(9) \qquad \sum_{i=1}^{n} (x_i - x_i^0)^2 \leq \epsilon^2$$

in x-space. The domain in which the initial values are prescribed is thus the "product domain" of a small sphere in the x-space and an

arbitrary domain G in y-space. Consider the solution u as a function of x, t with y as parameter. Then our prescribed values ψ determine the integrals of u over the surfaces of those spheres in x,t-space whose centers x_i, t lie in

$$t = 0, \qquad \sum_{i=1}^{n} (x_i - x_i^0)^2 \leq \epsilon^2,$$

and whose radii are not greater than the radius r^0 of the largest sphere about the point y in y-space which still lies entirely in G.

This follows immediately from the mean value theorem of §16 for $n > l$. If $n < l$, this mean value theorem at first yields only the integrals

$$\iint_{V_r} u(x, t) \left(r^2 - t^2 - \sum_{i=1}^{n} x_i^2\right)^{(l-n)/2} dx\, dt$$

over the interior V_r of every sphere in x,t-space with radius $r \leq r^0$ and center $x_1, \cdots, x_n, t = 0$; here $\sum_{i=1}^{n} (x_i - x_i^0)^2 \leq \epsilon^2$. If we denote the integral of u over the surface of such a sphere of radius r by $I(r)$, the integral above can be written as

$$\int_0^r I(\rho)(r^2 - \rho^2)^{(l-n)/2}\, d\rho.$$

But if this expression is known for $r < r^0$, then $I(r)$ is also uniquely determined for $r \leq r^0$. This follows from our earlier discussion (cf. §16) by solving an Abelian integral equation. Our assertion is thus proved also for $l > n$.

By article 1 the even function $u(x, y, t) + u(x, y, -t)$ is uniquely determined in the entire sphere

$$\sum_{i=1}^{n} (x_i - x_i^0)^2 + t^2 \leq (r^0)^2$$

by the given values of ψ. Similarly the even function

$$u_t(x, y, t) + u_t(x, y, -t)$$

is determined by ϕ. It follows immediately that $u(x, y, t)$ is determined uniquely. In particular, the initial values $u(x, y, 0)$ are determined for $t = 0$ in the sphere

$$(10) \qquad \sum_{i=1}^{n} (x_i - x_i^0)^2 \leq (r^0)^2$$

of the n-dimensional initial space R_n, and we therefore obtain the remarkable result:

If the initial values of a solution u of the ultrahyperbolic equation (8) are known for y in G and for x in an arbitrarily small sphere

$$\sum (x_i - x_i^0)^2 \leq \epsilon^2$$

(cf. article 1), *then the initial values are uniquely determined everywhere in the larger sphere*

$$\sum_{i=1}^{n} (x_i - x_i^0)^2 \leq (r^0)^2,$$

where r^0 is defined as above.

A consequence of this result is: *One cannot arbitrarily prescribe initial values $u(x, y, 0)$.*

For example, if, with given a, one prescribes initial values $u(y_1, y_2, x, 0)$ to solve the equation

$$u_{y_1y_1} + u_{y_2y_2} - u_{xx} - u_{tt} = 0 \qquad (11)$$

in a thin cylindrical disk

$$t = 0, \qquad (y_1 - y_1^0)^2 + (y_2 - y_2^0)^2 \leq a^2, \qquad |x - x^0| \leq \epsilon,$$

then $u(y_1, y_2, x, 0)$ is *a priori* uniquely determined in the double cone

$$t = 0, \qquad \sqrt{(y_1 - y_1^0)^2 + (y_2 - y_2^0)^2} + |x - x^0| \leq a.$$

See Figure 60.

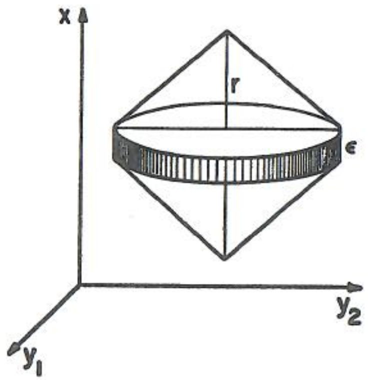

FIGURE 60

Similarly, consider the wave equation

$$u_{yy} - u_{x_1x_1} - y_{x_2x_2} - u_{tt} = 0 \qquad (12)$$

in which, however, the roles of the space variable y and the time variable t are interchanged. If the function $u(y, x_1, x_2, t)$ is prescribed in the thin cylinder

$$t = 0, \qquad (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 \leq \epsilon^2, \qquad |y - y^0| \leq a$$

parallel to the y-axis, the initial value $u(y, x_1, x_2, 0)$ is at once uniquely determined in the double cone

$$\sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2} + |y - y^0| \leq a.$$

See Figure 61.

Thus we see: *On a non-space-like plane, it is not possible to prescribe arbitrarily the initial values for the solution of the wave equation.*

If, in the case of the general equation (8), the initial value $u(y_1, y_2, \cdots, y_l; x_1, \cdots, x_n, 0)$ is prescribed for

$$\sum_{i=1}^{l} (y_i - y_i^0)^2 \leq a^2, \qquad \sum_{i=1}^{n} (x_i - x_i^0)^2 \leq \epsilon^2,$$

then it is *a priori* known for the region

$$\sqrt{\sum_{i=1}^{l} (y_i - y_i^0)^2} + \sqrt{\sum_{i=1}^{n} (x_i - x_i^0)^2} \leq a,$$

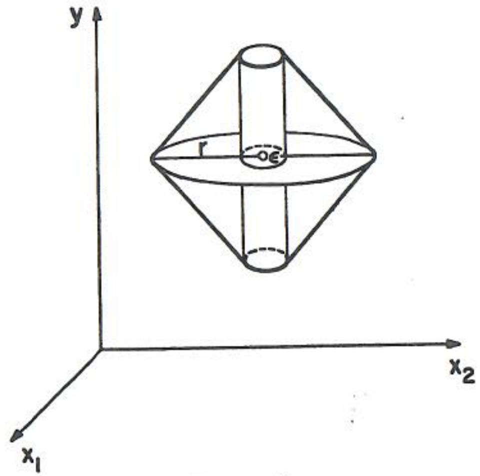

FIGURE 61

and the solution $u(y, x, t)$ is uniquely determined for

$$\sqrt{\sum_{i=1}^{l} (y_i - y_i^0)^2} + \sqrt{\sum_{i=1}^{n} (x_i - x_i^0)^2 + t^2} \leq a.$$

For the potential equation ($l = 0$)

$$(13) \qquad u_{x_1x_1} + \cdots + u_{x_nx_n} + u_{tt} = 0,$$

this means: *If a solution u is an even function of t, then the values $u(x, 0)$ in an arbitrarily small sphere*

$$\sum_{i=1}^{n} x_i^2 \leq \epsilon^2$$

uniquely determine for arbitrary a the values of the solution in the domain

$$\sum_{i=1}^{n} x_i^2 + t^2 \leq a^2.$$

In particular, for $t = 0$, the values $u(x, 0)$ uniquely determine the initial values of u. The statement for the initial values is again true without restriction to even solutions.

This result concerning the potential equation might have been expected from the analytic character of its solutions, which was already known to us. In the case of hyperbolic and ultrahyperbolic differential equations, however, the relations obtained above between the values of a solution on the initial plane are not so obvious. In fact, these initial functions may very well not even be analytic. Thus, in investigating the values of the solutions of hyperbolic and ultrahyperbolic differential equations along non-space-like planes, we are dealing with the remarkable phenomenon of functions which are not necessarily analytic, yet whose values in an arbitrarily small region determine the function in a substantially bigger domain.[1]

§18. *Remarks About Progressing Waves, Transmission of Signals and Huyghens' Principle*

1. Distortion-Free Progressing Waves. While the term "wave" was used in this book quite generally for any solution of a hyperbolic

[1] Cf. F. John [3] where, by a different method, even more extensive results are obtained for general linear equations with analytic coefficients.

problem,[1] there are certain specific classes of waves of particular interest, for example "standing waves", represented as products of a function of time and a function of the space variables. In the present section we want to comment further on the importance of another such class, the progressing waves, discussed in Chapter III for differential equations with constant coefficients and more generally in §4 of the present chapter. This concept is a key for the theory of transmission of signals, indeed a central subject in the theory of hyperbolic differential equations. For brevity we shall consider a single equation $L[u] = 0$.

In keeping with Ch. III, §3 we define a *family of undistorted progressing waves* as a family of solutions of $L[u] = 0$ depending on an *arbitrary function* $S(\phi)$ and having the form

$$(1) \qquad u = S(\phi(x, t)),$$

where S is called the *wave form* and $\phi(x, t)$ is a fixed *phase function* of the space variables x and the time $t = x_0$. Such a phase function might be

$$\phi(x, t) = \chi(x) - t.$$

The solution u represents the undistorted motion of the wave form S through space.

Using the arbitrariness of $S(\phi)$, we conclude that ϕ must satisfy

$$L[\phi] = 0,$$

and the characteristic equation

$$Q(D\phi) = 0.$$

The first equation is obtained by the special substitution $S(\phi) = \phi$; the second follows if we choose $S = \delta(\phi - c)$ with an arbitrary constant c (see §4). Thus we may state: The phase function ϕ is a characteristic function, i.e., the phase surfaces $\phi =$ const. are characteristic wave fronts.

In spite of this overdeterminacy of ϕ, some differential equations $L[u] = 0$ exist which do admit families of undistorted progressing waves. This is the case, e.g., for linear differential equations $L[u] = 0$ with constant coefficients containing only the highest order terms, in particular for the wave equation (see Ch. III, §3). However, in

[1] To avoid confusion we have consistently reserved the name "wave front" for surfaces of discontinuity which satisfy not the original differential equation but the associated characteristic equation of first order.

general the conditions for ϕ are not compatible. It is therefore appropriate to introduce the less restrictive concept of *"relatively undistorted" progressing wave families* having the form

$$u = g(x, t)S(\phi), \tag{2}$$

where again $S(\phi)$ is arbitrary and where not only the phase function $\phi(x, t)$, but also the distortion factor g, is specific. Such waves still can serve as suitable carriers of signals inasmuch as the factor g simply represents an attenuation. Spherical waves in three space dimensions, e.g., $\frac{S(t-r)}{r}$ or $\frac{S(t+r)}{r}$, are typical examples of such relatively undistorted wave families. Concentric spheres in space define the moving characteristic phase surfaces.

Again the conditions restrict (2) to a characteristic function; they imply an overdetermined system of differential equations for the distortion factor g. We recognize this simply, e.g., by substituting (2) in the differential equation and by realizing that the arbitrariness of S implies the vanishing of all the coefficients of $S, S', S'', \cdots$.

Hence, an equation $L[u] = 0$ has the desirable property of possessing relatively distortion-free families of solutions only in exceptional cases. Of course, if a differential equation $L[u] = 0$ does have this property, an entire class of equivalent differential equations possesses the same property. Two differential equations $L[u] = 0$ and $L^*[u^*] = 0$, for two functions $u(x)$ and $u^*(x)$, are called equivalent if they can be transformed into each other by a transformation of the form $x_i^* = \alpha_i(x_0, x_1, \cdots, x_n)$, $u^* = f(x)u$.

The question of determining all operators L which allow such families of solutions has hardly been touched.[1]

[1] A special fact, easily proved, is:
In the case of two variables $x_1 = x$, $t = x_0 = y$, the only differential equations of second order which admit relatively distortion-free progressing wave families in both space directions are $u_{xy} = 0$ and equations equivalent to it.

Certainly, the differential equation is equivalent to an equation of the form $2u_{xy} + Bu_x + Cu = 0$, where B and C are functions of x and y, where $x + y$ and $x - y$ represent the time and space coordinates, respectively, and where $x =$ const., $y =$ const. are the characteristics. The existence of the wave family $u = g(x, y)S(y)$ requires that $g_x = 0$ holds as well as $2g_{xy} + Bg_x + Cg = 0$ and hence $C = 0$. If, in addition, a wave family $u = h(x, y)S(x)$ advancing in another direction is to exist, then $2h_y + Bh = 0$ must be satisfied together with $2h_{xy} + Bh_x = 0$ so that $B_x = 0$ follows. But the equation

$$2u_{xy} + B(y)u_x = 0$$

is equivalent to the equation $u_{xy} = 0$.

There is a related problem for which a solution is known: Consider the wave equation with four independent variables. What are the possible wave fronts of relatively undistorted progressing waves? The answer is that all such wave fronts are cyclides of Dupin,[1] which include planes and spheres as special cases.

In general, in order to mitigate or to eliminate the overdeterminacy of the distortion factor g, one must introduce more such factors, as indeed we did in §4, defining progressing waves of higher degree, or even complete progressing waves. According to §§4 and 5 such waves provide an important step towards the construction of solutions, although they do represent a distortion of the initial shape of the signal.

2. Spherical Waves. The problem of transmission of signals is further clarified by the concept of "spherical waves" which generalize the spherical solutions of the three-dimensional wave equation. We confine ourselves to the case of linear differential equations $L[u] = 0$ of second order and consider a time-like[2] line Λ given in the form $x_i = \xi(\lambda)$ with a parameter λ. (The time variable is not emphasized here.) With the point $\xi(\lambda)$ as vertex we consider the characteristic conoid or spherical wave front $\Gamma(x; \xi) = 0$.

For given x, we may determine λ as a function of x from the equation $\Gamma(x, \xi(\lambda)) = 0$; we write $\lambda = \phi(x)$. The characteristic conoid with vertex $\xi(\phi(x))$ is given by the equation $\phi(x) =$ const. *A family of relatively undistorted spherical waves issuing from Λ* may then be defined as a solution u of the second order differential equation in the form

$$u(x) = g(x)S(\phi(x))$$

with specific g and arbitrary S.

Little is known about the scope of this concept, which obviously relates spherical waves to the problem of transmitting with perfect fidelity signals in all directions. All we can do here is to formulate a conjecture which will be given some support in article 3: *Families of spherical waves for arbitrary time-like lines Λ exist only in the case of two and four variables, and then only if the differential equation is equivalent to the wave equation.*

A proof of this conjecture would show that the four-dimensional

[1] See F. G. Friedlander [1] and M. Riesz [1].
[2] See §3, 7.

physical space-time world of classical physics enjoys an essential distinction.

Here we merely emphasize that for the wave equation using the t-axis as the time-like line Λ and with $r^2 = x^2 + y^2 + z^2$, we have such waves with $\phi = t - r$ and $g = 1/r$. For other straight time-like lines, spherical waves are obtained by Lorentz transformations.

In the case of an even number of independent variables $n + 1 = 2\nu + 4$ ($\nu = 1, 2, \cdots$), solutions exist in the form of families of progressing waves[1] of higher order. The explicit solutions given in §12, 4 or §15, 4 no longer enjoy the property of freedom from distortion, but still represent a progressing phenomenon.

As to equations of higher order, it is worth noting, as an example, that for all even values of $n + 1$ the $(n + 1)/2$-th iterated wave equation of order $n + 1$

$$L^{(n+1)/2}[u] = \left(\frac{\partial^2}{\partial t^2} - \Delta\right)^{(n+1)/2} u = 0,$$

possesses undistorted families of spherical waves

$$u = S(t - r), \qquad u = S(t + r),$$

although the wave equation itself

$$L[u] \equiv \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$$

does not. This fact is simply another interpretation of the theorem proved in §13, 4. It indicates that higher order equations admit various possibilities not existing for second order.

Finally it should be recalled that *individual progressing spherical waves* of higher degree with specific S, not necessarily families with arbitrary S, occur and are of importance (§4). In particular, the fundamental solution of §15, e.g., Hadamard's expression for the fundamental solution of single second order equations (§15, 6) is represented by such waves:

$$R = S(\Gamma)g(x, t) + S_1(\Gamma)g^1(x, t) + \cdots,$$

where $S(\Gamma)$ is a specific distribution.

3. Radiation and Huyghens' Principle. Huyghens' principle discussed on various occasions in this volume, stipulates that the solution at a point ξ, τ does not depend on the totality of initial data within the conoid of dependence (see §7) but only on data on the characteristic rays through ξ, τ. (Again we emphasize $x_0 = t$ and $\xi_0 = \tau$.) The principle is tantamount to the statement that the radiation matrix of §15 vanishes identically except on the rays through ξ, τ. Equivalently we may state: A sharp signal issued at the time τ and the location ξ is transmitted as a sharp signal along the rays and remains unnoticeable outside the ray conoid. The principle does not, however, state that signals are transmitted without distortion.

[1] The notation is slightly different from that above.

For single differential equations of second order with constant coefficients we have seen: Only for the wave equation in 3, 5, 7, $\cdots$ space dimensions, and for equivalent equations, is Huyghens' principle valid. For differential equations of second order with variable coefficients *Hadamard's conjecture*[1] states that the same theorem holds even if the coefficients are not constant. Examples to the contrary show that this conjecture cannot be completely true in this form,[2] although it is highly plausible that somehow it is essentially correct.[3]

Altogether, the question of Huyghens' principle for second order equations should be considered in the light of the much more comprehensive problem of the exact domains of dependence and influence for any hyperbolic problem (see §7), a problem which is still completely open.

Concerning the transmission of signals which not only remain sharp but are undistorted, the conjecture in article 2 stated that this phenomenon is possible only in three space dimensions. For an isotropic homogeneous medium, i.e., for constant coefficients (and second order equations), the proof of this conjecture is contained in the preceding discussions. Thus our actual physical world, in which acoustic or electromagnetic signals are the basis of communi-

[1] This famous conjecture in fact was not categorically asserted by Hadamard.

[2] An example to the contrary for seven space dimensions was recently given by K. L. Stellmacher [1].

[3] Hadamard has identified the condition of validity of Huyghens' principle with the vanishing of the logarithmical term in his expression of the fundamental solution for odd number n of space dimensions. In our version Huyghens' principle means that the series (44) in §15 does not contain terms with the Heaviside function and its integrals.

cation, seems to be singled out among other mathematically conceivable models by intrinsic simplicity and harmony.

Yet, at least in an approximate sense, any hyperbolic system implies preservation of sharpness of signals in the sense of the generalized Huyghens' principle (see §15, 3). This generalized principle is therefore significant for the mathematical understanding of transmission of signals. This is true all the more since the validity of Huyghens' principle is at best a highly unstable property of a differential operator; this property is destroyed by infinitesimal variations of the coefficients of the operator. It would seem, therefore, that the generalized Huyghens' principle should be considered as the proper expression of physical reality.

Appendix to Chapter VI

Ideal Functions or Distributions

§1. *Underlying Definitions and Concepts*

1. Introduction. In this appendix we shall discuss the concept of distributions or "ideal functions".[1] The specific use of these ideal functions in the preceding chapters will be justified within a more general framework.

It should be understood that "function" may mean a function-vector with k components. The functions involved may be complex-valued, but the independent variable x is always a real vector with n components.

Much of the substance of the theory has long played a role in physical literature and elsewhere.[2] But only since the publication

[1] The name "distributions" indicates that ideal functions, such as Dirac's delta-function and its derivatives, may be interpreted by mass distributions, dipole distributions, etc., concentrated in points, or along lines or on surfaces, etc. However, the term "ideal functions" seems much more indicative of the true role of this concept as it is used in connection with differential equations and in mathematical analysis generally. This role is indeed that of functions, almost as the role of real numbers is that of ordinary numbers.

[2] For example, attention might be called to a paper by S. L. Sobolev [1], which long preceded the present flurry of literature.

of Laurent Schwartz's comprehensive book on distributions,[1] has the topic been treated systematically in a multitude of monographs,[2] some of which go far in the direction of refinement.[3] The present appendix concentrates on the elementary core of the theory as far as it is relevant for our study of linear differential equations. We omit a detailed discussion of the much-treated applications to the theory of Fourier transformations. (See, however, §4, 4.)

2. Ideal Elements. "Distributions" are most appropriately introduced as ideal elements in function spaces. It is one of the very basic procedures of mathematics to extend a given set or "space" S of mathematical objects by additional new "ideal elements" not defined as entities in the original set S, and not defined descriptively but defined merely by relationships such that in the extended set $\bar{S}$ the original rules for basic operations are preserved. The purpose of the extension always is to remove restrictions prevailing in the original set S.

Thus, in projective geometry ideal "points at infinity" are defined by sets of parallel lines. In other cases, the ideal elements are introduced by completion of the original set S by "strong" limit processes with a suitable *norm*: For example, the real numbers are defined by[4] convergent sequences of rational numbers r_n for which the norm $|r_n - r_m|$ converges to zero if n and m tend to infinity. Lebesgue-integrable and also square-integrable functions similarly can be defined by sequences of continuous functions $f_n(x)$ for which in the respective x-domains the integrals $\int |f_n - f_m|\,dx$ and $\int |f_n - f_m|^2\,dx$ converge to zero.—Functions in Hilbert spaces are ideal elements represented by sequences of suitably smooth functions f_n for which underlying positive quadratic forms $Q(f_n - f_m)$ converge to zero.

In these examples, the extended spaces $\bar{S}$ are complete, that is,

[1] See L. Schwartz [1].

[2] See, e.g., I. M. Gelfand and G. E. Shilov [1].
Also, a recent short book by Lighthill should be mentioned especially for its emphasis on Fourier analysis. Lighthill's book partly follows a publication by G. Temple. See M. J. Lighthill [2] and G. Temple [1], and further literature quoted in these publications.

[3] See, e.g., a series of papers by L. Ehrenpreis [1].

[4] Often the desire for descriptive definitions of ideal elements has led to logical twists such as the assertion: "A real number *is* a Dedekind cut in the set of rational numbers." It seems that little is gained by attempts to avoid the need for defining ideal objects by relationships instead of substantive descriptions.