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A Note on the Convolution in the Mellin Sense with Generalized Functions

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Abstract. The classical convolution is given by the following equation

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t) dt$$

and is widely accessible in many literatures including its extension to generalized functions (Gelfand and Shilov [4], A. Zemanian [2]). Another form of convolution is given by the Mellin convolution (i.e. convolution in the Mellin sense) which is given by

$$h(x) = \left(f *_M g\right)(x) = \int_0^{\infty} f\left(\frac{x}{u}\right) g(u) \frac{du}{u} .$$

The theory of the convolution in the Mellin sense for Mellin transformable functions is well known (Butzer and Jansche [3], Srivastava and Buschman [6]). In this work we extend this setting to the generalized functions.

1. Introduction

First of all we review the Mellin transform and the convolution in the Mellin sense for Mellin transformable functions. The following definitions are held in [3].

Definition 1.1. *If* $f : R_+ \to C$ *is a function such that* $f(x)x^{s-1} \in L^1(R_+)$ *for some* $s \in C$, *where* $s = c + it$ *and* $c \in R$, $t \in R_+$ *the Mellin transform is defined by the following equation*

$$M(f) = F(s) = \int_0^{\infty} f(x)\; x^{s-1} dx \tag{1}$$

Definition 1.1 will lead us to the next definition.

Definition 1.2. *The space* X_c *for some* $c \in R$ *is defined as follows*

$$X_c := \left\{ f : R_+ \to C;\ f(x)\ x^{c-1} \in L^1(R_+) \right\}$$

and the norm defined on X_c ,

$$\| f \|_{X_c} := \left\| f(x)\ x^{c-1} \right\|_{L^1(R_+)} = \int_0^\infty \left| f(x)\ x^{c-1} dx \right|$$

Thus, X_c consists of all f for which the transform $M(f)$ exists for all $t \in R_+$ and some $c \in R$, which is operating on the vertical line $c \times iR$, with $\left| F(s) \right| \le \| f \|_{X_c}$. While the space $X_{(a,b)}$ is given for $a,\ b \in R$, $a < b$ by

$$X_{(a,b)} := \bigcap_{c \in (a,b)} X_c$$

Thus, the space $X_{(a,b)}$ consists of all f for which the transform $M(f)$ exists for all $t \in R_+$ and all $c \in (a,b)$, which is operating on an open strip $St(a,b) := (a,b) \times iR \subset C$.

Example 1.1. Let $f(x) = e^{-x} \in X_{(0,\infty)}$. Its Mellin transform is given by the well known the Gamma function**,**

$$M(e^{-x}) = \Gamma(s) = \int_0^\infty e^{-x}\ x^{s-1}\ dx\ ,\ \mathrm{Re}(s) > 0 \tag{2}$$

However, we can see that $e^{-x} \notin X_{[0,\infty)}$. Thus, convergence on $St(a,b)$ does not necessarily mean convergence on $St[a,b]$.

Definition 1.3. *The convolution in the Mellin sense* $f *_M g$ *of two given functions* $f, g : R_+ \to C$ *is defined by*

$$(f *_M g)(x) := \int_0^\infty f\left(\frac{x}{u}\right) g(u) \frac{du}{u} \tag{3}$$

Theorem 1.1. (Butzer and Jansche [3]) *If* $f, g \in X_c$, *then the convolution in the Mellin sense* $f *_M g$ *exists (a.e.) on* R_+. *Also the convolution product* $h(x) = (f *_M g)(x)$ *belongs to the space* X_c, *and one has*

$$\| f *_M g \|_{X_c} \le \| f \|_{X_c} \| g \|_{X_c} .$$

Theorem 1.2. (Srivastava, Buschman [6]) (Butzer and Jansche [3]) (Sasiela [5]) *If* $f, g \in X_c$, *then*

$$M(f *_M g) = F(s)\, G(s) \tag{4}$$

Equation (4) is known as the exchange formula.

Now we let $E_{p,q}$ be the space of infinitely differentiable functions that satisfies the following conditions.

Let ϕ be an infinitely differentiable arbitrary testing function in $E_{p,q}$. Then there exists any two real numbers p and q such that

$$\lim_{x \to 0} x^{k+1-p} \, \phi^{(k)}(x) \to 0$$

$$\lim_{x \to \infty} x^{k+1-q} \, \phi^{(k)}(x) \to 0$$

for $p < k+1 < q$ and $k = 0, 1, 2, \cdots$.

Let us define

$$h_{p,q}(x) = \begin{cases} x^{1-p} & ; \quad 0 < x < 1 \\ x^{1-q} & ; \quad x \ge 1 \end{cases}$$

where $p < 1 < q$, and

$$\gamma_{k,p,q}(\phi) = \sup_{x>0} \left\{ h_{p,q}(x)\, x^k \left| \phi^{(k)}(x) \right| \right\}$$

Every $\gamma_{k,p,q}(\phi)$ is bounded and positively defined. Moreover for particular case $k = 0$, the function $\gamma_{0,p,q}(\phi)$ is a norm. Next, we prove the following lemma.

Lemma 1.1. *Let* $\phi(uw)$ *be an infinitely differentiable function on* $(0, \infty)$. *For* $p < r + 1 < q$ *and* $p < k - r + 1 < q$ *;* $0 \le r \le k$ *and* $k,\ r = 0, 1, 2, \cdots,$ $\phi(uw)$ *is a member of* $E_{p,q}$.

Proof. In order to show that the $\phi(uw)$ is a member of $E_{p,q}$ we examine its partial derivatives evaluated at the origin and at infinity.

First of all, we fix w and treat $\phi(uw)$. Since, there exists two real numbers p and q such that $p < r + 1 < q$, for each partial derivative with respect to u , we have

$$\lim_{u \to 0} u^{1-(p-r)} \frac{\partial^r}{\partial u^r} \phi(uw) \to 0$$

$$\lim_{u \to \infty} u^{1-(q-r)} \frac{\partial^r}{\partial u^r} \phi(uw) \to 0$$

For each partial derivative with respect to u, that partial derivative possesses partial derivatives with respect to w, such that with the above mentioned real numbers p and q and $p < k - r + 1 < q$, similar to the above equation we have the following relation,

$$\lim_{w \to 0} w^{1-(p-(k-r))} \left(\frac{\partial^k}{\partial u^r \partial w^{k-r}} \phi(uw) \right) \to 0$$

$$\lim_{w \to \infty} w^{1-(q-(k-r))} \left(\frac{\partial^k}{\partial u^r \partial w^{k-r}} \phi(uw) \right) \to 0$$

For $p < 1 < q$, we define

$$n_{p,q}(u, w) = \begin{cases} (uw)^{1-p} & ; \quad 0 < u < 1,\ 0 < w < 1 \\ (uw)^{1-p} & ; \quad u \ge 1,\ w \ge 1 \end{cases}$$

and $\lambda_{k,p,q}(\phi) = \sup\limits_{u>0, w>0} \left\{ n_{p,q}(u, w)\, u^r w^{k-r} \left| \phi^{(k)}(uw) \right| \right\}$, where $\lambda_{k,p,q}(\phi)$ is bounded and positively defined and similar to the above particular case $k = 0$, $\lambda_{0,p,q}(\phi)$ is a norm.

Hence, $\phi(uw)$ is truly a member of $E_{p,q}$ with the above mentioned properties. We note that the same can be derived if we fix the variable u first and treat $\phi(uw)$ as a function of only w. Then we state easily that if $\phi(uw)$ is a member of $E_{p,q}$ for any generalized function $f \in E'_{p,q}$, the $\Theta(u) = \langle f(w), \phi(uw) \rangle$ is also a testing function in $E_{p,q}$.

Now we consider the space $E'_{p,q}$ which is the linear space of continuous linear functionals on $E_{p,q}$ which is zero on the interval $(-\infty, 0)$. In fact the space $E'_{p,q}$ is the dual space of $E_{p,q}$. That is, for every $f \in E'_{p,q}$ if and only if the following conditions are satisfied.
For any $\phi \in E_{p,q}$,

1. $\langle f(x), \phi(x) \rangle$ is defined.
2. $\langle f(x), \phi(x) \rangle = 0$ if $\phi(x) = 0$ for $x > 0$
3. $\langle f(x), \alpha\phi_1(x) + \beta\phi_2(x) \rangle = \langle f(x), \alpha\phi_1(x) \rangle + \langle f(x), \beta\phi_2(x) \rangle$ for $\alpha, \beta \in R$
4. $\langle f(x), \phi_n(x) \rangle \to 0$ if $\phi_n(x) \to 0$ as $n \to \infty$

If $f \in E'_{p,q}$ and $\phi \in E_{p,q}$ $\langle f(x), \phi(x) \rangle$ is defined by the following integral,

$$\left| \langle f(x), \phi(x) \rangle \right| = \left| \int_0^\infty f(x)\, \phi(x)\, dx \right| \le \gamma_{0,p,q}(\phi) \int_0^\infty \left| \frac{f(x)}{h_{p,q}(x)} \right| dx \qquad (4)$$

where the right hand-side of (4) exists.

Since the theory of generalized functions is a linear theory, we can extend some operations which are valid for ordinary functions to $E'_{p,q}$. Such operations are called regular operations such as addition and multiplication by scalar .

For example if $f, g \in E'_{p,q}$ and $\alpha \in C$, for any $\phi \in E_{p,q}$ then easily one can have,

1. $\langle f(x) + g(x), \phi(x) \rangle = \langle f(x), \phi(x) \rangle + \langle g(x), \phi(x) \rangle$
2. $\langle \alpha f(x), \phi(x) \rangle = \alpha \langle f(x), \phi(x) \rangle$

2. Convolution of generalized functions in the Mellin sense

Definition 2.1. *If* $f,\ g \in E'_{p,q}$ *we define the convolution in the Mellin sense as*

$$\langle h,\phi\rangle = \langle f *_M g,\phi\rangle = \langle g(u),\langle f(w),\phi(uw)\rangle\rangle \tag{5}$$

and for $\phi \in E_{p,q}$.

Theorem 2.1. *If* $f, g \in E'_{p,q}$, *then* $f *_M g \in E'_{p,q}$ *for every* $\phi \in E_{p,q}$ *when* $p < r+1 < q$ *where* $r = 0, 1, 2, \cdots$.

Proof: The convolution of generalized functions in $E'_{p,q}$ in the Mellin sense is as given by (5). Let $\Theta(u) = \langle f(w),\ \phi(uw)\rangle$. For $r = 0,1,2,\cdots$ we can have the following,

$$\lim_{u\to 0}\left|\Theta^{(r)}(u)\right| \le \lambda_{0,\,p,q}(\phi)\int_0^\infty \left|\frac{f(w)}{n_{p,q}(u,w)}\right| dw \to 0$$

$$\lim_{u\to \infty}\left|\Theta^{(r)}(u)\right| \le \lambda_{0,\,p,q}(\phi)\int_0^\infty \left|\frac{f(w)}{n_{p,q}(u,w)}\right| dw \to 0$$

where $\lambda_{0,\,p,q}(\phi)$ and $n_{p,q}(u,w)$ is as given in Lemma 1.1. Thus, $\Theta(u)$ is a testing function in $E_{p,q}$. And since, $g \in E'_{p,q}$, the right hand-side of equation (5) exists.

Next, let us define the following testing function,

$$\phi(uw) = \begin{cases} (uw)^{s-1} & ;\quad u > 0 \quad , \quad w > 0 \\ 0 & ;\quad u \le 0 \quad , \quad w \le 0 \\ 0 & ;\quad u \to \infty \quad , \quad w \to \infty \end{cases} \tag{6}$$

From the equation (6) it follows that $\phi(uw)$ is a member of $E_{p,q}$ for $p < r + s < q$ and $p < k - r + s < q$ with the semi norms imposed on it as given in Lemma 1.1.

Now we give the Mellin transform of the convolution of generalized functions in $E'_{p,q}$ in the Mellin sense.

Theorem 2.2. *Let* $f(x)$ *and* $g(x)$ *be Mellin transformable generalized functions in* $E'_{p,q}$ *and let,*

$$M[f] = F(s) \text{ for } p_1 < \text{Re}(s) < q_1$$

and

$$M[g] = G(s) \text{ for } p_2 < \text{Re}(s) < q_2 .$$

Then $f *_M g$ *exists, and is also a Mellin transformable generalized function in* $E'_{p,q}$ *and,*

$$M\left[f *_M g\right] = F(s)G(s) \tag{7}$$

for $\text{Re}(s) \in (p_1, q_1) \cap (p_2, q_2)$, *where* $(p_1, q_1) \cap (p_2, q_2)$ *is not empty and* $p < r + s < q$ *and* $p < k - r + s < q$ *for* $0 \le r \le k$ *and* $k, r = 0, 1, 2, \cdots$.

Proof. Since $p < r + s < q$ and $p < k - r + s < q$, then $\left\langle f(x) *_M g(x), \phi(x)\right\rangle$ exists for any $\phi \in E_{p,q}$. The Mellin transform of the generalized function given by $h(x) = f *_M g(x)$ is given by the equation

$$M(h) = M\left(f *_M g\right) = \left\langle g(u) \times_M f(w), (uw)^{s-1}\right\rangle = \left\langle g(u), \left\langle f(w), (uw)^{s-1}\right\rangle\right\rangle \tag{8}$$

On using the equation (6) and since $p < r + s < q$ and $p < k - r + s < q$, the equation (8) exists. Thus it follows that,

$$M\left(f *_M g\right) = \left\langle g(u), u^{s-1}\right\rangle \left\langle f(w), w^{s-1}\right\rangle = F(s)\ G(s)$$

for $\text{Re}(s) \in (p_1, q_1) \cap (p_2, q_2)$, where $(p_1, q_1) \cap (p_2, q_2)$ is not empty, such that (7) is established.

Corollary 2.1. *The convolution of generalized functions in* $E'_{p,q}$ *in the Mellin sense is commutative and distributive. That is, for* $f, g \in E'_{p,q}$ *and* $\phi \in E_{p,q}$, $f *_M g = g *_M f$ *and* $h *_M [f + g] = (h *_M f) + (h *_M g)$.

References

1. A.I. Zayed, *Function and Generalized Functions Transformations*, CRC Press, 1996.
2. A. Zemanian, *Distribution Theory and Transform Analysis*, McGraw Hill, New York, 1965.
3. Butzer, Jansche, Mellin transform theory and the role of its differential and integral operators, *Proc. Con. On Transform Methods and Special Functions, Varna*, 1996.
4. I.M. Gelfand and G.E. Shilov, *Generalized Functions Vol. I*, Academic Press, 1964.
5. R.J. Sasiela, *Electromagnetic Wave Propagation In Turbulence Evaluation and Application of Mellin Transform*, Springer-Verlag, 1994.
6. H.M. Srivastava and R.G Buschman, *Theory and Applications of Convolution Integral Equations*, Kluwer Academic Publishers, 1992.