

ESSAY

MATS315 FOURIER ANALYSIS

The Hilbert Transform

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Abstract

This essay is written for the mathematics course on Fourier analysis lectured in the winter of 2015 at the University of Jyväskylä by Esa Vesalainen. The main goal of this work is to introduce the concept of Hilbert transformation and its basic properties. In addition, the Hilbert transforms of a few common functions are calculated to illustrate the use of this transformation.

1 Introduction

This work is a brief survey to the mathematical theory of Hilbert transforms. Historically the study of these transforms, named after the German mathematician David Hilbert, arose for the first time in the early 20th century within the study of functions analytic on the unit disc [1]. Since then the theory has expanded widely and found many applications in both mathematics and physics.

Even though the topics discussed in this essay do not follow any specific source, the main influence behind many of them is the first volume of the grand work *Hilbert Transforms* written by Frederick W. King [1]. We start by defining the Hilbert transform, and by reviewing the notion of the Cauchy principal value needed in the definition. After this we proceed to examine some basic properties of the Hilbert transformation, most of which will be proven in detail. The last section of this essay is devoted to the calculation of the Hilbert transform of some functions to get acquainted with its use.

Throughout this work our convention for the Fourier transform of a real-valued function f will be

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx.$$

Correspondingly the inverse Fourier transform will be given by

$$\mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{\infty} f(\xi)e^{2\pi i \xi x} d\xi.$$

We shall also need the convolution of two real-valued functions f and g given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy.$$

2 The Hilbert transform

2.1 Definition

The *Hilbert transform* of a real-valued function f is defined as

$$\mathcal{H}(f)(t) \doteq \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau,$$

provided that the above expression exists [1]. One immediately sees that there is a possible divergence at $t = \tau$, which is why the integral is considered as a *Cauchy principal value*, denoted by *P.V.* It is also readily noted that one can think of the Hilbert transform as the convolution of f with the function $(\pi t)^{-1}$. Before considering the properties of the Hilbert transform, let us briefly review the notion of the Cauchy principal value.

2.2 The Cauchy principal value

Let us consider a real-valued function f and its integral over an interval $[a, b]$:

$$\int_a^b f(x) dx.$$

Suppose that for some $x_0 \in [a, b]$ f is unbounded, i.e.

$$\limsup_{x \rightarrow x_0} |f(x)| = \infty.$$

Then one usually considers the integral as

$$\int_a^b f(x) dx = \lim_{\varepsilon_1 \rightarrow 0} \int_a^{x_0 - \varepsilon_1} f(x) dx + \lim_{\varepsilon_2 \rightarrow 0} \int_{x_0 + \varepsilon_2}^b f(x) dx,$$

where the limits are taken independently of each other. These limits may however not exist. Another possibility is to look at a symmetric limit,

called the Cauchy principal value of the integral,

$$\lim_{\varepsilon \rightarrow 0} \left[\int_a^{x_0 - \varepsilon} f(x) dx + \int_{x_0 + \varepsilon}^b f(x) dx \right] \doteq P.V. \int_a^b f(x) dx,$$

which may exist even if the individual limits do not. This is due to the fact that using a single parameter in the limit allows for cancellations between the integrals. The Cauchy principal value of an improper integral $\int_{-\infty}^{\infty} f(x) dx$ is defined in a similar fashion:

$$P.V. \int_{-\infty}^{\infty} f(x) dx \doteq \lim_{C \rightarrow \infty} \int_{-C}^C f(x) dx.$$

It should be noted that even if both the Cauchy principal value and the non-symmetric limit of an integral exist, they do not need to yield the same result. [2]

The Cauchy principal value is a useful tool that enables one to extract finite and meaningful quantities from otherwise ill-defined expressions. Consider for example the integral

$$\int_0^4 \frac{1}{3-x} dx,$$

where the integrand has a pole at $x = 3$. The standard non-symmetric limiting procedure results in

$$\begin{aligned} \int_0^4 \frac{1}{3-x} dx &= \lim_{\varepsilon_1 \rightarrow 0} \int_0^{3-\varepsilon_1} \frac{1}{3-x} dx + \lim_{\varepsilon_2 \rightarrow 0} \int_{3+\varepsilon_2}^4 \frac{1}{3-x} dx \\ &= -\lim_{\varepsilon_1 \rightarrow 0} \left[\ln \left(\frac{\varepsilon_1}{3} \right) \right] - \lim_{\varepsilon_2 \rightarrow 0} \left[\ln \left(\frac{1}{\varepsilon_2} \right) \right], \end{aligned}$$

an expression formally of the form $\infty - \infty$ and hence not defined. The Cauchy principal value of the integral gives on the other hand the neat result

$$\begin{aligned} P.V. \int_0^4 \frac{1}{3-x} dx &= \lim_{\varepsilon \rightarrow 0} \left(\int_0^{3-\varepsilon} \frac{1}{3-x} dx + \int_{3+\varepsilon}^4 \frac{1}{3-x} dx \right) \\ &= -\lim_{\varepsilon \rightarrow 0} \left[\ln \left(\frac{\varepsilon}{3} \right) + \ln \left(\frac{1}{\varepsilon} \right) \right] = -\lim_{\varepsilon \rightarrow 0} \left[\ln \left(\frac{1}{3} \right) \right] \\ &= \ln 3. \end{aligned}$$

The symmetry of the limit is indeed important for the consistency of the results; if the upper limit of the first integral in the above calculation would have been $3 - 2\varepsilon$ while the other integral still had $3 + \varepsilon$ as its lower limit, the result would have been $\ln \frac{2}{3}$, which has really no meaning at all.

2.3 Properties

In this chapter we shall study some of the main properties of the Hilbert transform. Many of the results can be generalized to distributions, but for simplicity we state the properties for functions on \mathbb{R} . It is also assumed that the functions behave nicely in such a way that results from Fourier analysis can be used when needed.

We start with a simple result concerning the character of the Hilbert transform as an operator.

Theorem 1. *The Hilbert transformation is linear.*

Proof. Let f and g be real-valued functions and $\alpha, \beta \in \mathbb{R}$. Straight from the definition we obtain

$$\begin{aligned} \mathcal{H}(\alpha f + \beta g)(t) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\alpha f(\tau) + \beta g(\tau)}{t - \tau} d\tau \\ &= \frac{\alpha}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau + \frac{\beta}{\pi} P.V. \int_{-\infty}^{\infty} \frac{g(\tau)}{t - \tau} d\tau \\ &= \alpha \mathcal{H}(f)(t) + \beta \mathcal{H}(g)(t). \end{aligned}$$

□

Obviously it would be nice to know how the Hilbert transform \mathcal{H} is related to the Fourier transform \mathcal{F} . This information is depicted in the next important theorem.

Theorem 2. *The Hilbert and Fourier transforms are related to each other by*

$$\mathcal{F}(\mathcal{H}(f))(\xi) = -i \operatorname{sgn}(\xi) \mathcal{F}(f)(\xi)$$

where sgn is the signum function

$$\text{sgn}(x) \doteq \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Proof. Let us first recall that the Hilbert transformation can be written as a convolution:

$$\mathcal{H}(f)(t) = (f * g)(t), \quad (1)$$

where $g(x) = (\pi x)^{-1}$. It should be kept in mind that integrals are still considered as Cauchy principal values. Fourier transforming both sides of equation (1) and using the fact that the Fourier transform of a convolution equals the product of the Fourier transformed functions¹ yields

$$\begin{aligned} \mathcal{F}(\mathcal{H}(f))(\xi) &= \mathcal{F}(f * g)(\xi) = [\mathcal{F}(f)(\xi)] [\mathcal{F}(g)(\xi)] \\ &= \frac{1}{\pi} \mathcal{F}\left(\frac{1}{x}\right)(\xi) \mathcal{F}(f)(\xi). \end{aligned} \quad (2)$$

We are left to calculate the Fourier transform of $\frac{1}{x}$. This can be written as

$$\begin{aligned} \mathcal{F}\left(\frac{1}{x}\right)(\xi) &= P.V. \int_{-\infty}^{\infty} \frac{1}{x} e^{-2\pi i x \xi} dx = P.V. \int_{-\infty}^{\infty} \frac{\cos(2\pi x \xi) - i \sin(2\pi x \xi)}{x} dx \\ &= P.V. \int_{-\infty}^{\infty} \frac{\cos(2\pi x \xi)}{x} dx - i P.V. \int_{-\infty}^{\infty} \frac{\sin(2\pi x \xi)}{x} dx. \end{aligned}$$

The first integral vanishes as a principal value, because the integrand is an odd function. The latter one has no divergences since the integrand approaches the value 1 near the origin, and so the principal value plays no role. Instead, it has to be considered separately for different values of ξ . For $\xi = 0$ the integral vanishes trivially. Assuming that $\xi < 0$ and making a change of variables $z \doteq 2\pi x \xi$ the integral becomes

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi x \xi)}{x} dx = - \int_{-\infty}^{\infty} \frac{\sin(z)}{z} dz = - \int_{-\infty}^{\infty} \text{sinc}(z) dz = -\pi,$$

¹Strictly speaking, in order to use this result here one should verify that it really holds also for principal-valued integrals.

where we used the well-known value for the integral of the sinc function. For $\xi > 0$ the same calculation yields

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi x \xi)}{x} dx = \pi.$$

Combining all these results we can conclude that

$$\mathcal{F}\left(\frac{1}{x}\right)(\xi) = -i\pi \operatorname{sgn}(\xi). \quad (3)$$

Substituting this into equation (2) gives us the desired result:

$$\mathcal{F}(\mathcal{H}(f))(\xi) = \frac{1}{\pi} [-i\pi \operatorname{sgn}(\xi)] \mathcal{F}(f)(\xi) = -i \operatorname{sgn}(\xi) \mathcal{F}(f)(\xi).$$

□

Now that we know how the Fourier and Hilbert transforms behave together, we can prove the following theorem effortlessly.

Theorem 3. *Applying the Hilbert transform twice to the same function gives the function back with a negative sign, i.e.*

$$\mathcal{H}(\mathcal{H}(f))(t) = -f(t).$$

Proof. Taking the inverse Fourier transform of the result in Theorem 2 gives

$$\mathcal{H}(f)(t) = -i\mathcal{F}^{-1}(\operatorname{sgn}(\xi)\mathcal{F}(f)(\xi))(t).$$

Two successive Hilbert transforms yield then

$$\begin{aligned} \mathcal{H}(\mathcal{H}(f))(t) &= -i\mathcal{F}^{-1}\left(\operatorname{sgn}(\omega)\mathcal{F}\left(-i\mathcal{F}^{-1}(\operatorname{sgn}(\xi)\mathcal{F}(f)(\xi))\right)(\omega)\right)(t) \\ &= -\mathcal{F}^{-1}\underbrace{(\operatorname{sgn}(\omega)\operatorname{sgn}(\omega))}_{= 1 \text{ when } \omega \neq 0}\mathcal{F}(f)(\omega)(t) = -f(t). \end{aligned}$$

□

The result obtained above can be symbolically written as $\mathcal{H}^2 = -\mathbb{1}$, where $\mathbb{1}$ indicates the identity operator. This implies formally that $\mathcal{H} = -\mathcal{H}^{-1}$, i.e. $-\mathcal{H} = \mathcal{H}^3$ serves as the inverse Hilbert transform. A function $f(t)$ can then be written as

$$f(t) = \mathcal{H}^{-1}(\mathcal{H}(f))(t) = -\mathcal{H}(\mathcal{H}(f))(t) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\mathcal{H}(f)(\tau)}{t - \tau} d\tau.$$

We shall end this section with a result concerning the boundedness of the Hilbert transform. We won't give a proof here, but address the reader to the original source.

Theorem 4. *Let $f \in L^p(\mathbb{R})$, where $p > 1$. Then there exists a constant M_p such that*

$$\|\mathcal{H}(f)\|_p \leq M_p \|f\|_p.$$

Proof. See VII in [3]. □

3 Examples

In this chapter we will explicitly compute the Hilbert transforms of certain functions to demonstrate its use.

3.1 A constant function

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a constant function, $f(x) = c \in \mathbb{R}$. The Hilbert transform of f is then

$$\begin{aligned} \mathcal{H}(f)(t) &= \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{c}{t - \tau} d\tau = \frac{c}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{t-\frac{1}{\varepsilon}}^{t-\varepsilon} \frac{1}{t - \tau} d\tau + \int_{t+\varepsilon}^{t+\frac{1}{\varepsilon}} \frac{1}{t - \tau} d\tau \right) \\ &= -\frac{c}{\pi} \lim_{\varepsilon \rightarrow 0} \left[\ln(\varepsilon) - \ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\frac{1}{\varepsilon}\right) - \ln(\varepsilon) \right] = 0. \end{aligned}$$

Note that here we had to deal with two kinds of singularities: the one due to the pole at $\tau = t$ and the one coming from the integral diverging at $\pm\infty$. Since this calculation holds for any $t \in \mathbb{R}$, we conclude that the Hilbert transform of a constant function vanishes.

3.2 The Dirac delta function

In this section we look into the Hilbert transform of the Dirac delta function $\delta(x)$. This is a little delicate issue, since the Dirac delta function is not really a function in the traditional sense; it should be rather considered as a distribution or a measure.² Therefore it is not obvious at first sight that our earlier-presented techniques are compatible at all with this creature. This is why the presentation given below should be thought of as an outline of a more rigorous procedure, which is left outside the scope of this work.

We start by noting that according to Theorem 2, the Hilbert transform of δ is (assuming that it exists) given by

$$\mathcal{H}(\delta)(t) = -i\mathcal{F}^{-1}(\operatorname{sgn}(\xi)\mathcal{F}(\delta)(\xi))(t). \quad (4)$$

So the only thing we actually need to know about the Dirac delta function is its Fourier transform, which is well known to be (see e.g. [4])

$$\mathcal{F}(\delta)(\xi) = \int_{-\infty}^{\infty} \delta(x)e^{-2\pi i x \xi} dx = 1.$$

Substituting this into equation (4) gives

$$\mathcal{H}(\delta)(t) = -i\mathcal{F}^{-1}(\operatorname{sgn}(\xi))(t), \quad (5)$$

so we are done if we are just able to find the inverse Fourier transform of the signum function.

We could just calculate this explicitly; there are standard techniques to do this. However, we can use a result obtained in the proof of Theorem 2, namely equation (3), which states that the Fourier transform of $\frac{1}{x}$ is

²Intuitively one can think of the Dirac delta function as a point mass in the origin; its value is zero everywhere except at the origin where it is infinite.

$-i\pi \operatorname{sgn}(\xi)$. Taking the inverse Fourier transform and using its linearity in equation (3) gives us

$$\mathcal{F}^{-1}(\operatorname{sgn}(\xi))(t) = -\frac{1}{i\pi t}.$$

This together with equation (5) gives us the Hilbert transform of the Dirac delta function:

$$\mathcal{H}(\delta)(t) = \frac{1}{\pi t}.$$

3.3 A Gaussian function

We shall end this section by calculating the Hilbert transform of a Gaussian function. So let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = e^{-\alpha x^2}$, where $\alpha \in \mathbb{R}$, $\alpha > 0$. Using again Theorem 2 we know that the Hilbert transform of f is given by

$$\mathcal{H}(f)(t) = -i\mathcal{F}^{-1}\left(\operatorname{sgn}(\xi)\mathcal{F}\left(e^{-\alpha x^2}\right)(\xi)\right)(t). \quad (6)$$

The Fourier transform of a Gaussian function is just another Gaussian, in our conventions

$$\mathcal{F}\left(e^{-\alpha x^2}\right)(\xi) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\pi^2 \xi^2}{\alpha}}.$$

Hence equation (6) becomes

$$\mathcal{H}(f)(t) = -i\sqrt{\frac{\pi}{\alpha}} \mathcal{F}^{-1}\left(\operatorname{sgn}(\xi)e^{-\frac{\pi^2 \xi^2}{\alpha}}\right)(t). \quad (7)$$

The inverse Fourier transform above is

$$\begin{aligned} \mathcal{F}^{-1}\left(\operatorname{sgn}(\xi)e^{-\frac{\pi^2 \xi^2}{\alpha}}\right)(t) &= \int_{-\infty}^{\infty} \operatorname{sgn}(\xi) e^{-\frac{\pi^2 \xi^2}{\alpha}} e^{2\pi i \xi t} d\xi \\ &= \int_{-\infty}^{\infty} \operatorname{sgn}(\xi) e^{-\frac{\pi^2 \xi^2}{\alpha}} [\cos(2\pi \xi t) + i \sin(2\pi \xi t)] d\xi \\ &= \int_{-\infty}^{\infty} \operatorname{sgn}(\xi) e^{-\frac{\pi^2 \xi^2}{\alpha}} \cos(2\pi \xi t) d\xi + i \int_{-\infty}^{\infty} \operatorname{sgn}(\xi) e^{-\frac{\pi^2 \xi^2}{\alpha}} \sin(2\pi \xi t) d\xi. \end{aligned}$$

In order to simplify the above expression we note that signum and sine are odd functions, while cosine is even. The first integral has thus an odd integrand, and it vanishes. The second integrand is even, and we can perform the integration over the positive real numbers and multiply the result by two to obtain the same result. All in all we obtain

$$\mathcal{F}^{-1} \left(\text{sgn}(\xi) e^{-\frac{\pi^2 \xi^2}{\alpha}} \right) (t) = 2i \int_0^{\infty} e^{-\frac{\pi^2 \xi^2}{\alpha}} \sin(2\pi \xi t) d\xi,$$

where we have abandoned the signum function since it no longer plays any role. If we further introduce a change of variables $\eta \doteq \frac{\pi \xi}{\sqrt{\alpha}}$ our result becomes

$$\mathcal{F}^{-1} \left(\text{sgn}(\xi) e^{-\frac{\pi^2 \xi^2}{\alpha}} \right) (t) = \frac{i2\sqrt{\alpha}}{\pi} \int_0^{\infty} e^{-\eta^2} \sin(2\sqrt{\alpha}t\eta) d\eta. \quad (8)$$

To this end we can use the relation

$$\int_0^{\infty} e^{-u^2} \sin(2xu) du = D(x),$$

where $D(x)$ is the so-called *Dawson's integral* defined by

$$D(x) \doteq e^{-x^2} \int_0^x e^{u^2} du,$$

also called the *Dawson's function* [5]. This allows us to write the inverse Fourier transform (8) as

$$\mathcal{F}^{-1} \left(\text{sgn}(\xi) e^{-\frac{\pi^2 \xi^2}{\alpha}} \right) (t) = \frac{i2\sqrt{\alpha}}{\pi} D(\sqrt{\alpha}t)$$

which together with equation (7) gives us the Hilbert transform of our Gaussian function:

$$\mathcal{H}(f)(t) = \frac{2}{\sqrt{\pi}} D(\sqrt{\alpha}t).$$

This final result can just as well be given in terms of the *error function* erf defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds,$$

yielding

$$\mathcal{H}(f)(t) = -ie^{-ax^2} \operatorname{erf}(i\sqrt{at}).$$

Yet another form can be attained using *Kummer's confluent hypergeometric functions*, see [1] for further details.

References

- [1] F. W. King, *Hilbert Transforms, Volume 1*, Encyclopedia of Mathematics and Its Applications 124, Cambridge University Press, New York, 2009.
- [2] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists, 4th Edition*, Academic Press, San Diego, 1995.
- [3] M. Riesz, *Sur les fonctions conjuguées*, *Mathematische Zeitschrift* **27** (1928) 218-244, doi: 10.1007/BF01171098.
- [4] P. Szekeres, *A Course In Modern Mathematical Physics: Groups, Hilbert Space and Differential Geometry*, Cambridge University Press, New York, 2012.
- [5] E. Carneiro and F. Littmann, *Bandlimited Approximations to the Truncated Gaussian and Applications*, *Constructive Approximation* **38** (2013) 19-57, doi: 10.1007/s00365-012-9177-8.