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Zeta functions: formulas and applications

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Abstract

The existence conditions of the zeta function of a pseudodifferential operator and the definition of determinant thereby obtained are reviewed, as well as the concept of multiplicative anomaly associated with the determinant and its calculation by means of the Wodzicki residue. Exponentially fast convergent formulas – valid in the whole of the complex plane and yielding the pole positions and residua – that extend the ones by Chowla and Selberg for the Epstein zeta function (quadratic form) and by Barnes (affine form) are then given. After briefly recalling the zeta function regularization procedure in quantum field theory, some applications of these expressions in physics are described. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The regularization and renormalization procedures are essential issues of contemporary physics – without which it would simply not exist, at least in the form we know it. Among the different methods, zeta function regularization – which is obtained by analytical continuation in the complex plane of the zeta function of the relevant physical operator in each case – may be the most attractive of all. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system (with constraints of any kind, in principle). Assuming the corresponding Hamiltonian operator, H, has a spectral decomposition of the form (think, as simplest case, in a quantum

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harmonic oscillator): $\{\lambda_i, \varphi_i\}_{i\in I}$, I being some set of indices (which can be discrete, continuous, mixed, multiple, etc.), then the quantum vacuum energy is obtained as follows [21]:

$$\sum_{i\in I}(\varphi_i, H\varphi_i) = \operatorname{tr} H = \sum_{i\in I}\lambda_i = \sum_{i\in I}\lambda_i^{-s}\Big|_{s=-1} = \zeta_H(-1), \tag{1}$$

where ζ_H is the zeta function corresponding to the operator H. Note that the formal sum over the eigenvalues is usually ill defined, and that the last step involves analytic continuation, inherent with the definition of the zeta function itself. These mathematically simple-looking relations involve very deep physical concepts (no wonder that understanding them took several decades in the recent history of quantum field theory, QFT). The zeta function method is unchallenged at the one-loop level, where it is rigorously defined and where many calculations of QFT reduce basically (from a mathematical point of view) to the computation of determinants of elliptic pseudodifferential operators (ΨDOs) [32]. It is thus not surprising that the preferred definition of determinant for such operators is obtained through the corresponding zeta function (see the next section).

When one comes to specific calculations, the zeta function regularization method relies on the existence of simple formulas for obtaining the analytical continuation above. They consist of the reflection formula of the corresponding zeta function in each case, together with some other fundamental expressions, as the Jacobi theta function identity, Poisson's resummation formula and the famous Chowla–Selberg formula. However, some of these formulae are restricted to very specific zeta functions, and it often turned out that for some physically important cases the corresponding formulas did not exist in the literature. This has required a painful process (it has taken over a decade already) of generalization of previous results and derivation of new expressions of this sort. Here we will provide some of them, obtained recently and not to be found in the new books [21,14].

It has also been realized, during the last couple of years, that the calculation of effective actions through the corresponding determinants is not free from dangers – that were overlooked till very recently by the physics community – stemming from the existence of a multiplicative anomaly, e.g., the fact that the determinant of a product of operators is not equal, in general, to the product of the determinants. The difference proves to be physically relevant, since only in some particular cases can it be simply absorbed into the renormalized constants. This issue will be described in detail in the next sections, with several examples and physical applications. Besides guiding the reader through the relevant bibliography of published books and papers on these subjects, we will obtain in what follows a number of new results.

2. The zeta function of a ΨDO and its associated determinant

2.1. The zeta function

Let A be a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}$, acting on the space of smooth sections of E, an n-dimensional vector bundle over M, a closed n-dimensional manifold. The *zeta function* ζ_A is defined as

$$\zeta_A(s) = \operatorname{tr} A^{-s} = \sum_j \lambda_j^{-s}, \qquad \operatorname{Re} s > \frac{n}{m} \equiv s_0, \tag{2}$$

where $s_0 = \dim M/\mathrm{ord}\,A$ is called the *abscissa of convergence* of $\zeta_A(s)$. Under these conditions, it can be proved that $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s=0$), provided that the principal symbol of A (that is $a_m(x,\xi)$) admits a *spectral cut*: $L_\theta = \{\lambda \in \mathbb{C}; \mathrm{Arg}\,\lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\mathrm{Spec}\,A \cap L_\theta = \emptyset$ (Agmon–Nirenberg condition). The definition of $\zeta_A(s)$ depends on the position of the cut L_θ. The only possible singularities of $\zeta_A(s)$ are *simple poles* at $s_k = (n-k)/m,\ k = 0,1,2,\dots,n-1,n+1,\dots$. Kontsevich and Vishik [30] have managed to extend this definition to the case when $m \in \mathbb{C}$ (no spectral cut exists).

2.2. The zeta determinant

Let A be a ΨDO operator with a spectral decomposition: $\{\varphi_i, \lambda_i\}_{i\in I}$, where I is some set of indices. The definition of determinant starts by trying to make sense of the product $\prod_{i\in I}\lambda_i$, which can be easily transformed into a "sum": $\ln \prod_{i\in I}\lambda_i = \sum_{i\in I}\ln\lambda_i$. From the definition of the zeta function of A: $\zeta_A(s) = \sum_{i\in I}\lambda_i^{-s}$, by taking the derivative at $s=0$: $\zeta_A'(0) = -\sum_{i\in I}\ln\lambda_i$, we arrive at the following definition of determinant of A [33–35]: $\det_\zeta A = \exp[-\zeta_A'(0)]$. An older definition (due to Weierstrass) is obtained by subtracting in the series above (when it is such) the leading behavior of λ_i as a function of i, as $i \to \infty$, until the series $\sum_{i\in I}\ln\lambda_i$ is made to converge. A shortcoming here is – for physical applications – that these additional terms turn out to be *nonlocal* and, thus, are non admissible in any physical renormalization procedure.

In algebraic QFT, in order to write down an action in operator language one needs a functional that replaces integration. For the Yang–Mills theory this is the Dixmier trace, which is the *unique* extension of the usual trace to the ideal $\mathscr{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum: $\sigma_N(T) \equiv \sum_{j=0}^{N-1}\mu_j = \mathcal{O}(\log N), \mu_0 \geqslant \mu_1 \geqslant \cdots$. The definition of the Dixmier trace of T is $\mathrm{Dtr}\,T = \lim_{N\to\infty}[1/(\log N)]\sigma_N(T)$, provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N\to\infty$ (recall that $M(f)(\lambda) = (1/\ln\lambda)\int_1^\lambda f(u)\mathrm{d}u/u$). Then, the Hardy–Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s=1$ (see [8]): $\mathrm{Dtr}\,T = \lim_{s\to 1^+}(s-1)\zeta_{T^{-1}}(s)$.

The Wodzicki (or noncommutative) residue [38] is the *only* extension of the Dixmier trace to the ΨDOs which are not in $\mathscr{L}^{(1,\infty)}$. It is the *only* trace one can define in the algebra of ΨDOs (up to a multiplicative constant), its definition being: res $A = 2\,\mathrm{Res}_{s=0}\,\mathrm{tr}(A\Delta^{-s})$, with Δ the Laplacian. It satisfies the trace condition: $\mathrm{res}(AB) = \mathrm{res}(BA)$. A very important property is that it can be expressed as an integral (local form) $\mathrm{res}\,A = \int_{S^*M}\mathrm{tr}\,a_{-n}(x,\xi)\,\mathrm{d}\xi$ with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler–Manin residue).

If $\dim M = n = -\mathrm{ord}\,A$ (M compact Riemann, A elliptic, $n\in\mathbb{N}$) it coincides with the Dixmier trace, and one has $\mathrm{Res}_{s=1}\,\zeta_A(s) = (1/n)\mathrm{res}\,A^{-1}$. The Wodzicki residue continues to make sense for ΨDOs of arbitrary order and, even if the symbols $a_j(x,\xi)$, $j<m$, are not invariant under coordinate choice, their integral is, and defines a trace. All residua at poles of the zeta function of a ΨDO can be easily obtained from the Wodzciki residue [15].

2.3. The multiplicative anomaly and its implications

Given A, B and AB ΨDOs, even if ζ_A, ζ_B and ζ_{AB} exist, it turns out that, in general, $\det_\zeta(AB) \neq \det_\zeta A\det_\zeta B$. The multiplicative (or noncommutative, or determinant) anomaly is defined as

$$\delta(A,B)=\ln\left[\frac{\det_\zeta(AB)}{\det_\zeta A\det_\zeta B}\right]=-\zeta'_{AB}(0)+\zeta'_A(0)+\zeta'_B(0). \tag{3}$$

Wodzicki's formula for the multiplicative anomaly reads [38,28]:

$$\delta(A,B)=\frac{\operatorname{res}\{[\ln\sigma(A,B)]^2\}}{2\operatorname{ord}A\operatorname{ord}B(\operatorname{ord}A+\operatorname{ord}B)},\quad \sigma(A,B):=A^{\operatorname{ord}B}B^{-\operatorname{ord}A}. \tag{4}$$

At the level of quantum mechanics (QM), where it was originally introduced by Feynman, the path-integral approach is just an alternative formulation of the theory. In QFT it is much more than this, being in many occassions *the* actual formulation of QFT [32]. In short, consider the Gaussian functional integration

$$\int[\mathrm{d}\Phi]\exp\left\{-\int\mathrm{d}^Dx[\Phi^\dagger(x)(\quad)\Phi(x)+\cdots]\right\}\to\det(\quad)^\pm \tag{5}$$

and assume that the operator matrix has the following simple structure (being each A_i an operator on its own):

$$\begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}\to\begin{pmatrix}A & \\ & B\end{pmatrix}, \tag{6}$$

where the last expression is the result of diagonalizing the operator matrix. A question now arises. What is the determinant of the operator matrix: $\det(AB)$ or $\det A\cdot\det B$? This has been very much on discussion during the last months [26,9,18,19]. There is agreement in that: (i) In a situation where a superselection rule exists, AB has no sense (much less its determinant), and then the answer must be $\det A\cdot\det B$. (ii) If the diagonal form is obtained after a change of basis (diagonalization process), then the quantity that is preserved by such transformations is the value of $\det(AB)$ and *not* the product of the individual determinants (there are counterexamples supporting this viewpoint [17,31,6]).

3. On the explicit calculation of ζ_A and $\det_\zeta A$

A fundamental property shared by zeta functions is the existence of a reflection formula. For the Riemann zeta function: $\Gamma(s/2)\zeta(s)=\pi^{s-1/2}\Gamma(1-s/2)\zeta(1-s)$. For a generic zeta function, $Z(s)$, it has the form: $Z(\omega-s)=F(\omega,s)Z(s)$, and allows for its analytic continuation in a very easy way – what is, as advanced above, the whole story of the zeta function regularization procedure (or at least the main part of it). But the analytically continued expression thus obtained is just another series, which has again a slow convergence behavior, of power series type [3] (actually the same that the original series had, in its own domain of validity). Some years ago, Chowla and Selberg found a formula, for the Epstein zeta function in the two-dimensional case [7], that yields *exponentially quick convergence, and not only in the reflected domain*. They were very proud of that formula – as one can appreciate by reading the original paper, where actually no hint about its derivation was given [7]. In Ref. [11], I generalized this expression to inhomogeneous zeta functions (very important for physical applications), but staying always in *two* dimensions, for this was commonly believed to be an unsurmountable restriction of the original formula (see, for instance, Ref. [27]). Recently, I have obtained an extension to an *arbitrary* number of dimensions [16], both in the homogeneous (quadratic form) and nonhomogeneous (quadratic plus affine form) cases.

In short, for the following zeta functions (that correspond to the general quadratic – plus affine – case and to the general affine case, in any number of dimensions, d) explicit formulas of the CS type have been obtained [16]: $\zeta_1(s) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} [Q(\boldsymbol{n}) + A(\boldsymbol{n})]^{-s}$ and $\zeta_2(s) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} A(\boldsymbol{n})^{-s}$, where Q is a nonnegative quadratic form and A a general affine one, in d dimensions (giving rise to Epstein and Barnes zeta functions, respectively). Moreover, some expressions for the much more difficult cases when the summation ranges are interchanged, that is: $\zeta_3(s) = \sum_{\boldsymbol{n}\in\mathbb{N}^d} [Q(\boldsymbol{n}) + A(\boldsymbol{n})]^{-s}$ and $\zeta_4(s) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} A(\boldsymbol{n})^{-s}$, have also been constructed [16]. For lack of space, we cannot give all the details of these derivations here.

3.1. Quadratic (plus affine) form

3.1.1. Extended Epstein zeta function in p dimensions

The starting point is *Poisson's resummation formula* in p dimensions, which arises from the distribution identity $\sum_{\boldsymbol{n}\in\mathbb{Z}^p} \delta(\boldsymbol{x}-\boldsymbol{n}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^p} \mathrm{e}^{\mathrm{i}2\pi\boldsymbol{m}\cdot\boldsymbol{x}}$. (We shall indistinctly write $\boldsymbol{m}\cdot\boldsymbol{x} \equiv \boldsymbol{m}^{\mathrm{T}}\boldsymbol{x}$ in what follows.) Applying this identity to the function $f(\boldsymbol{x}) = \exp(-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x})$, with A an invertible $p\times p$ matrix, and integrating over $\boldsymbol{x}\in\mathbb{R}^p$, we get

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^p} \exp\left(-\frac{1}{2}\boldsymbol{n}^{\mathrm{T}}A\boldsymbol{n} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{n}\right) = \frac{(2\pi)^{p/2}}{\sqrt{\det A}} \sum_{\boldsymbol{m}\in\mathbb{Z}^p} \exp\left[\frac{1}{2}(\boldsymbol{b}+2\pi\mathrm{i}\boldsymbol{m})^{\mathrm{T}}A^{-1}(\boldsymbol{b}+2\pi\mathrm{i}\boldsymbol{m})\right].$$

Consider now the following zeta function ($\Re s > p/2$):

$$\zeta_{A,\boldsymbol{c},q}(s) = \sum_{\boldsymbol{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}(\boldsymbol{n}+\boldsymbol{c})^{\mathrm{T}}A(\boldsymbol{n}+\boldsymbol{c}) + q\right]^{-s} \equiv \sum_{\boldsymbol{n}\in\mathbb{Z}^p}{}' [Q(\boldsymbol{n}+\boldsymbol{c}) + q]^{-s}$$

(the prime on the summatories will always mean that the point $\boldsymbol{n} = \boldsymbol{0}$ is to be excluded from the sum). The aim is to obtain a formula that gives (the analytic continuation of) this multidimensional zeta function in terms of an exponentially convergent multiseries and which is valid in the whole complex plane, exhibiting the singularities (simple poles) of the meromorphic continuation – with the corresponding residua – explicitly. The only condition on the matrix A is that it corresponds to a (nonnegative) quadratic form, which we call Q. The vector $\boldsymbol{c}$ is arbitrary, while q will (for the moment) be a positive constant. Use of the Poisson resummation formula yields

$$\zeta_{A,\boldsymbol{c},q}(s) = \frac{(2\pi)^{p/2}q^{p/2-s}}{\sqrt{\det A}}\frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)} \times \sum_{\boldsymbol{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\boldsymbol{m}\cdot\boldsymbol{c})(\boldsymbol{m}^{\mathrm{T}}A^{-1}\boldsymbol{m})^{s/2-p/4}K_{p/2-s}(2\pi\sqrt{2q\boldsymbol{m}^{\mathrm{T}}A^{-1}\boldsymbol{m}}), \quad (7)$$

where K_ν is the modified Bessel function of the second kind and the subindex $\frac{1}{2}$ in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\boldsymbol{m}\in\mathbb{Z}^p$ intervene in the sum. That is, if we take an $\boldsymbol{m}\in\mathbb{Z}^p$ we must then exclude $-\boldsymbol{m}$ (as the simple criterion one can, for instance, select those vectors in $\mathbb{Z}^p\setminus\{\boldsymbol{0}\}$ whose first nonzero component is positive). Eq. (7) fulfills *all* the requirements demanded before. It is worth noting how the only pole of this inhomogeneous Epstein zeta function appears explicitly at

$s=p/2$, where it belongs. Its residue is given by $\text{Res}_{s=p/2}\,\zeta_{A,\boldsymbol{c},q}(s)=(2\pi)^{p/2}/(\sqrt{\det A}\,\Gamma(p/2))$. With a little of care, it is relatively simple to obtain the limit of expression (7) as $q\to 0$.

When $q=0$ there is *no* way to use the Poisson formula on all p indices of $\boldsymbol{n}$. However, one can still use it on *some* of the p indices $\boldsymbol{n}$ only, say on just one of them, n_1. Poisson's formula on one index reduces to the celebrated Jacobi identity for the θ_3 function, that can be written as $\sum_{n=-\infty}^{\infty} \mathrm{e}^{-(n+z)^2 t}=\sqrt{(\pi/t)}[1+\sum_{n=1}^{\infty}\mathrm{e}^{-\pi^2 n^2/t}\cos(2\pi nz)]$. Here z and t are arbitrary complex numbers, $z,t\in\mathbb{C}$, with the only restriction that $\Re\, t>0$. Applying this last formula to the first component, n_1, we obtain the following recurrent formula (for the sake of simplicity we set $\boldsymbol{c}=\boldsymbol{0}$, but the result can be easily generalized to $\boldsymbol{c}\neq\boldsymbol{0}$):

$$\zeta_{A,\boldsymbol{0},q}(s)=\zeta_{a,\boldsymbol{0},q}(s)+\sqrt{\frac{\pi}{a}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{\Delta_{p-1},\boldsymbol{0},q}(s-\tfrac{1}{2})+\frac{4\pi^s}{a^{s/2+1/4}\Gamma(s)}\sum_{\boldsymbol{n}_2\in\mathbb{Z}^{p-1}}{}' \times\sum_{n_1=1}^{\infty}\cos\left(\frac{\pi n_1}{a}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{n}_2\right)n_1^{s-1/2}(\boldsymbol{n}_2^{\mathrm{T}}\Delta_{p-1}\boldsymbol{n}_2+q)^{1/4-s/2}K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a}}\sqrt{\boldsymbol{n}_2^{\mathrm{T}}\Delta_{p-1}\boldsymbol{n}_2+q}\right). \quad (8)$$

This is a recurrent formula in p, the number of dimensions, the first term of the recurrence being [14]

$$\zeta_{a,\boldsymbol{0},q}(s)=2\sum_{n=1}^{\infty}(an^2+q)^{-s}=q^{-s}+\sqrt{\frac{\pi}{a}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}q^{1/2-s} +\frac{4\pi^s}{\Gamma(s)}a^{-1/4-s/2}q^{1/4-s/2}\sum_{n=1}^{\infty}n^{s-1/2}K_{s-1/2}\left(2\pi n\sqrt{\frac{q}{a}}\right). \quad (9)$$

To take in these expressions the limit $q\to 0$ is immediate:

$$\zeta_{A,\boldsymbol{0},0}(s)=2a^{-s}\zeta(2s)+\sqrt{\frac{\pi}{a}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{\Delta_{p-1},\boldsymbol{0},0}(s-\tfrac{1}{2})+\frac{4\pi^s}{a^{s/2+1/4}\Gamma(s)}\sum_{\boldsymbol{n}_2\in\mathbb{Z}^{p-1}}{}' \times\sum_{n_1=1}^{\infty}\cos\left(\frac{\pi n_1}{a}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{n}_2\right)n_1^{s-1/2}(\boldsymbol{n}_2^{\mathrm{T}}\Delta_{p-1}\boldsymbol{n}_2)^{1/4-s/2}K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a}}\sqrt{\boldsymbol{n}_2^{\mathrm{T}}\Delta_{p-1}\boldsymbol{n}_2}\right). \quad (10)$$

In the above formulas, A is a $p\times p$ symmetric matrix $A=(a_{ij})_{i,j=1,2,\dots,p}=A^{\mathrm{T}}$, A_{p-1} the $(p-1)\times(p-1)$ reduced matrix $A_{p-1}=(a_{ij})_{i,j=2,\dots,p}$, a the component $a=a_{11}$, $\boldsymbol{b}$ the $p-1$ vector $\boldsymbol{b}=(a_{21},\dots,a_{p1})^{\mathrm{T}}=(a_{12},\dots,a_{1p})^{\mathrm{T}}$, and Δ_{p-1} is the following $(p-1)\times(p-1)$ matrix: $\Delta_{p-1}=A_{p-1}-(1/4a)\boldsymbol{b}\otimes\boldsymbol{b}$.

It turns out that the limit as $q\to 0$ of Eq. (7) is again the recurrent formula (10). More precisely, what is obtained in the limit is the reflected formula, which one gets after using the Epstein zeta function reflection $\Gamma(s)Z(s;A)=[\pi^{2s-p/2}/\sqrt{\det A}]\,\Gamma(p/2-s)Z(p/2-s;A^{-1})$, $Z(s;A)$ being the Epstein zeta function [24,25]. This result is easy to understand after some thinking. Summing up, we have thus checked that Eq. (7) is valid for *any* $q\geqslant 0$, since it contains in a hidden way, for $q=0$, the recurrent expression (10).

The formulas here can be considered as generalizations of the Chowla–Selberg formula. All share the same properties that are so much appreciated by number theoretists as pertaining to the CS formula. In a way, these expressions can be viewed as improved reflection formulas for zeta functions; they are in fact much better than those in several aspects: while a reflection formula connects one region of the complex plane with a complementary region (with some intersection) by analytical continuation, the CS formula and the formulas above are valid on the *whole* complex plane, exhibiting the poles of the zeta function and the corresponding residua *explicitly*. Even more important, while a reflection formula is intended to replace the initial expression of the zeta function – a power series whose convergence can be extremely slow – by another power series with the same type of convergence, it turns out that the expressions here obtained give the meromorphic extension of the zeta function, on the whole complex s-plane, in terms of an *exponentially decreasing* power series (as was the case with the CS formula, that one being its most precious property). Actually, exponential convergence strictly holds under the condition that $q\geqslant 0$. However, the formulas themselves are valid for $q<0$ or even complex. What is not guaranteed for general $q\in\mathbb{C}$ is the exponential convergence of the series. Those analytical continuations in q must be dealt with specifically. The physical example of a field theory with a chemical potential falls clearly into this class.

3.1.2. Generalized Epstein zeta function in two dimensions

For completeness, let us write down the corresponding series when $p=2$ explicitly. They are, with $q>0$,

$$\zeta_E(s;a,b,c;q) = -q^{-s} + \frac{2\pi q^{1-s}}{(s-1)\sqrt{\Delta}} + \frac{4}{\Gamma(s)}\left[\left(\frac{q}{a}\right)^{1/4}\left(\frac{\pi}{\sqrt{qa}}\right)^{s}\right.$$
$$\times\sum_{n=1}^{\infty} n^{s-1/2}K_{s-1/2}\left(2\pi n\sqrt{\frac{q}{a}}\right) + \sqrt{\frac{q}{a}}\left(2\pi\sqrt{\frac{a}{q\Delta}}\right)^{s}\sum_{n=1}^{\infty} n^{s-1}K_{s-1}\left(4\pi n\sqrt{\frac{aq}{\Delta}}\right)$$
$$+\sqrt{\frac{2}{a}}(2\pi)^s\sum_{n=1}^{\infty} n^{s-1/2}\cos(\pi n b/a)\sum_{d|n} d^{1-2s}\left(\Delta+\frac{4aq}{d^2}\right)^{1/4-s/2}$$
$$\left.\times K_{s-1/2}\left(\frac{\pi n}{a}\sqrt{\Delta+\frac{4aq}{d^2}}\right)\right], \tag{11}$$

where $\Delta = 4ac-b^2>0$, and, with $q=0$, the CS formula [7]

$$\zeta_E(s;a,b,c;0) = 2\zeta(2s)\,a^{-s} + \frac{2^{2s}\sqrt{\pi}\,a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\,\Gamma(s-\tfrac{1}{2})\zeta(2s-1)$$
$$+\frac{2^{s+5/2}\pi^s}{\Gamma(s)\Delta^{s/2-1/4}\sqrt{a}}\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\cos(\pi n b/a)\,K_{s-1/2}\left(\frac{\pi n}{a}\sqrt{\Delta}\right), \tag{12}$$

where $\sigma_s(n)\equiv\sum_{d|n} d^s$, sum over the s-powers of the divisors of n. We observe that the r.h.s. of (11) and (12) exhibit a simple pole at $s=1$, with a common residue: $\mathrm{Res}_{s=1}\,\zeta_E(s;a,b,c;q) = 2\pi/\sqrt{\Delta} = \mathrm{Res}_{s=1}\,\zeta_E(s;a,b,c;0)$.

3.1.3. Truncated Epstein zeta function in two dimensions

The most involved case in the family of Epstein-like zeta functions corresponds to having to deal with a *truncated* range. This comes about when one imposes boundary conditions of the usual Dirichlet or Neumann type [21,14]. Jacobi's theta function identity and Poisson's summation formula are then *useless* and no expression in terms of a convergent series for the analytical continuation to values of $\Re s$ below the abscissa of convergence can be obtained. The method must use then the zeta function regularization theorem [10,22,12] and the best one gets is an *asymptotic* series. The issue of extending the CS formula, or the most general expression we have obtained before, to this situation is not an easy one (see, however, Ref. [16]). This problem has seldom (if ever) been properly addressed in the literature.

As an example, let us consider the following series in one dimension: $\zeta_G(s;a,c;q) \equiv \sum_{n=-\infty}^{\infty} [a(n+c)^2+q]^{-s}$, $\Re s > \frac{1}{2}$. Associated with this zeta functions, but considerably more difficult to treat, is the truncated series, with indices running from 0 to ∞

$$\zeta_{G_t}(s;a,c;q) \equiv \sum_{n=0}^{\infty} [a(n+c)^2+q]^{-s}, \quad \Re s > \tfrac{1}{2}. \tag{13}$$

In this case the Jacobi identity is of no use. The way to proceed is employing specific techniques of analytic continuation of zeta functions [21,4]. There is no place to describe them here in detail. The usual method involves three steps [10,22,12]. The first step is easy: to write the initial series as a Mellin transform $\sum_{n=0}^{\infty} [a(n+c)^2+q]^{-s} = (1/\Gamma(s)) \sum_{n=0}^{\infty} \int_0^{\infty} \mathrm{d}t\, t^{s-1} \exp\{-[a(n+c)^2+q]t\}$. The second, to expand in power series part of the exponential, leaving a converging factor: $\sum_{n=0}^{\infty} [a(n+c)^2 + q]^{-s} = (1/\Gamma(s)) \sum_{n=0}^{\infty} \int_0^{\infty} \mathrm{d}t \sum_{m=0}^{\infty} ((-a)^m/m!)(n+c)^{2m} t^{s+m-1} \mathrm{e}^{-qt}$. The third, and most difficult, step is to interchange the order of the two summations – with the aim to obtain a series of zeta functions – which means transforming the second series into an integral along a path on the complex plane, that has to be closed into a circuit (the sum over poles inside reproduces the original series), with a part of it being sent to infinity. Usually, after interchanging the first series and the integral, there is a contribution of this part of the circuit at infinity, what provides in the end an *additional* contribution to the trivial commutation (given by the zeta function regularization theorem). More important, what one obtains in general through this process is *not* a convergent series of zeta functions, but an asymptotic series [21,14]. That is, in our example,

$$\sum_{n=0}^{\infty} [a(n+c)^2+q]^{-s} \sim \sum_{m=0}^{\infty} \frac{(-a)^m \Gamma(m+s)}{m!\,\Gamma(s) q^{m+s}} \zeta_H(-2m,c) + \text{additional terms.}$$

Being more precise, as an outcome of the whole process we obtain the following result for the analytic continuation of the zeta function [13, p. 6100]:

$$\zeta_{G_t}(s;a,c;q) \sim \left(\frac{1}{2}-c\right) q^{-s} + \frac{q^{-s}}{\Gamma(s)} \sum_{m=1}^{\infty} \frac{(-1)^m \Gamma(m+s)}{m!} \left(\frac{q}{a}\right)^{-m} \zeta_H(-2m,c)$$
$$+\sqrt{\frac{\pi}{a}} \frac{\Gamma(s-\frac{1}{2})}{2\Gamma(s)} q^{1/2-s} + \frac{2\pi^s}{\Gamma(s)} a^{-1/4-s/2} q^{1/4-s/2} \sum_{n=1}^{\infty} n^{s-1/2} \cos(2\pi nc) K_{s-1/2}(2\pi n\sqrt{q/a}). \tag{14}$$

(Note that this expression reduces to Eq. (9) in the limit $c \to 0$.) The first series on the r.h.s. is asymptotic [10,22,12,13, p. 3308]. Observe, on the other hand, the singularity structure of this zeta function. Apart from the pole at $s=\frac{1}{2}$, there is a whole sequence of poles at the negative real axis, for $s=-\frac{1}{2},-\frac{3}{2},\ldots$, with residua: $\mathrm{Res}_{s=1/2-j}\, \zeta_{G_t}(s;a,c;q) = (2j-1)!!\, q^j/j!\, 2^j \sqrt{a}$, $j=0,1,2,\ldots$. The generalization of this to p dimensions can be found in Ref. [16].

3.2. Affine form

3.2.1. Barnes and related zeta functions in two dimensions

Consider the Barnes zeta function in two dimensions [1,2]

$$\zeta_B(s;a|\boldsymbol{r}) = \sum_{n_1,n_2=0}^{\infty} (a + r_1 n_1 + r_2 n_2)^{-s}, \quad \Re s > 2,\ r_1, r_2 > 0, \tag{15}$$

and the related zeta function

$$\zeta(s;a|\boldsymbol{r}) = \sum_{n_1,n_2\in Z}^{\infty}{}'(a + r_1 n_1 + r_2 n_2)^{-s}, \tag{16}$$

where the prime means that the term with $n_1 = n_2 = 0$ is absent from the sum (actually, we could have defined the Barnes zeta function in this way too, in order to allow for the particular case $a = 0$, but that would not be the usual definition). The two functions are related, in fact: $\zeta(s;a|\boldsymbol{r}) = \zeta_B(s;a|\boldsymbol{r}) + \zeta_B(s;a|-\boldsymbol{r}) + \zeta_B(s;a|(r_1,-r_2)) + \zeta_B(s;a|(-r_1,r_2)) - \sum_{i=1}^{2}[r_i^{-s}\zeta_H(s,a/r_i) + (-r_i)^{-s}\zeta_H(s,-a/r_i)]$, being ζ_H the Hurwitz zeta function. Recall Hermite's formula for the Hurwitz zeta function

$$\zeta_H(s,a) = \frac{a^{1-s}}{s-1} + \frac{a^{-s}}{2} + 2\int_0^\infty \mathrm{d}y\, \frac{(a^2+y^2)^{-s/2}\sin[s\arctan(y/a)]}{\mathrm{e}^{2\pi y}-1}.$$

This expression exhibits the singularity structure of $\zeta_H(s,a)$ explicitly, the integral being an analytic function of s, $\forall s \in \mathbb{C}$ (it is uniformly convergent for $|s| \leqslant R$, for any $R > 0$). If we expand $\zeta_H(s,a)$ around $s = 0$, we obtain, at $s = 0$, the formula $\zeta_B(0;a|\boldsymbol{r}) = (r_1/2r_2)B_2(a/r_1) + \frac{1}{2}(\frac{1}{2} - (a/r_1)) + (r_2/12r_1)$. Taking the derivative with respect to s, at $s = 0$, we obtain

$$\begin{aligned}\zeta_B'(0;a|\boldsymbol{r}) = {} & -\zeta_B(0;a|\boldsymbol{r})\log r_2 + \left(1 - \log\frac{r_1}{r_2}\right)\frac{r_1}{2r_2}B_2(a/r_1) - \frac{r_1}{r_2}\zeta_H'(-1,a/r_1) \\ & + \left(\frac{a}{2r_1} - \frac{1}{4}\right)\log\frac{r_1}{r_2} + \frac{1}{2}\log\Gamma(a/r_1) - \frac{1}{4}\log(2\pi) \\ & - \frac{r_2}{12r_1}[\log(r_1/r_2) + \psi(a/r_1)] + R(a|\boldsymbol{r}),\end{aligned} \tag{17}$$

where

$$R(a|\boldsymbol{r}) \begin{cases} = \dfrac{2r_1}{r_2}\displaystyle\sum_{n=0}^{\infty}(n + a/r_1)\left[\int_0^\infty \frac{\arctan x\,\mathrm{d}x}{\exp[2\pi(r_1/r_2)x(n + a/r_1)] - 1}\right. \\ \left. -\dfrac{1}{24}\left(\dfrac{a}{r_2} + \dfrac{r_1}{r_2}n\right)^{-2}\right], \\ \sim \dfrac{1}{\pi}\displaystyle\sum_{k=1}^{\infty}(-1)^k(2k)!\,\zeta(2k+2)\left(\frac{r_2}{2\pi r_1}\right)^{2k+1}\zeta_H(2k+1,a/r_1). \end{cases}$$

These are the two possible expressions for the remainder term $R(a|\boldsymbol{r})$. The first is valid for $r_2 \leqslant r_1 \leqslant a$. It is very quickly convergent in this region and, therefore, appropriate for numerical calculations. The second is an explicit asymptotic series for $r_2 \ll 2\pi a$.

For the other two-dimensional zeta function considered at the beginning:

$$\zeta'(0;a|\boldsymbol{r}) = \sum_{\alpha=1}^{4} \zeta_B'(0;a|\boldsymbol{r}_\alpha) + \sum_{i=1}^{2} \left[\left(\frac{1}{2} - \frac{a}{r_i}\right) \log r_i + \left(\frac{1}{2} + \frac{a}{r_i}\right) \log(-r_i) \right.$$
$$\left. -\log \Gamma\left(\frac{a}{r_i}\right) - \log \Gamma\left(\frac{a}{-r_i}\right) + \log(2\pi) \right], \tag{18}$$

where $\boldsymbol{r}_1 = (r_1, r_2)$, $\boldsymbol{r}_2 = (r_1, -r_2)$, $\boldsymbol{r}_3 = (-r_1, r_2)$, $\boldsymbol{r}_4 = (-r_1, -r_2)$. Note that this evaluation involves an analytic continuation on the values of r_1 and r_2 – which are strictly positive in the case of the Barnes zeta function – to negative and, in general, complex values $r_1, r_2 \in \mathbb{C}$. In this way, we elude the apparently unsurmountable problems that a direct interpretation of the sum in Eq. (16) poses, even for positive $r_1, r_2 \in \mathbb{R}$. In particular, for $r_1 = r_2$ it develops an infinite number of zero and constant modes, for an infinite number of constants, what renders a direct interpretation of the series extremely problematic. Naive analytical continuation followed by a natural transportation of the negative signs of the indices n_i to the parameters r_i, yields

$$\zeta'(0;a|\boldsymbol{r}) = -\log\left[\Gamma\left(\frac{a}{r_2}\right)\Gamma\left(\frac{a}{-r_2}\right)\right] + \frac{a\pi \mathrm{i}}{r_2} - \log\frac{a}{r_2} \pm \mathrm{i}\frac{\pi}{2} + \log 2\pi. \tag{19}$$

Note that the breaking of the initial symmetry under interchange of r_1 and r_2 has finally resulted in the disappearance of one of the two parameters from the end result. The fact that the final formula actually preserves the modular invariance of the inital one, under the parameter change on the other variable, here r_2, is however remarkable,

$$a \to a + kr_2, \quad k \in Z,$$

since it transforms as

$$\zeta'(0;a|\boldsymbol{r}) \to \zeta'(0;a|\boldsymbol{r}) + 2\pi k,$$

so that the modular symmetry of a with respect to r_2 is preserved. The r_1 dependence has disappeared completely, the reason for this being the following. The formula for the Barnes zeta function for positive r_1 and r_2 obtained in the preceding section cannot be analytically continued to both r_1 and r_2 negative in this naive way, because, for any value of s (and a, r_1, r_2 fixed), once we break the symmetry and do the analytical continuation say in r_2, then the analytical continuation in r_1 is restricted to the only possibility of n_1 being zero. In fact, in other words, when we try to pull out of the r_1 real axis, for any value of $n_1 \neq 0$ we encounter a pole of the initial zeta function, taking n_2 big enough. Now, it is immediate to see that one can interprete the final formula (19) as the true analytical continuation for the restriction $n_1 = 0$. In principle, there is a way to perform a true analytic continuation in the two variables taking into account the contribution of the poles when crossing the axis (a whole series of them appear). This issue is under investigation and we cannot yet give a final result.

3.2.2. Barnes and related zeta functions in d dimensions

Using once more Hermite's formula, we obtain the following recurrence for the Barnes zeta function in d dimensions:

$$\zeta_B^{(d)}(s;a|\boldsymbol{r}) = \sum_{\boldsymbol{n}_d=\boldsymbol{0}}^{\infty} (a + \boldsymbol{r}_d \cdot \boldsymbol{n}_d)^{-s}, \quad \Re\, s > d \tag{20}$$

in terms of the corresponding ones in $d-1$ dimensions

$$\zeta_B^{(d)}(s;a|\boldsymbol{r}) = \frac{r_d^{-1}}{s-1}\zeta_B^{(d-1)}(s-1;a|\boldsymbol{r}_{d-1}) + \frac{1}{2}\zeta_B^{(d-1)}(s;a|\boldsymbol{r}_{d-1}) + r_d^{-1}\sum_{\boldsymbol{n}_{d-1}=\boldsymbol{0}}^{\infty}(a+\boldsymbol{r}_{d-1}\cdot\boldsymbol{n}_{d-1})^{1-s}\int_0^\infty \mathrm{d}x\frac{(1+x^2)^{-s/2}\sin(s\arctan x)}{\mathrm{e}^{2\pi r_d^{-1}(a+\boldsymbol{r}_{d-1}\cdot\boldsymbol{n}_{d-1})x}-1}. \tag{21}$$

Proceeding as before, we take the derivative at $s=0$, with the result

$$\zeta_B^{(d)\prime}(0;a|\boldsymbol{r}) = -r_d^{-1}\zeta_B^{(d-1)}(-1;a|\boldsymbol{r}_{d-1}) - r_d^{-1}\zeta_B^{(d-1)\prime}(-1;a|\boldsymbol{r}_{d-1}) + \frac{1}{2}\zeta_B^{(d-1)\prime}(0;a|\boldsymbol{r}_{d-1}) + \frac{1}{2\pi}\sum_{k=0}^{[d/2-1]}(-1)^k\left(\frac{r_d}{2\pi}\right)^{2k+1}(2k)!\,\zeta(2k+2)\ \mathrm{FP}[\zeta_B^{(d-1)}(2k+1;a|\boldsymbol{r}_{d-1})] + R_d(0;a|\boldsymbol{r}), \tag{22}$$

where FP[ ] means the finite part of, and for the remainder term we get the two different expressions:

$$R_d(0;a|\boldsymbol{r})\begin{cases} = r_d^{-1}\displaystyle\sum_{\boldsymbol{n}_{d-1}=\boldsymbol{0}}^{\infty}(a+\boldsymbol{r}_{d-1}\cdot\boldsymbol{n}_{d-1})\left[\int_0^\infty \mathrm{d}x\frac{\arctan x}{\mathrm{e}^{2\pi r_d^{-1}(a+\boldsymbol{r}_{d-1}\cdot\boldsymbol{n}_{d-1})x}-1}\right. \\ \quad \left. -\displaystyle\sum_{k=0}^{[d/2-1]}(-1)^k\left(\frac{r_d}{2\pi}\right)^{2k+2}(2k)!\,(a+\boldsymbol{r}_{d-1}\cdot\boldsymbol{n}_{d-1})^{-2(k+1)}\right], \\ \sim \dfrac{1}{2\pi}\displaystyle\sum_{k=[d/2]}^{\infty}(-1)^k\left(\frac{r_d}{2\pi}\right)^{2k+1}(2k)!\,\zeta(2k+2)\zeta_B^{(d-1)}(2k+1;a|\boldsymbol{r}_{d-1}). \end{cases}$$

In order to obtain $\zeta_B^{(d)\prime}(0;a|\boldsymbol{r})$ we need two derivatives of the $d-1$ zeta function, namely $\zeta_B^{(d-1)\prime}(0;a|\boldsymbol{r}_{d-1})$ and $\zeta_B^{(d-1)\prime}(-1;a|\boldsymbol{r}_{d-1})$. The derivative at $s=-1$ can be calculated in the same way, just by performing first the expansion of the Barnes zeta function around $s=-1$. For the case $d=2$ (needed in order to initiate the recurrence), the result is the following:

$$\zeta_B^{(2)\prime}(-1;a|\boldsymbol{r}) = -r_1\log r_1\ \zeta_B^{(2)}(-1;a|\boldsymbol{r}) + r_1\left[-\frac{r_1}{4r_2}\zeta_H(-2,a/r_1) - \frac{r_1}{2r_2}\zeta_H'(-2,a/r_1) + \frac{1}{2}\zeta_H'(-1,a/r_1)\frac{r_2}{12r_1}\zeta_H(0,a/r_1) - \frac{r_2}{12r_1}\zeta_H'(0,a/r_1)\right] + R_1(-1;a|\boldsymbol{r}),$$

the remainder $R_1(-1;a|\boldsymbol{r})$ being given by the asymptotic expansion

$$R_1(-1;a|\boldsymbol{r}) \sim \frac{r_1}{\pi}\sum_{k=1}^{\infty}(-1)^k(2k-1)!\zeta(2k+2)\left(\frac{r_2}{2\pi r_1}\right)^{2k+1}\zeta_H(2k,a/r_1),$$

or by the corresponding expression as the difference of an integral and the term $k=0$ of the series (as above). This allows for the calculation of any derivative of the Barnes zeta function in d dimensions recursively (with an accurate numerical determination of the remainder).

4. Examples

4.1. Dirac-like operator in one dimension

Consider the square root of the harmonic oscillator obtained by Delbourgo. This example has potentially some interesting physical applications, for it is well known that a fermion in an external constant electromagnetic field has a similar spectrum (Landau spectrum). Exactly in the same way as when going from the Klein–Gordon to the Dirac equation and at the same price of doubling the number of components (e.g., introducing spin), Delbourgo has constructed a model for which there exists a square root of its Hamiltonian, which is very close to the one for the harmonic oscillator. The main difference lies in the introduction of the parity operator, Q. Whereas creation and destruction operators for the harmonic oscillator, $a^{\pm}=P\pm \mathrm{i}X$, are nonhermitian, the combinations $D^{\pm}=P\pm \mathrm{i}QX$ are hermitian and $H^{\pm}\equiv (D^{\pm})^2=P^2+X^2\mp Q=2H_{\mathrm{osc}}\mp Q$. Note that Q commutes with H_{osc}. Doubling the components (σ_i are Pauli matrices): $P=-\mathrm{i}\sigma_1(\partial/\partial x)$, $X=\sigma_1 x$, $Q=\sigma_2$, while $D^{\pm}=-\mathrm{i}\sigma_1(\partial/\partial x)\pm\sigma_3 x$. They have for eigenfunctions and eigenvalues,

$$\psi_n^{\pm}(x)=\frac{-\mathrm{i}\mathrm{e}^{-x^2/2}}{\sqrt{2^{n+1}\,(n-1)!\sqrt{\pi}}}\begin{pmatrix}-\mathrm{i}[H_{n-1}(x)\pm H_n(x)/\sqrt{2n}]\\ [H_{n-1}(x)\mp H_n(x)/\sqrt{2n}]\end{pmatrix},$$
$$\lambda_n=\pm\sqrt{2n},\ n\geqslant 1,\quad \psi_0(x)=\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\sqrt{\pi}}}\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix},\quad \lambda_0=0 \tag{23}$$

with $H_n(x)$ Hermite polynomials. Let us call now $D\equiv D^+$.

The two operators for the calculation of the anomaly will be D and $D_V=D+V$, V being a real, constant potential with $\|V\|<\sqrt{2}$, that goes multiplied with the identity matrix in the two (spinorial) dimensions (implicit here). Notice that D and D_V are hermitian, commuting operators. The zeta function is $\zeta_D(s)=\sum_i \lambda_i^{-s}=\sum_{n=1}^{\infty}[1+(-1)^{-s}](\sqrt{2n})^{-s}=[1+(-1)^{-s}]2^{-s/2}\zeta_R(s/2)$, $\zeta_R(s)$ being the Riemann zeta function, which has a simple pole at $s=1$. The manifold is not compact, but we get

$$a(D,D_V)=\frac{V^2}{2}-\ln V, \tag{24}$$

in accordance with Wodzicki's formula. The logarithmic term is due to the presence of a zero mode. We have here the first and the most simple example of appearance of a nontrivial anomaly for operators of degree one in a space of dimension one (spinorial, however).

4.2. Harmonic oscillator in d dimensions

Let us consider the harmonic oscillator in d dimensions, with angular frequencies $(\omega_1,\dots,\omega_d)$. The eigenvalues read $\lambda_{\bar n}=\bar n\cdot\bar\omega+b$, $\bar n\equiv(n_1,\dots,n_d)$, $\bar\omega\equiv(\omega_1,\dots,\omega_d)$, $b=\frac{1}{2}\sum_{k=1}^{d}\omega_k$, and the related zeta function is the Barnes one, $\zeta_d(s,b|\bar\omega)$, whose poles are to be found at the points $s=k$ $(k=d,d-1,\dots,1)$. Their corresponding residua can be expressed in terms of generalized Bernoulli polynomials $B_{d-k}^{(d)}(b|\bar{\omega})$, defined by

$$\frac{t^d \mathrm{e}^{-at}}{\prod_{i=1}^d (1-\mathrm{e}^{-b_i t})} = \frac{1}{\prod_{i=1}^d b_i} \sum_{n=0}^{\infty} B_n^{(d)}(a|b_i) \frac{(-t)^n}{n!}.$$

The residua of the Barnes zeta function are

$$\operatorname{Res} \zeta_d(k, b|\bar{\omega}) = \frac{(-1)^{d+k}}{(k-1)!(d-k)! \prod_{j=1}^d \omega_j} B_{d-k}^{(d)}(b|\bar{\omega}), \quad k = d, d-1, \dots .$$

Now, being V is a constant potential, we obtain

$$a(H, H_V) = \frac{(-1)^d}{2\prod_{j=1}^d \omega_j} \sum_{k=1}^{[d/2]} \frac{[\gamma + \psi(d-2k)]\, B_{2k}^{(d)}(b|\bar{\omega})}{(2k)!\,(d-2k)!} V^{2k}. \tag{25}$$

Here the generalized Bernoulli polynomials of odd order vanish. In spite of the manifold being noncompact, we confirm again the validity of the Wodzicki formula. On the other hand, the remaining generalized Bernoulli polynomials are *never* zero, in fact,

$$B_0^{(d)}(b|\bar{\omega}) = 1, \qquad B_2^{(d)}(b|\bar{\omega}) = -\frac{1}{12} \sum_{i=1}^d \omega_i^2,$$

$$B_4^{(d)}(b|\bar{\omega}) = \frac{1}{24} \left[\frac{7}{10} \sum_{i=1}^d \omega_i^4 + \sum_{i<j} \omega_i^2 \omega_j^2 \right],$$

$$B_6^{(d)}(b|\bar{\omega}) = -\frac{5}{96} \left[\frac{31}{70} \sum_{i=1}^d \omega_i^6 + \frac{7}{10} \sum_{i \neq j} \omega_i^4 \omega_j^2 + \sum_{i<j<k} \omega_i^2 \omega_j^2 \omega_k^2 \right], \dots . \tag{26}$$

So, the anomaly does not vanish for d odd or $d = 2$, whatever the frequencies ω_i be, only even powers of the potential V appear.

4.3. Bosonic model at finite T with chemical potential

The physical implications of the issue for self-interacting theories at zero and finite temperature, eventually with a chemical potential, have been investigated recently. For lack of space, the reader is addressed again to the relevant Refs. [17,31,6,23]. Here we only sketch one example: the noninteracting bosonic model at finite temperature with a nonvanishing chemical potential. The partition function for a free charged field in $\mathbb{R}^d$, described by two real components ϕ_i with Euclidean action: $S = \int \mathrm{d}x^d [\phi_i(-\Delta_d + m^2)\phi_i]$, where $\phi^2 = \phi_k \phi_k$ is O(2) invariant, may be written as [11] $\ln Z_\beta(\mu) = -\frac{1}{2} \ln \det \|A_{ik}/M^2\| = -\frac{1}{2} \ln \det((L_+/M^2)(L_-/M^2))$, where $L_\pm = L_\tau + (\sqrt{L_N} \pm e\mu)^2$, with $L_\tau = -\partial_\tau^2$ (of discrete spectrum $\omega_n^2 = (4\pi^2 n^2/\beta^2)$) and $L_N = -\Delta_N + m^2$. But another possible factorization is $K_\pm = L_N + (\sqrt{L_\tau} \pm \mathrm{i}e\mu)^2$. Here we have $L_+ L_- = K_+ K_-$ and, in both cases, we deal with a couple of ΨDOs, L_+ and L_- being also formally self-adjoint. The partition function, chosen a factorization, say L_+L_-, is

$$\ln Z_\beta(L_+,L_-) = -\frac{1}{2}\ln\det\left\|\frac{L_{ik}}{M^2}\right\|$$
$$= -\frac{1}{2}\ln\det\left(\frac{L_+}{M^2}\frac{L_-}{M^2}\right)$$
$$= \frac{1}{2}\zeta'_{L_+}(0) + \frac{1}{2}\zeta'_{L_-}(0) + \frac{\ln M^2}{2}[\zeta_{L_+}(0) + \zeta_{L_-}(0)] - \frac{1}{2}a_d(L_+,L_-),$$

and similarly for K_+K_-. One can see the crucial role of the multiplicative anomaly: as, in the factorization, different operators may enter, the multiplicative anomaly is necessary in order to have the same regularized partition function in both cases, namely $\ln Z_\beta(L_+,L_-) = \ln Z_\beta(K_+,K_-)$, which is an obvious physical requirement (the existence of the anomaly is obvious since $\det K_+ \det K_- \neq \det L_+ \det L_-$). For the first factorization, we get

$$\zeta_{L_\pm}(s) = \frac{\beta}{2\sqrt{\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta_{(\sqrt{L_N}\pm e\mu)^2}(s-\tfrac{1}{2}) - 2s\sum_i \ln(1-\mathrm{e}^{-\beta(\sqrt{\lambda_i}\pm e\mu)}) + \mathrm{O}(s^2).$$

For d odd, taking the derivative with respect to s and the limit $s\to 0$, the two factorizations give

$$\ln Z_{\beta,\mu}(L_+,L_-) = \frac{\beta V_N m^d}{(4\pi)^{d/2}}\Gamma\left(-\frac{d}{2}\right) + S(\beta,\mu) - \frac{1}{2}a_d(L_+,L_-),$$

$$\ln Z_{\beta,\mu}(K_+,K_-) = \frac{\beta V_N m^d}{(4\pi)^{d/2}}\Gamma\left(-\frac{d}{2}\right) + S(\beta,\mu) - \frac{1}{2}a_d(K_+,K_-).$$

It should be noted that one gets $\ln Z_{\beta,\mu}(L_+,L_-) = \ln Z_{\beta,\mu}(K_+,K_-)$ and thus the standard textbook result as soon as one is able to prove that the two multiplicative anomalies are vanishing for d odd, which indeed happens.

For d even, the situation is different because, within the first factorization, the vacuum sector depends explicitly on the chemical potential μ. For $d=2$ the two factorizations give

$$\ln Z_{\beta,\mu}(L_+,L_-) = \frac{\beta V_1 m^2}{4\pi}\left(\ln\frac{m^2}{M^2} - 1\right) + S(\beta,\mu) + \frac{V_1\beta}{2\pi}e^2\mu^2 - \frac{1}{2}a_2(L_+,L_-),$$

$$\ln Z_{\beta,\mu}(K_+,K_-) = \frac{\beta V_1 m^2}{4\pi}\left(\ln\frac{m^2}{M^2} - 1\right) + S(\beta,\mu) - \frac{1}{2}a_2(K_+,K_-),$$

respectively, while for $d=4$ one has

$$\ln Z_{\beta,\mu}(L_+,L_-) = \frac{\beta V_3}{32\pi^2}m^4\left(\ln\frac{m^2}{M^2} - 3/2\right) + \frac{\beta V_3}{8\pi^2}\left(\frac{e^4\mu^4}{3} - e^2\mu^2 m^2\right)$$
$$+ S(\beta,\mu) - \frac{1}{2}a_4(L_+,L_-),$$

$$\ln Z_{\beta,\mu}(K_+,K_-) = \frac{\beta V_3}{32\pi^2}m^4\left(\ln\frac{m^2}{M^2} - 3/2\right) + S(\beta,\mu) - \frac{1}{2}a_4(K_+,K_-).$$

In general, for d even: $\ln Z_{\beta,\mu}(L_+,L_-) = -\beta V_N \mathscr{E}_V(m,M) + \beta V_N \mathscr{E}_d(m,\mu) + S(\beta,\mu) - \frac{1}{2}a_d(L_+,L_-)$, and $\ln Z_{\beta,\mu}(K_+,K_-) = -\beta V_N \mathscr{E}_V(m,M) + S(\beta,\mu) - \frac{1}{2}a_d(K_+,K_-)$, where $\mathscr{E}_V(m,M)$ is the naive vacuum energy density, $d=2Q$ and $\mathscr{E}_d(m,\mu) = \sum_{r=1}^{Q} c_{Q,r}(e\mu)^{2r} m^{d-2r}$, what reflects the presence of the anomaly terms. Here the $c_{Q,r}$ are computable coefficients, given by:

$$c_{Q,r} = \frac{\Gamma(2r-1)}{\Gamma(2r+1)\pi^{N/2}} \frac{2^{1-N}}{\Gamma(r-\frac{1}{2})} \frac{(-1)^{Q-r}}{(Q-r)!}.$$

In the cases $d = 2, 4$ and 6 one obtains

$$\mathscr{E}_2(m,\mu) = \frac{e^2\mu^2}{2\pi}, \qquad \mathscr{E}_4(m,\mu) = -\frac{e^2\mu^2}{8\pi^2}\left(m^2 - \frac{e^2\mu^2}{3}\right),$$
$$\mathscr{E}_6(m,\mu) = -\frac{e^2\mu^2}{16\pi^3}\left(-\frac{1}{4}m^4 + \frac{e^2\mu^2 m^2}{6} - \frac{2e^4\mu^4}{45}\right). \tag{27}$$

It could be argued that the presence of a multiplicative anomaly is strictly linked with the zeta-function regularization employed and, thus, that it might be an artifact of it. But it has been proved in [17,31,6] that the multiplicative anomaly is present indeed in a large class of regularizations of functional determinants appearing in the one-loop effective action.

4.4. Other physical applications

4.4.1. Calculation of the trace anomaly

In a paper by Bousso and Hawking [5], where the *trace anomaly* of a dilaton coupled scalar in two dimensions is calculated, the zeta function method is employed for obtaining the one-loop effective action, W, which is given by the well-known expression $W = \frac{1}{2}[\zeta_A(0)\ln\mu^2 + \zeta_A'(0)]$. In conformal field theory and in a Euclidean background manifold of toroidal topology, the eigenvalues of A are found perturbatively (see [5]), what leads one to consider the following zeta function: $\zeta_A(s) = \sum_{k,l=-\infty}^{\infty}(\Lambda_{kl})^{-s}$, with the eigenvalues Λ_{kl} being given by $\Lambda_{kl} = k^2 + l^2 + (\varepsilon^2/2) + (\varepsilon^2/2(4l^2-1))$, where ε is a perturbation parameter. It can be shown that the integral of the trace anomaly is given by the value of the zeta function at $s=0$. One barely needs to follow the several pages long discussion in [5], leading to the calculation of this value, in order to appreciate the power of the formulae of the preceding section. In fact, to begin with, no mass term needs to be introduced to arrive at the result and no limit mass $\to 0$ needs to be taken later. Using binomial expansion (the same as in Ref. [5]), one gets

$$\zeta(s) = \sum_{k,l=-\infty}^{\infty}\left(k^2 + l^2 + \frac{\varepsilon^2}{2}\right)^{-s} - \frac{\varepsilon^2 s}{2}\sum_{k,l=-\infty}^{\infty}\left(k^2 + l^2 + \frac{\varepsilon^2}{2}\right)^{-1-s}(4l^2-1)^{-1}.$$

From Eq. (11) above, the first zeta function gives, at $s=0$, *exactly*: $-\pi\varepsilon^2/2$. And this is the whole result (which coincides with the one obtained in [5]), since the second term has no pole at $s=0$ and provides no contribution.

4.4.2. Calculation of the Casimir energy density

Another direct application is the calculation of the *Casimir energy density* corresponding to a massive scalar field on a general, d-dimensional toroidal manifold (see [29]). In the spacetime

$\mathscr{M}=\mathbb{R}\times\Sigma$, with $\Sigma=[0,1]^d/\sim$, which is topologically equivalent to the d torus, the Casimir energy density for a massive scalar field is given directly by Eq. (7) at $s=-\frac{1}{2}$, with $q=m^2$ (mass of the field), $\boldsymbol{b}=\boldsymbol{0}$, and A being the matrix of the metric g on Σ, the general d-torus: $E^C_{\mathscr{M},m}=\zeta_{g,\boldsymbol{0},m^2}\ (s=-\frac{1}{2})$. The components of g are, in fact, the coefficients of the different terms of the Laplacian, which is the relevant operator in the Klein–Gordon field equation. The massless case is also obtained, with the same specifications, from the corresponding formula Eq. (10). In both cases no extra calculation needs to be done, and the physical results follow from a mere *identification* of the components of the matrix A with those of the metric tensor of the manifold in question [24]. Very much related with this application but more involved and ambitious is the calculation of vacuum energy densities corresponding to spherical configurations and the bag model (see [4,36,37] and references therein).

4.4.3. Calculation of the effective action

A third application consists in calculating the *determinant* of a differential operator, say the Laplacian on a general p-dimensional torus. A very important problem related with this issue is that of the multiplicative anomaly discussed before [23,20]. To this end the derivative of the zeta function at $s=0$ has to be obtained. From Eq. (7), we get

$$\zeta'_{A,\boldsymbol{c},q}(0)=\frac{4(2q)^{p/4}}{\sqrt{\det A}}\sum_{\boldsymbol{m}\in\mathbb{Z}^p_{1/2}}{}' \frac{\cos(2\pi\boldsymbol{m}\cdot\boldsymbol{c})}{(\boldsymbol{m}^{\mathrm{T}}A^{-1}\boldsymbol{m})^{p/4}}K_{p/2}(2\pi\sqrt{2q\boldsymbol{m}^{\mathrm{T}}A^{-1}\boldsymbol{m}})$$
$$+\begin{cases}\dfrac{(2\pi)^{p/2}\Gamma(-p/2)q^{p/2}}{\sqrt{\det A}}, & p \text{ odd},\\[2ex] \dfrac{(-1)^k(2\pi)^k q^k}{k!\sqrt{\det A}}[\Psi(k+1)+\gamma-\ln q], & p=2k \text{ even},\end{cases}$$

and, from here, $\det A=\exp-\zeta'_A(0)$. For $p=2$, we have explicitly

$$\det A(a,b,c;q)=\mathrm{e}^{2\pi(q-\ln q)/\sqrt{\Delta}}(1-e^{-2\pi\sqrt{q/a}})\exp\left\{-4\sum_{n=1}^{\infty}\frac{1}{n}\left[\sqrt{\frac{a}{q}}K_1\left(4\pi n\sqrt{\frac{aq}{\Delta}}\right)\right.\right.$$
$$\left.\left.+\cos(\pi nb/a)\sum_{d|n}d\exp\left(-\frac{\pi n}{a}\sqrt{\Delta+\frac{4aq}{d^2}}\right)\right]\right\}.$$

In the homogeneous case (CS formula) we obtain for the determinant

$$\det A(a,b,c)=\frac{1}{a}\exp\left[-4\zeta'(0)-\frac{\pi\sqrt{\Delta}}{6a}-4\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}\cos(\pi nb/a)\mathrm{e}^{-\pi n\sqrt{\Delta}/a}\right],$$

or, in terms of the Teichmüller coefficients, τ_1 and τ_2, of the metric tensor (for the metric, A, corresponding to the general torus in two dimensions):

$$\det A(\tau_1,\tau_2)=\frac{\tau_2}{4\pi^2|\tau|^2}\exp\left[-4\zeta'(0)-\frac{\pi\tau_2}{3|\tau|^2}-4\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n}\cos\left(\frac{2\pi n\tau_1}{|\tau|^2}\right)\mathrm{e}^{-\pi n\tau_2/|\tau|^2}\right].$$

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