Theorems from G. Polyá

G. Polyá obtained the following general theorem about zeros of the Fourier transform of a real function:

Theorem 1: Let $0 \le a < b \le \infty$ and let $g(x)$ be a strictly positive continuous function on (a,b) and differentiable there, except possible at finitely many points. Suppose that

$$\alpha \le -x\frac{g'(x)}{g(x)} \le \beta$$

at every point of (a,b) where $g(x)$ is differentiable. Suppose further that the integral

$$G(s) := \int_0^\infty x^s g(x)\frac{dx}{x}$$

is convergent for $a^* < \mathrm{Re}(s) < \beta^*$. Then all zeros ρ of $G(s)$ in this stripe satisfy $\alpha \le \mathrm{Re}(\rho) \le \beta$.

Let $$g(s) := \frac{\int_1^\infty x^{-s} d\mu}{\int_1^\infty x^{-s} dx}$$.

Theorem 2 (G. Polya): If φ is a polynomial which has all its roots on the imaginary axis, or if φ is an entire function which can be written in a suitable way as limit of such polynomials, then

If (*) $\int_0^\infty u^{1-s}F(u)\frac{du}{u}$ has all its zeros on the critical line, so does $\int_0^\infty u^{1-s}F(u)\varphi(\log u)\frac{du}{u}$.

Modern version: An operator which takes an even function $q(\nu)$ and replaces it by $\frac{q(\nu+1)-q(\nu-1)}{\nu}$ has the property of moving the zeros of a function closer on the imaginary axis, and so an eigenfunction of this operator should have its zeros on the imaginary axis.

Theorem 3 (G. Polya): If $\phi_m(x)$ is a polynomial which has all its roots on the imaginary axis, or if it is an entire function which can be written in a suitable way as a limit of such polynomials, then

if $\int_0^\infty x^{-s}F(x)dx$ has all its zeros on the critical axis, so does

$$\int_0^\infty x^{-s}F(x)\phi(\log x)dx = \int_{-\infty}^\infty e^{-ys}F(e^y)\phi(y)dy \cdot$$

The Müntz Formula

Theorem (**Müntz' formula**): For $\omega(x), \omega'(x)$ continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ and $\omega(x) = o(x^{-\beta})$ for $x \to \infty$ and $\alpha, \beta > 1$ it holds

$$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s \left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0 < \mathrm{Re}(s) < 1.$$

Proof: because $\omega(x)$ is continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ it holds

$$\sum_1^\infty \frac{1}{n^s}\left|\int_0^\infty x^{s-1}\omega(x)\right|dx \quad \text{exists for} \quad 1 < \sigma < \alpha \ ,$$

i.e. the inversion leading to the left hand side of (4.3) is justified.

ii) $$\sum_1^\infty \omega(nx) - \int_0^\infty \omega(xt)dt = x\int_0^\infty \omega'(t)(t - [t])dt = x\int_0^{1/x} O(1)dt + x\int_{1/x}^\infty O((xt)^{-\beta})dt = O(1)$$

The first summand is justified, because $\omega(x)$ is continuous and bounded in any finite interval the second summand is justified, because $\omega(x) = o(x^{-\alpha})$, i.e. it holds

$$\sum_1^\infty \omega(nx) = O(1) + \frac{c}{x} \quad \text{with} \quad c := \int_0^\infty \omega(t)dt .$$

Hence

$$\int_0^\infty x^s \sum_1^\infty \omega(nx) + \frac{dx}{x} = \int_0^1 x^s \left[\sum_1^\infty \omega(nx) - \frac{c}{x}\right]\frac{dx}{x} + \int_1^\infty x^s \sum_1^\infty \omega(nx)\frac{dx}{x} + \frac{c}{s-1}$$

for $\sigma > 0$ except $s = 1$. Also

$$-c\int_1^\infty x^{s-2}dx = \frac{c}{s-1} \qquad \text{for } \sigma < 1$$

and therefore the result for $0 < \sigma = \mathrm{Re}(s) < 1$ •

Ikehara's Theorem

If the measure $d\mu$ is positive and the function $g(s)$ fulfills

i) $g(s)$ is properly defined for $\mathrm{Re}(s) > 0$

ii) $\lim g(s)$ exists for $s \to 1^+$ and is written as $g(1)$

iii) $\frac{g(s)-g(1)}{(s-1)}$ has a continuous extension from the open halfplane $\mathrm{Re}(s) > 1$, (whereby it is necessarily defined and analytical) to the closed halfplane $\mathrm{Re}(s) \geq 1$,

then

If $\lim \frac{\int_1^\infty x^{-s} d\mu}{\int_1^\infty x^{-s} dx} = 1$ for $s \to 1^+$ then $\lim \frac{\int_0^A d\mu}{\int_o^A dx} = 1$ for $A \to \infty$

i.e. roughly speaking $d\mu \approx dx$ in the sense above. The function

$$g(s) := (s-1)\left[-\frac{\zeta'(s)}{\zeta(s)}\right]$$

gives the prime number theorem. The Siegel formula (see below) might give the link to the Stieltjes density above:

$$g(s) := (s-1)(\zeta(s) \approx 1 + (s-1)/2 + ... \quad .$$

From [LGa] we recall the two versions of Ikehara theorem:

Lemma (Ikehara version 1): Let μ be a monotone nondecreasing function on $(0,\infty)$ and let

$$F(s) = \int_1^\infty x^{1-s} \frac{d\mu(x)}{x} \quad .$$

If the integral converges absolutely for $\mathrm{Re}(s) > 1$ and there is a constant A such that

$$F(s) - \frac{A}{s-1}$$

extents to a continuous function in $\mathrm{Re}(s) \geq 1$ then

$$\mu(x) \approx Ax \quad .$$

Lemma (Ikehara version 2): Let the Dirichlets series

$$F(s)=\sum_{1}^{\infty}\frac{c_n}{n^s}$$

be convergent for $\mathrm{Re}(s)>1$. If there exists a constant A such that

$$F(s)-\frac{A}{s-1}$$

admits a continuous extension to the line $\mathrm{Re}(s)\geq 1$, then

$$\sum_{1}^{N} c_n \approx A*N \quad \text{as} \quad N\to\infty \; .$$

Wiener's Tauberian theorem

The closed linear hull of the translates of a function

$$f(x)\in L_1(R)$$

is the whole space $L_1(R)$ if and only if its Fourier transform

$$\hat{f}(x):=\int_R e^{-itx} f(t)dt$$

never vanishes. Note that the close linear hull in question contains all convolutions

$$f*g(x):=\int_R f(x-y)g(y)dy \; .$$

Ramanujan's Master Theorem

Ramanujan's Master Theorem: In the neighborhood of $x = 0$ for

$$F(x) = \sum_{0}^{\infty} \frac{\varphi(k)}{k!}(-x)^k$$

the following representation holds true

$$\int_0^\infty F(x)x^{s-1}dx = \Gamma(s)\varphi(-s) \cdot$$

Ramanujan motivated his formula with the following wordings ([1] B. C. Berndt, chapter 4, Entry 8):

"*Statement: If two functions of* x *be equal, then a general theorem can be formed by simply writing* $\varphi(n)$ *instead of* x^n *in the original theorem*

Solution: "Put $x=1$ *and multiply it by* $f(0)$ *then change* x *to* $x, x^2, x^3, x^4....$ *and multiply* $\frac{f'(0)}{1!}, \frac{f''(0)}{2!}, \frac{f'''(0)}{3!},...$ *respectively and add up all the results. Then instead of* x^n *we have* $f(x^n)$ *for positive as well as for negative values of* n *. Changing* $f(x^n)$ *to* $\varphi(n)$ *we can get the result*."

Example:
$$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$$

Ramanujan's building process:

$$f(0)[\arctan 1 + \arctan 1] = \frac{\pi}{2} f(0) \ ,$$

$$\frac{f'(0)}{1!}\left[\arctan x + \arctan \frac{1}{x}\right] = \frac{f'(0)}{1!}\frac{\pi}{2} \ ,$$

$$\frac{f''(0)}{2!}\left[\arctan x + \arctan \frac{1}{x}\right] = \frac{f''(0)}{2!}\frac{\pi}{2}$$

...

Replace $\arctan z$ *by its Maclaurin series in* z *, where* z *is any integral power of* x *. Now add all the equalities above. On the left side one obtains two double series. Invert the order of summation in each double series to find that*

$$\sum_0^\infty (-1)^n \frac{f(x^{2n+1}) + f(x^{-2n-1})}{2n+1} = \frac{\pi}{2} f(1) \ .$$

Replace $f(x^n)$ *by* $\varphi(n)$ *to conclude that*

$$\sum_0^\infty (-1)^n \frac{\varphi(2n+1) + \varphi(-2n-1)}{2n+1} = \frac{\pi}{2} \varphi(0) \ .$$

Of course, this formal procedure is fraught with numerous difficulties, but the theorem was finally correctly proved by G.H. Hardy.

The link to differential form is given by the Pfaff form $\omega = -\mu_y(x)dx + \mu_x(y)dy$.

Let

$$U := R^2 - \{(0,0)\} \quad \text{and} \quad V := R^2 - \{(x,0) | x \le 0\}$$

In case of the (non-star formed) domain U there is no “integral” for the differential, but this is the case for the domain V . In this case the “integral” of ω is related to one of the “core” functions used by Ramanujan $\arctan(y(x)$, which is

$$F(x,y) = \int_{(1,0)}^{(x,y)} \omega = \varphi = \begin{cases} \arctan \frac{y}{x} & x > 0 \\ \frac{\pi}{2} - \arctan \frac{y}{x} & y > 0 \\ -\frac{\pi}{2} - \arctan \frac{y}{x} & y < 0 \end{cases}$$

It holds

$$dF(x,y) = 0 \quad , \quad \frac{\partial}{\partial y}(-\mu_y(x)) = \frac{\partial}{\partial x}(-\mu_x(y)) \ .$$

Remark Putting

$$\varphi(k) := \frac{1}{\zeta(2k+1)}$$

the Hardy/Littlewood resp. the Riesz equivalence criteria of the Riemann Hypothesis are

(HL) RH holds if and only if $F(x) = \sum_0^\infty \frac{\varphi(k)}{k!}(-x)^k = O(x^{-1/4})$.

(R) RH holds if and only if $\sum_1^\infty \frac{(-1)^{k+1}}{(k-1)!\zeta(2k)} x^k = O(x^{1/4+\varepsilon})$.

Bagchi's Nyman criterion formulation

Let H denote the weighted l^2 – space consisting of all sequences $a=\{a_n | n \in N\}$ of complex numbers such that

$$\sum_{n=1}^{\infty} \omega_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \le \omega_n \le \frac{c_2}{n^2} \; .$$

Let $\gamma := \{1,1,1,1,......\}$, $\gamma_k := \left\{ \rho(\frac{n}{k}) \middle| n = 1,2,3,.... \right\} \in H$ for $k = 1,2,3,...$

and Γ_k be the closed linear span of γ_k . Then the Nyman criterion states

$$\text{The Riemann Hypothesis is true} \quad \Leftrightarrow \quad \gamma \in \overline{\Gamma}_k \; .$$

Integral Equation from Theodorsen

For a complex valued function 2π – periodic function $f(\varphi) = u(\varphi) + iv(\varphi)$ its conjugated function can be represented by ([DGa], 1.1, 1.2)

$$\bar{f}(\varphi) = -\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{\varepsilon,\pi} f(\varphi+\vartheta) - f(\varphi-\vartheta) \cot\frac{\vartheta}{2} d\vartheta = \frac{1}{2\pi i} \oint_{0,2\pi} f(\vartheta) \cot\frac{\varphi-\vartheta}{2} d\vartheta \; .$$

Let $a_0; a_n, b_n$ be the Fourier coefficients of f . Then $0; -b_n, a_n$ are the Fourier coefficients of its conjugate and it holds

$$\frac{1}{\pi} \int_0^{2\pi} f^2(\varphi) d\varphi = \frac{a_0^2}{2} + \frac{1}{\pi} \int_0^{2\pi} \bar{f}^2(\varphi) d\varphi \quad \text{resp.} \quad \frac{1}{\pi} \int_0^{2\pi} \bar{f}^2(\varphi) d\varphi = \sum_1^n a_n^2 + b_n^2 \; .$$

Conformal mapping problem: Let C be a star-shaped Jordan curve of the w – plane with respect to $w = 0$, let $\rho = \rho(\theta)$ its representation by polar coordinates. Let D denote the inner region of C and $w = f(z)$ the conformal mapping function with domain $|z| < 1$ onto D, which is normalized by $f(0) = 0$ and $f'(0) > 0$. Then the unknown "boundary function" $\rho = \rho(\vartheta)$ is the solution of the non-linear, singular integral equation from Theodorsen:

$$\theta(\varphi) = \varphi + \frac{1}{2\pi i} \oint_{0,2\pi} \log \rho(\theta(\vartheta)) \cot\frac{\varphi-\vartheta}{2} d\vartheta \; .$$

This results in the following representation of the conformal mapping function

$$f(z) = z * \exp\left[\frac{1}{2\pi i} \oint_{0,2\pi} \log \rho(\theta(\vartheta)) \frac{e^{i\vartheta}+z}{e^{i\vartheta}-z} d\vartheta \right] \qquad |z| < 1 \; .$$

Theorem of Frullani

Lemma (***Theorem of Frullani***) Let $f(x)$ be a continuous, integrable function over any interval $0 \le A \le x \le B < \infty$. Then, for $0 < b < a$,

$$\int_0^\infty [f(ax) - f(bx)]\frac{dx}{x} = [f(\infty) - F(0)]\log\frac{a}{b}$$

where $f(0) = \lim f(x)$ for $x \to 0^+$ and $f(\infty) = \lim f(x)$ for $x \to \infty$.

We mention a generalization of this lemma, due to Hardy (Quart. J. Math. 33 (1902) p. 113-144) in the form

$$\int_0^\infty [\varphi(ax^m) - \psi(bx^n)](\log x)^p \frac{dx}{x} \cdot$$

The Hardy Theorem

The Gauss-Weierstrass density function

$$\omega_1(x) := f_\alpha(x) := e^{-\pi\alpha x^2} \quad \text{with} \quad \alpha := 1$$

gives the Jacobi's $\vartheta-$relation ([HEd] 1.6ff.)

$$\vartheta(x^2) := G(x) := \sum_{-\infty}^{\infty} f(nx) = G(1/x)/x =: 1 + 2\psi(x^2) \cdot$$

A modified integral operator representation ([HEd] 11.1) in the form

$$\frac{2\xi(s)}{s(s-1)} = \int_0^\infty x^{1-s}\left[G(x) - 1 - \frac{1}{x}\right]\frac{dx}{x}$$

is used to prove the ***Hardy theorem*** ([HEd] 11.1), i.e. that *there are infinitely many roots of* $\xi(s)=0$ *on the line* $\mathrm{Re}(s)=1/2$. If the integral operator would be self-adjoint all zeros have to be on the critical line.

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