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Existence and Uniqueness of Nonnegative Eigenfunctions of the Boltzmann Operator*

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1. INTRODUCTION

One of the basic problems of neutron transport theory requires to determine the time-evolution of a given initial neutron distribution in a bounded sourceless medium. The problem is described by the transport equation

$$\frac{\partial N}{\partial t} = -\mathbf{v}\operatorname{grad} N - v\Sigma(\mathbf{r}, \mathbf{v})\, N + \int v'\Sigma_s(\mathbf{r}, \mathbf{v}' \to \mathbf{v})\, N(\mathbf{r}, \mathbf{v}', t)\, d\omega'. \qquad (1)$$

Here $\mathbf{r}$ is the position and $\mathbf{v}$ the velocity vector; $N(\mathbf{r}, \mathbf{v}, t)$ represents the neutron density per unit velocity space at the time t; $\Sigma_s(\mathbf{r}, \mathbf{v}' \to \mathbf{v})$ is the differential macroscopic scattering cross section, and $\Sigma(\mathbf{r}, \mathbf{v})$ the total macroscopic scattering cross section. The integral on the right extends over all velocity space ω. We assume that the medium where the diffusion process occurs is a finite convex body V with no neutrons coming from outside. Therefore, the distribution function $N(\mathbf{r}, \mathbf{v}, t)$ satisfies also the boundary condition: $N(\mathbf{r}, \mathbf{v}, t) = 0$ for $\mathbf{r}$ on the surface of V and for ingoing $\mathbf{v}$.

Let us denote by A the linear operator defined by the integro-differential expression on the right of (1) and by the same boundary condition. The exact domain of definition of A is dependent on the space of functions where A acts, and will be described in Section 2. A is sometimes called the Boltzmann operator.

The initial-value problem requires to find a solution $N(\mathbf{r}, \mathbf{v}, t)$ of (1) satisfying the prescribed boundary condition and such that for $t \to 0$ it approaches a given initial distribution $N_0(\mathbf{r}, \mathbf{v})$. For the proof of existence and uniqueness of such a solution see [1], [2], [3], [4].

A solution of (1) exponentially dependent on t, i.e., of the form $N(\mathbf{r}, \mathbf{v}, t) = N(\mathbf{r}, \mathbf{v})\, e^{\lambda t}$, is called a decay mode. Evidently such a solution exists if and only if the decay constant λ is an eigenvalue of the Boltzmann operator A. If the corresponding eigenfunction $N(\mathbf{r}, \mathbf{v})$ is nonnegative, then

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the distribution function $N(\mathbf{r}, \mathbf{v})\, e^{\lambda t}$ has a physical significance and is called a fundamental mode of decay. It does not always exist.

Since the Laplace transform technique is often useful for solving the transport equation (1), it is important to know the resolvent of the operator A and also its spectrum $\sigma(A)$. Many papers have been devoted to the study of this spectrum. Lehner and Wing, Bednarz, Albertoni and Montagnini, Mika and other authors ([3], [5], [6], [7], [8]) have proved the following facts under some restrictive assumptions, in particular iosotropic scattering: Put $\lambda^* = \min[v\Sigma(\mathbf{r}, \mathbf{v})]$. Then (i) the whole half-plane $\operatorname{Re}(\lambda) \leqslant -\lambda^*$ belongs to the spectrum $\sigma(A)$, (ii) there exists a number c such that the half-plane $\operatorname{Re}(\lambda) > c$ lies in the resolvent set of A, and (iii) the points of the strip $-\lambda^* < \operatorname{Re}(\lambda) \leqslant c$ also belong to the resolvent set $\rho(A)$ with the exception of a set of isolated eigenvalues λ_k. This set may also be empty if the body V is sufficiently small. In all cases considered so far the decay constants λ_k are finite in number and real.

The main purpose of this paper is to prove that there exists a fundamental mode of decay and is unique up to a constant factor if the set of the decay constants λ_k is not empty. The existence of a nonnegative eigenfunction of A was shown for several cases by computations.

The spectrum of A may depend upon the linear space X where the Boltzmann operator A acts. Denote by ω the velocity space. Since $N(\mathbf{r}, \mathbf{v}, t)$ means the neutron density per unit velocity space, and since the total number of neutrons at a given instant is finite, it would be natural to take for X the Banach space of all bounded measures in the six-dimensional space $V \times \omega$. But the space of measures is not convenient for our considerations since it is not reflexive and since the set of continuous functions is not dense in it. Hence we shall take for X the Banach space L_p of all measurable functions $\psi(\mathbf{r}, \mathbf{v})$ whose pth power is integrable over the space $V \times \omega$. Here p can be any finite number $\geqslant 1$. Sometimes it is also convenient to introduce a suitably chosen weight function $\rho(\mathbf{v})$, so that the norm of a function $\psi(\mathbf{r}, \mathbf{v}) \in X$ is now defined by

$$\| \psi \|^p = \int_V \int_\omega | \psi(\mathbf{r}, \mathbf{v}) |^p \rho(\mathbf{v})\, dV\, d\omega.$$

For $p = 2$ we have a Hilbert space where the inner product is defined with the weight function $\rho(\mathbf{v})$. Since the space L_1 is not reflexive, we must suppose $p > 1$ in order to prove the uniqueness of the nonnegative eigenfunction of A.

Let us denote the product $v'\Sigma_s(\mathbf{r}, \mathbf{v}' \to \mathbf{v})$ by $K(\mathbf{r}, \mathbf{v}, \mathbf{v}')$ and $v\Sigma(\mathbf{r}, \mathbf{v})$ by $k(\mathbf{r}, \mathbf{v})$. We shall assume that $K(\mathbf{r}, \mathbf{v}, \mathbf{v}')$ is a measurable function and such that the integral operator

$$K\psi = \int_\omega K(\mathbf{r}, \mathbf{v}, \mathbf{v}')\, \psi(\mathbf{r}, \mathbf{v}')\, d\omega' \tag{2}$$

is bounded. (This seems to hold for nonmultiplying systems in thermal equilibrium if we take the space L_2 with the weight function $[M(v)]^{-1}$, where $M(v)$ is the Maxwellian distribution.) Next we shall assume that the measurable function $k(\mathbf{r}, \mathbf{v})$ is such that the product $k(\mathbf{r}, \mathbf{v})\, \psi(\mathbf{r}, \mathbf{v})$ belongs to the space X for all continuous functions $\psi(\mathbf{r}, \mathbf{v})$ with a compact support, and such that $k(\mathbf{r} - t\mathbf{v}, \mathbf{v})$ is a piecewise continuous function of the real variable t where it is defined. Finally we shall suppose that almost everywhere $K(\mathbf{r}, \mathbf{v}, \mathbf{v}') > 0$ and $k(\mathbf{r}, \mathbf{v}) > 0$.

2. The Boltzmann Operator as the Infinitesimal Generator of a Semigroup

The operator A can be written as the sum $A = T + K$, where K is the integral operator (2) and

$$T\psi = -\mathbf{v} \operatorname{grad} \psi - k(\mathbf{r}, \mathbf{v})\, \psi,$$

the boundary conditions for T being the same as for A. We shall show that the domain of definition of T can be chosen in such a manner that T becomes the infinitesimal generator of a strongly continuous semigroup of bounded operators.

Take any $\psi(\mathbf{r}, \mathbf{v}) \in X$ and consider $\psi(\mathbf{r} - t\mathbf{v}, \mathbf{v}) = f(t)$ as a function of t. Put also $f(t) = 0$ if $t > 0$ and $\mathbf{r} - t\mathbf{v} \notin V$, so that $f(t)$ is defined in an interval (a, ∞) containing the point $t = 0$. Now, $-\mathbf{v} \operatorname{grad} \psi$ is the derivative of $f(t)$ at the point $t = 0$, if this derivative exists, hence $-\mathbf{v} \operatorname{grad} \psi = f'(0)$. Denote by D_0 the set of functions $\psi(\mathbf{r}, \mathbf{v}) \in X$ satisfying the conditions: (i) the corresponding function $f(t)$ is continuous and piecewise continuously differentiable for all $\mathbf{r}$, $\mathbf{v}$, and (ii) the expression $-\mathbf{v} \operatorname{grad} \psi - k(\mathbf{r}, \mathbf{v})\, \psi$ is a function of the space X. The set D_0 is evidently a linear manifold dense in X, and each $\psi(\mathbf{r}, \mathbf{v}) \in D_0$ satisfies the prescribed boundary condition of T. We shall call T_0 the operator defined on D_0 and such that

$$T_0\psi = -\mathbf{v} \operatorname{grad} \psi - k(\mathbf{r}, \mathbf{v})\, \psi \qquad \text{for} \qquad \psi \in D_0 .$$

It will be proved that T_0 is closable. We shall define T as the closure of T_0, hence $T = \bar{T}_0$.

Consider the family of operators $G(t)$, $t \geqslant 0$, defined as follows:

$$G(t)\, \psi = \psi(\mathbf{r} - t\mathbf{v}, \mathbf{v}) \exp\left[-\int_0^t k(\mathbf{r} - \tau\mathbf{v}, \mathbf{v})\, d\tau\right], \tag{3}$$

where on the right $\psi(\mathbf{r} - t\mathbf{v}, \mathbf{v})$ is our function $f(t)$, which is zero if $t > 0$ and $\mathbf{r} - t\mathbf{v} \notin V$. Since the body V is by assumption convex, $\mathbf{r} \in V$ and

$\mathbf{r} - t\mathbf{v} \in V$ imply $\mathbf{r} - \tau\mathbf{v} \in V$ for $0 \leqslant \tau \leqslant t$. Thus the expression on the right of (3) is well defined.

Since

$$\min k(\mathbf{r}, \mathbf{v}) = \min[v\Sigma(|\,\mathbf{r}\,|, \mathbf{v})] = \lambda^* \geqslant 0,$$

$$\exp\left(-\int_0^t k(\mathbf{r} - \tau\mathbf{v}, \mathbf{v})\, d\tau\right) \leqslant \exp(-\lambda^* t)$$

if $\mathbf{r}, \mathbf{r} - t\mathbf{v} \in V$. Hence, we have $\| G(t)\,\psi \| \leqslant e^{-\lambda^* t} \| \psi \|$. Therefore, the operators $G(t)$ are bounded and

$$\| G(t) \| \leqslant e^{-\lambda^* t}, \qquad t \geqslant 0. \tag{4}$$

It is not difficult to verify that the family $[G(t)]$ is a semigroup, i.e., $G(t + s) = G(t)\,G(s)$ for all t, $s \geqslant 0$. If the function $\psi(\mathbf{r}, \mathbf{v})$ is continuous with a compact support, then evidently $\lim_{t\to+0} \| G(t)\,\psi - \psi \| = 0$. Since the continuous functions with a compact support are dense in the Banach space $X = L_p$ and since the family of operators $G(t)$ is uniformly bounded, we conclude that the semigroup $[G(t)]$ is strongly continuous. Its infinitesimal generator is therefore a densely defined and closed operator.

Let $\psi(\mathbf{r}, \mathbf{v})$ be any function of X such that $\psi(\mathbf{r} - t\mathbf{v}, \mathbf{v})$ is continuously dependent on t for $t \geqslant 0$. Put $\varphi(\mathbf{r}, \mathbf{v}) = \int_0^h G(s)\,\psi(\mathbf{r}, \mathbf{v})\, ds$ where $h > 0$ is arbitrary. It is well known ([9], p. 620) that $\varphi(\mathbf{r}, \mathbf{v})$ lies in the domain of definition of the infinitesimal generator of $[G(t)]$. Let us denote by T_1 the generator restricted to the set of functions $\varphi(\mathbf{r}, \mathbf{v})$ constructed in this way. Then ([9], p. 620)

$$T_1\varphi = G(h)\,\psi - \psi.$$

The infinitesimal generator of $[G(t)]$ is $\bar{T}_1$, the closure of T_1.

After a short calculation we obtain

$$\varphi(\mathbf{r} - t\mathbf{v}, \mathbf{v}) = \int_t^{t+h} \psi(\mathbf{r} - s\mathbf{v}, \mathbf{v}) \exp\left(-\int_t^s k(\mathbf{r} - \tau\mathbf{v}, \mathbf{v})\, d\tau\right) ds.$$

Here we must replace $\psi(\mathbf{r} - s\mathbf{v}, \mathbf{v})$ by 0 if $s > 0$ and $\mathbf{r} - s\mathbf{v} \notin V$. It is immediately seen from this expression that $\varphi(\mathbf{r}, \mathbf{v}) = 0$ if $\mathbf{r}$ lies on the surface of the body V and the vector $\mathbf{v}$ points inwards. The function $\varphi(\mathbf{r} - t\mathbf{v}, \mathbf{v})$ is also piecewise continuously differentiable on t and its derivative for $t = 0$ is equal to

$$-\,\mathbf{v} \operatorname{grad} \varphi = G(h)\,\psi - \psi + k(\mathbf{r}, \mathbf{v})\,\varphi.$$

Hence,

$$-\,\mathbf{v} \operatorname{grad} \varphi - k(\mathbf{r}, \mathbf{v})\,\varphi = G(h)\,\psi - \psi \in X$$

so that $\varphi \in D_0$ and

$$T_1\varphi = -\mathbf{v}\,\mathrm{grad}\,\varphi - k(\mathbf{r}, \mathbf{v})\,\varphi = T_0\varphi.$$

Therefore, $T_1 \subset T_0$.

Let $\varphi(\mathbf{r}, \mathbf{v}) \in D_0$. A straightforward calculation shows that

$$G(h)\,\varphi - \varphi = \int_0^h G(s)\,T_0\varphi\,ds,$$

where $h > 0$ is arbitrary. This implies ([10], Thereom 10.5.2) that T_0 is a restriction of the infinitesimal generator $\bar{T}_1$ of the semigroup $[G(t)]$. Hence T_0 is closable and $T = \bar{T}_0 = \bar{T}_1$. Thus we have proved that T exists and is the infinitesimal generator of the semigroup $[G(t)]$.

The inequality (4) implies that the open half-plane $\mathrm{Re}\,(\lambda) > -\lambda^*$ belongs to the resolvent set of T. Hence $(\lambda I - T)^{-1}$, where I is the identity operator, exists as a bounded and everywhere defined operator if $\mathrm{Re}\,(\lambda) > -\lambda^*$. We can express $(\lambda I - T)^{-1}$ as the Laplace transform of the semigroup $[G(t)]$:

$$\begin{aligned}(\lambda I - T)^{-1}\,\psi &= \int_0^\infty e^{-\lambda t}\,G(t)\,\psi\,dt \\ &= \int_0^\infty \psi(\mathbf{r} - t\mathbf{v}, \mathbf{v}) \exp\left[-\lambda t - \int_0^t k(\mathbf{r} - \tau\mathbf{v}, \mathbf{v})\,d\tau\right] dt. \qquad (5)\end{aligned}$$

It follows from (4) that

$$\|\,(\lambda I - T)^{-1}\,\| \leqslant \int_0^\infty e^{-\sigma t}\,\|\,G(t)\,\|\,dt \leqslant \frac{1}{\sigma + \lambda^*}, \qquad (6)$$

where $\sigma = \mathrm{Re}\,(\lambda)$.

Since the integral operator K is bounded, the sum $T + K = A$ is a densely defined and closed operator with the domain of definition $D(A) = D(T)$. According to a well known theorem ([10], Theorem 13.2.1), the Boltzmann operator A is also the infinitesimal generator of a strongly continuous semigroup of bounded operators $[\Gamma(t)]$, $t \geqslant 0$. This implies that the time-dependent transport equation, i.e., the equation

$$\frac{\partial N}{\partial t} = AN,$$

has a unique solution $N(\mathbf{r}, \mathbf{v}, t)$, provided the initial distribution $N_0(\mathbf{r}, \mathbf{v})$ belongs to $D(A)$. We have $N(\mathbf{r}, \mathbf{v}, t) = \Gamma(t)\,N_0(\mathbf{r}, \mathbf{v})$. For any fixed $t \geqslant 0$ the distribution function $N(\mathbf{r}, \mathbf{v}, t) \in D(A)$.

The operators $G(t)$ are evidently positive in the sense that they map nonnegative functions of the space X into nonnegative functions. Since this holds also for the integral operator K, whose kernel is a nonnegative function, the sum $T + K = A$ is the infinitesimal generator of a semigroup of positive operators ([10], Theorem 13.4.2). Thus we have proved

THEOREM I. *The Boltzmann operator A is the infinitesimal generator of a strongly continuous semigroup of bounded positive operators.*

3. THE SPECTRUM OF A IN THE HALF-PLANE $\mathrm{Re}(\lambda) > -\lambda^*$

Let λ be any complex number with $\mathrm{Re}(\lambda) > -\lambda^*$. Then it belongs to the resolvent set of T so that $(\lambda I - T)^{-1}$ exists.

Write $A_\lambda = \lambda I - A = \lambda I - T - K$. We have

$$(\lambda I - T)^{-1} A_\lambda \subset I - (\lambda I - T)^{-1} K \tag{7}$$

since the operator on the right is everywhere defined and on the left is not. From this it follows

$$K(\lambda I - T)^{-1} A_\lambda \subset K - K(\lambda I - T)^{-1} K.$$

THEOREM II. *Suppose the operator $K(\lambda I - T)^{-1} K$ is compact for any complex λ with* $\mathrm{Re}(\lambda) > -\lambda^*$. *Then the part of the spectrum of the Boltzmann operator A lying in the half-plane* $\mathrm{Re}(\lambda) > -\lambda^*$, *consists at most of a countable set of isolated points λ_k. Each λ_k is an eigenvalue of finite multiplicity and is a pole for the resolvent $(\lambda I - A)^{-1}$.*

PROOF. Let us write $B_\lambda = (\lambda I - T)^{-1} K$. Since $K(\lambda I - T)^{-1} K$ is compact if $\mathrm{Re}(\lambda) > -\lambda^*$, the same holds for $B_\lambda^2 = (\lambda I - T)^{-1} K(\lambda I - T)^{-1} K$. Now we shall apply a theorem of Šmul'yan [11], which asserts: Let $C(\lambda)$ be an analytic operator valued function on a region D of the complex λ plane whose values are compact operators on the Banach space X. Then for any $\mu \neq 0$, one of the two possibilities must hold: (a) For every $\lambda \in D$, μ is an eigenvalue of $C(\lambda)$, or (b) Except for a discrete set of values $\lambda_k \in D$, the operator $\mu I - C(\lambda)$ has a bounded everywhere defined inverse, while $(\mu I - C(\lambda))^{-1}$ has a pole at each of the points λ_k.

The function $\lambda \to B_\lambda^2$ is regular analytic in the half-plane $\mathrm{Re}(\lambda) > -\lambda^*$ and its values B_λ^2 are by assumption compact operators. Hence Šmul'yan's theorem applies. Since, by (6), we have

$$\| B_\lambda^2 \| \leqslant \| K \|^2 \| (\lambda I - T)^{-1} \|^2 \leqslant \frac{\| K \|^2}{(\sigma + \lambda^*)^2},$$

B_λ^2 tends to zero if $\mathrm{Re}(\lambda) = \sigma \to \infty$. Therefore, $\mu = 1$ is not an eigenvalue for all operators B_λ^2. Thus the possibility (b) must hold. Hence $(I - B_\lambda^2)^{-1}$ exists as a bounded everywhere defined operator for all λ in the half-plane $\mathrm{Re}(\lambda) > -\lambda^*$, except for a discrete set of values λ_k, where the function $(I - B_\lambda^2)^{-1}$ has a pole. Since $(I - B_\lambda)^{-1} = (I + B_\lambda)(I - B_\lambda^2)^{-1}$, the function $(I - B_\lambda)^{-1}$ has a similar behavior as $(I - B_\lambda^2)^{-1}$ in the half-plane $\mathrm{Re}(\lambda) > -\lambda^*$.

Let λ be such that $(I - B)_\lambda^{-1}$ exists. Put

$$R_\lambda = (I - B_\lambda)^{-1}(\lambda I - T)^{-1}.$$

From (7) we obtain $R_\lambda A_\lambda \subset I$. Now it is well known that the existence of

$$(I - (\lambda I - T)^{-1} K)^{-1} = (I - B_\lambda)^{-1}$$

implies the existence of $(I - K(\lambda I - T)^{-1})^{-1}$. Since $(I - B_\lambda)^{-1}(\lambda I - T)^{-1} = (\lambda I - T)^{-1}.\ (I - K(\lambda I - T)^{-1})^{-1}$, we have $A_\lambda R_\lambda = I$. Hence A_λ^{-1} exists for such a λ as a bounded everywhere defined operator and is equal to R_λ. Consequently, the resolvent $(\lambda I - A)^{-1} = R_\lambda = (I - B_\lambda)^{-1}(\lambda I - T)^{-1}$ is an analytic function of λ in the half-plane $\mathrm{Re}(\lambda) > -\lambda^*$ with the exception of a discrete set of values λ_k, where R_λ has a pole.

Any pole λ_k of R_λ is an eigenvalue of A. A corresponding eigenfunction $\psi(\mathbf{r}, \mathbf{v})$ satisfies, according to (7), the equation $B_{\lambda_k}\varphi = \varphi$, or

$$B_{\lambda_k}^2 \varphi = B_{\lambda_k}\varphi = \varphi.$$

The operator $B_{\lambda_k}^2$ being compact, the space of solutions of this equation is finite dimensional. This implies that the space of eigenfunctions of A corresponding to the eigenvalue λ_k is finite dimensional too. The Theorem II is now completely proved.

Since

$$\| B_\lambda \| \leqslant \| K \| \, \| (\lambda I - T)^{-1} \| \leqslant \frac{\| K \|}{\sigma + \lambda^*}, \qquad \sigma = \mathrm{Re}(\lambda), \qquad \sigma > -\lambda^*,$$

it follows that $\| B_\lambda \| < 1$ in the open half-plane $\mathrm{Re}(\lambda) > \| K \| - \lambda^*$. Hence $(I - B_\lambda)^{-1}$ exists and so this half-plane belongs to the resolvent set of the operator A. The strip $-\lambda^* < \mathrm{Re}(\lambda) \leqslant \| K \| - \lambda^*$ contains at most isolated points of $\sigma(A)$.

The product $K(\lambda I - T)^{-1} K$ is a rather complicated integral operator in the general case. Let us calculate it for a homogeneous body. Then the functions $\Sigma_s(\mathbf{r}, \mathbf{v}' \to \mathbf{v})$ and $\Sigma(\mathbf{r}, \mathbf{v})$ do not depend on the position vector $\mathbf{r}$. Thus we have

$$K(\lambda I - T)^{-1} K\psi$$

$$= \int_\omega K(\mathbf{v}, \mathbf{v}'')\, d\omega'' \int_0^\infty e^{-(\lambda + k(\mathbf{v}''))t}\, dt \int_\omega K(\mathbf{v}'', \mathbf{v}')\, \psi(\mathbf{r} - t\mathbf{v}'', \mathbf{v}')\, d\omega'.$$

If we change the order of integration and introduce a new variable $\mathbf{r}' = \mathbf{r} - t\mathbf{v}''$, we obtain an integral operator of the form

$$K(\lambda I - T)^{-1} K\psi = \int_V \int_\omega L(\mathbf{r}, \mathbf{r}', \mathbf{v}, \mathbf{v}')\, \psi(\mathbf{r}', \mathbf{v}')\, dV'\, d\omega',$$

where the kernel L is

$$L(\mathbf{r}, \mathbf{r}', \mathbf{v}, \mathbf{v}') = \int_0^\infty t^{-3} \exp\left\{- \lambda t - tk\left[\frac{\mathbf{r} - \mathbf{r}'}{t}\right]\right\}$$
$$\times K\left(\mathbf{v}, \frac{\mathbf{r} - \mathbf{r}'}{t}\right) K\left(\frac{\mathbf{r} - \mathbf{r}'}{t}, \mathbf{v}'\right) dt.$$

We get a more simple integral operator in the case of a homogeneous body and isotropic scattering, when the functions $\Sigma_s(\mathbf{v}' \to \mathbf{v})$ and $\Sigma(\mathbf{v})$ depend only on the absolute velocities v, v'. The operator K is then a product of two bounded commuting operators S and P, where S means the integration over v' with the kernel depending on $\Sigma_s(v' \to v)$, and P the integration over the solid angle, i.e., $P = (4\pi)^{-1} \int \cdot\, d\Omega$. The operator P is a projection. If $PB_\lambda = P(\lambda I - T)^{-1} K$ is a compact operator for all λ in the half-plane $\mathrm{Re}\,(\lambda) > -\lambda^*$, then the operator $SPB_\lambda = K(\lambda I - T)^{-1} K$ is compact too and the Theorem II can be applied. The product PB_λ is in essence the so called Peierls operator [6]. It acts in the subspace consisting of functions $\psi(\mathbf{r}, \mathbf{v})$ which are dependent on the opsition vector $\mathbf{r}$ and on the absolute velocity v.

4. Existence and Uniqueness of Nonnegative Eigenfunctions

The Boltzmann operator A is the infinitesimal generator of a strongly continuous semigroup of bounded positive operators in the Banach space X. According to the Theorem 11.7.2 of [10], the resolvent $(\lambda I - A)^{-1}$ is also a positive operator for any positive number $\lambda > \| K \| - \lambda^*$.

Theorem III. *Suppose that $K(\lambda I - T)^{-1} K$ is a compact operator for all λ in the open half-plane* $\mathrm{Re}\,(\lambda) > -\lambda^*$, *and that the intersection of the set*

$\sigma(A)$ with the strip $-\lambda^* < \operatorname{Re}(\lambda) \leqslant \|K\| - \lambda^*$ *is not empty. Then there exists a real eigenvalue λ_0 of A with a corresponding nonnegative eigenfunction. The half-plane* $\operatorname{Re}(\lambda) > \lambda_0$ *belongs to the resolvent set of A.*

If the space X is L_p, where $p > 1$, the nonnegative eigenfunction is uniquely determined up to a positive constant factor.

PROOF. Take any positive number $\alpha > \|K\| - \lambda^*$. Then $R_\alpha = (\alpha I - A)^{-1}$ is a bounded positive operator. We obtain the spectrum of R_α in the μ plane if we map the set $\sigma(A)$ from the λ plane into the μ plane by the mapping function

$$\mu = \frac{1}{\alpha - \lambda}. \tag{8}$$

Since the half-plane $\operatorname{Re}(\lambda) > \|K\| - \lambda^*$ belongs to the set $\rho(A)$, the spectrum of R_α lies entirely in the closed disc $|2(\alpha + \lambda^* - \|K\|)\mu - 1| \leqslant 1$ of the μ plane. The image of the half-plane $\operatorname{Re}(\lambda) > -\lambda^*$ is the exterior of the circle $|2(\alpha + \lambda^*)\mu - 1| = 1$. Hence, according to Theorem II, there are only isolated points of the spectrum of R_α in this exterior.

Let us consider the outside of the circle $|\mu| = 1/(\alpha + \lambda^*)$. This region is the image of the open disc $|\lambda - \alpha| < \alpha + \lambda^*$ in the λ plane. The circle $|\lambda - \alpha| = \alpha + \lambda^*$ passes through the point $\lambda = -\lambda^*$ and meets perpendicularly the real axis. By assumption, there is in the strip

$$-\lambda^* < \operatorname{Re}(\lambda) \leqslant \|K\| - \lambda^*$$

at least one point λ_1 of $\sigma(A)$. If α is sufficiently large, the point λ_1 lies on the disc $|\lambda - \alpha| < \alpha + \lambda^*$. Hence, for such α, there is outside the circle $|\mu| = 1/(\alpha + \lambda^*)$ at least one eigenvalue of $R_\alpha = (\alpha I - A)^{-1}$. But only a finite number of eigenvalues lies in the outside of the circle $|\mu| = r$ for any $r > 1/(\alpha + \lambda^*)$. Therefore, there exists an eigenvalue μ_0 of R_α with the largest absolute value.

Now, Theorem II tells us that μ_0 is an isolated point of the spectrum and that the resolvent $(\mu I - R_\alpha)^{-1}$ has there a pole. Hence, we have in a neighborhood of μ_0 the following expansion

$$(\mu I - R_\alpha)^{-1} = \sum_{k=-n}^{\infty} (\mu - \mu_0)^k C_k. \tag{9}$$

Since R_α is a positive operator, we can apply a theorem of M. G. Krein and M. A. Rutman ([12], Theorem 6.1) on the existence of positive eigenvectors of positive operators. Though this theorem is proved in [12] only for compact positive operators, the proof is valid verbatim also for such bounded positive operators whose spectral points with the largest absolute value are all isolated and the resolvent has at these points an expansion of the type (9). Hence the operator R_α has a positive eigenvalue μ_0 with a nonnegative corresponding eigenfunction. All points of $\sigma(R_\alpha)$ lie on the closed disc $|\mu| \leqslant \mu_0$. It follows that the open disc $|\lambda - \alpha| < 1/\mu_0$, the image in the λ plane of the exterior of $|\mu| = \mu_0$, belongs to the resolvent set of the operator A. The number $\lambda_0 = \alpha - 1/\mu_0$ is an eigenvalue of A and among the corresponding eigenfunctions there is at least one nonnegative.

We obtain this eigenvalue λ_0 for any real α which is sufficiently large. Since the points of $\sigma(A)$ are isolated in the strip $-\lambda^* < \mathrm{Re}(\lambda) \leqslant \|K\| - \lambda^*$ and since the circle $|\lambda - \alpha| = \alpha - \lambda_0$ can be made arbitrarily large, it follows that the spectrum of A belongs to the closed half-plean $\mathrm{Re}(\lambda) \leqslant \lambda_0$.

Suppose now that the Banach space X where A acts is L_p with $p > 1$. We shall prove the uniqueness of the nonnegative solution as in [2]. Let us denote by $\varphi_0(\mathbf{r}, \mathbf{v})$ a nonnegative eigenfunction of A corresponding to the eigenvalue λ_0. It satisfies the equation

$$\varphi_0 = B_{\lambda_0}\varphi_0 = (\lambda_0 I - T)^{-1} K\varphi_0,$$

or

$$\varphi_0(\mathbf{r}, \mathbf{v}) = \int_0^\infty dt \int_\omega K(\mathbf{r} - t\mathbf{v}, \mathbf{v}, \mathbf{v}') \exp\left\{-\lambda_0 t - \int_0^t k(\mathbf{r} - \tau\mathbf{v}, \mathbf{v})\, d\tau\right\} \times \varphi_0(\mathbf{r} - t\mathbf{v}, \mathbf{v}')\, d\omega'. \tag{10}$$

Since $K(\mathbf{r}, \mathbf{v}, \mathbf{v}') > 0$ almost everywhere, the kernel of this integral equation is positive. We conclude that a nonnegative solution $\varphi_0(\mathbf{r}, \mathbf{v})$ vanishes at most in a set of measure zero.

The adjoint space of $X = L_p$, i.e., the space X^* of all bounded semilinear functionals, is L_q, $q = p/(p-1)$, $1 < q < \infty$. We can represent any element ψ of X^* by a measurable function $\psi(\mathbf{r}, \mathbf{v})$ of L_q. The value of the functional $\psi \in X^*$ on the vector $\varphi(\mathbf{r}, \mathbf{v}) \in X$ is equal to the integral

$$\int_V \int_\omega \overline{\varphi(\mathbf{r}, \mathbf{v})}\, \psi(\mathbf{r}, \mathbf{v})\, dV\, d\omega = (\psi, \varphi)$$

If the norm in L_p is defined with a weight function $\rho(\mathbf{v})$, the integral on the left must be correspondingly modified.

The adjoint operator T^* of an operator T acts on the functions $\psi(\mathbf{r}, \mathbf{v}) \in X^*$ and we have $(\psi, T\varphi) = (T^*\psi, \varphi)$ for all $\varphi \in D(T)$ and $\psi \in D(T^*)$.

Since the space L_p, $p > 1$, is reflexive, the adjoint operator A^* of the Boltzmann operator A is closed and densely defined. Let the weight function $\rho(\mathbf{v}) = 1$. It is not difficult to verify that in this case the value $A^*\psi$ is determined for all sufficiently smooth functions $\psi(\mathbf{r}, \mathbf{v})$ by the integro-differential expression

$$A^*\psi = \mathbf{v}\,\mathrm{grad}\,\psi - k(\mathbf{r}, \mathbf{v})\,\psi + \int_\omega K(\mathbf{r}, \mathbf{v}', \mathbf{v})\,\psi(\mathbf{r}, \mathbf{v}')\, d\omega'$$

and by the adjoint boundary condition: $\psi(\mathbf{r}, \mathbf{v}) = 0$ if $\mathbf{r}$ lies on the surface of V and for outgoing $\mathbf{v}$. The operator A^* is the generator of a strongly continuous semigroup of bounded positive operators. This is the adjoint semigroup $[\Gamma^*(t)]$ of the semigroup $[\Gamma(t)]$ generated by A.

If $K(\lambda I - T)^{-1}$ is compact for all λ with $\operatorname{Re}(\lambda) > -\lambda^*$, then the resolvent $(\lambda I - A)^{-1}$ is a regular analytic function in the half-plane $\operatorname{Re}(\lambda) > -\lambda^*$ except for the points λ_k where it has a pole. Since $(\lambda I - A^*)^{-1} = [(\bar{\lambda} I - A)^{-1}]^*$, the same holds for the resolvent $(\lambda I - A^*)^{-1}$ of the adjoint operator A^*. It follows that λ_0 is an eigenvalue of A^* and that at least one of the corresponding eigenfunctions is positive almost everywhere. Denote this function by $\psi_0(\mathbf{r}, \mathbf{v})$. It is easily seen that $\psi_0(\mathbf{r}, \mathbf{v})$ satisfies the equation $(\lambda_0 I - T^*)^{-1} K^* \psi_0 = \psi_0$. Put $\psi_1 = K^* \psi_0$, where $\psi_1(\mathbf{r}, \mathbf{v}) > 0$ almost everywhere. Since $B^*_{\lambda_0} = K^*(\lambda_0 I - T^*)^{-1}$, we have

$$B^*_{\lambda_0} \psi_1 = \psi_1$$

Let $\varphi(\mathbf{r}, \mathbf{v})$ be any eigenfunction of A corresponding to an eigenvalue $\lambda \neq \lambda_0$. Since $A\varphi = \lambda\varphi$, we have

$$\lambda(\psi_0, \varphi) = (\psi_0, A\varphi) = (A^*\psi_0, \varphi) = \lambda_0(\psi_0, \varphi).$$

Hence $(\psi_0, \varphi) = 0$. Since $\psi_0(\mathbf{r}, \mathbf{v}) > 0$ a.e., there exists a set of positive measure where $\varphi(\mathbf{r}, \mathbf{v}) > 0$, and a set of positive measure where $\varphi(\mathbf{r}, \mathbf{v}) < 0$. Thus $\varphi(\mathbf{r}, \mathbf{v})$ is not proportional to a nonnegative eigenfunction of A.

Suppose now that there exists an eigenfunction $\varphi(\mathbf{r}, \mathbf{v})$ of A corresponding to the eigenvalue λ_0 and linearly independent on φ_0. We can assume that $(\psi_0, \varphi) = 0$. (If this equality does not hold, we replace φ by a linear combination $\varphi + \alpha\varphi_0$, where the constant α is suitably chosen.) It follows from $(\psi_0, \varphi) = 0$ that $\varphi(\mathbf{r}, \mathbf{v})$ is neither nonnegative nor nonpositive almost everywhere. Therefore, $|\varphi(\mathbf{r}, \mathbf{v})| > \varphi(\mathbf{r}, \mathbf{v})$ on a set of positive measure. Since B_{λ_0} is an integral operator with a positive kernel and $B_{\lambda_0}\varphi = \varphi$, it follows that $|\varphi(\mathbf{r}, \mathbf{v})| < (B_{\lambda_0} |\varphi|)(\mathbf{r}, \mathbf{v})$. This implies

$$0 < (\psi_1, |\varphi|) < (\psi_1, B_{\lambda_0} |\varphi|) = (B^*_{\lambda_0}\psi_1, |\varphi|) = (\psi_1, |\varphi|),$$

which is a contradiction. Therefore, λ_0 is a simple eigenvalue of A and the nonnegative eigenfunction $\varphi_0(\mathbf{r}, \mathbf{v})$ is determined up to a positive constant factor.

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