Kernels and point processes associated with Whittaker functions

Gordon Blower* **and Yang Chen**

Department of Mathematics and Statistics, Lancaster University, LA1 4YF, England, UK,

and

Department of Mathematics, University of Macau, Avenda da Universidade, Taipa, Macau, China

* Corresponding author's email: `g.blower@lancaster.ac.uk`

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Abstract. This article considers Whittaker's confluent hypergeometric function $W_{\kappa,\mu}$ where κ is real and μ is real or purely imaginary. Then $\varphi(x) = x^{-\mu-1/2}W_{\kappa,\mu}(x)$ arises as the scattering function of a continuous time linear system with state space $L^2(1/2,\infty)$ and input and output spaces $\mathbf{C}$. The Hankel operator Γ_φ on $L^2(0,\infty)$ is expressed as a matrix with respect to the Laguerre basis and gives the Hankel matrix of moments of a Jacobi weight $w_0(x) = x^b(1-x)^a$. The operation of translating φ is equivalent to deforming w_0 to give $w_t(x) = e^{-t/x}x^b(1-x)^a$. The determinant of the Hankel matrix of moments of w_ε satisfies the σ form of Painlevé's transcendental differential equation PV. It is shown that Γ_φ gives rise to the Whittaker kernel from random matrix theory, as studied by Borodin and Olshanski (Comm. Math. Phys. 211 (2000), 335–358). Whittaker kernels are closely related to systems of orthogonal polynomials for a Pollaczek–Jacobi type weight lying outside the usual Szegö class.

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1. INTRODUCTION

The Whittaker function $W_{\kappa,\mu}$ is the solution of the second order linear differential equation

$$y'' + \Big(\frac{-1}{4} + \frac{\kappa}{x} + \frac{1/4-\mu^2}{x^2}\Big)y = 0 \tag{1}$$

that is asymptotic to $x^\kappa e^{-x/2}$ as $x \to \infty$ through real values, and possibly has a logarithmic singularity at $x = 0$. The general solution of Eqn (1) is given by linear combinations of the Whittaker functions $M_{\kappa,\pm\mu}$. See Refs 15, p.264; 39 p. 343 for basic definitions and properties, such as $W_{\kappa,\mu} = W_{\kappa,-\mu}$. We consider the case in which κ is real, and μ is either real or purely imaginary; hence $W_{\kappa,\mu}(x)$ is real for all $x > 0$. In random matrix theory, as in Refs 6, 7, 8, 25, kernels such as

$$K(x,y) = \big((\kappa-1/2)^2-\mu^2\big)\sqrt{xy}\,\frac{W_{\kappa-1,\mu}(x)W_{\kappa,\mu}(y) - W_{\kappa,\mu}(x)W_{\kappa-1,\mu}(y)}{x-y}, \tag{2}$$

provide self-adjoint integral operators on $L^2((0,\infty);\mathbf{C})$, and are associated with determinantal random point fields.

The purpose of this note is to provide some transparent proofs of some basic properties of these kernels and their associated determinantal random point fields. The new contribution is showing that the Whittaker kernels are closely related to systems of orthogonal polynomials for a Pollaczek–Jacobi type weight

$$w_t(x) = e^{-t/x}x^b(1-x)^a \qquad (0 < x < 1) \tag{3}$$

for $t > 0, a > -1$ and $b \in \mathbf{R}$. In Ref 26, the authors consider physical applications of w_t in quantum mechanical models of highly excited states. We compare w_t with some classical weights.

An integrable weight $w : [0,1] \to [0,\infty)$ belongs to the Szegö class if

$$\int_0^1 \frac{\log w(x)}{\sqrt{x(1-x)}}\,dx > -\infty. \tag{4}$$

Szegö's fundamental approximation theorem for orthogonal polynomials uses such weights as in page 157 of Ref 33.

The classical Jacobi weight function as in Ref 33 can be translated onto $[0,1]$ to become

$$w_0(x) = x^b(1-x)^a \qquad (0 < x < 1), \tag{5}$$

where $a, b > -1$. Clearly the Jacobi weight w_0 satisfies (4), and Szegö uses this condition to derive properties of Jacobi orthogonal polynomials in Ref 33. See also Ref 2 for the deformations of the Jacobi weight by e^{-tx}, which gives a weight which resembles the Laguerre weight.

Thus the weight w_t may be regarded as a deformation of the Jacobi weight w_0 by multiplication by $e^{-t/x}$ where $t > 0$ is the deformation parameter. However, the weight w_t vanishes to infinite order as $x \to 0+$ for all $t > 0$ and $b \in \mathbf{R}$, and $\log w_t(x)$ is not integrable with respect to $dx/\sqrt{x(1-x)}$, violating condition (4); so the properties of orthogonal polynomials with respect to w_t can be rather different from the Jacobi orthogonal polynomials. In this paper, we establish a remarkably strong connection between the weights w_t for $t > 0$ and the Whittaker kernel K, which in our view provides a clear insight into the properties of K as an operator, and explains why K has similar properties to other kernels which arise in random matrix theory. In applications to random matrix theory, one is primarily interested in the determinants associated with the operators and weights, which are defined as follows.

Definition With $a, b > -1$ and $t \geq 0$, we introduce

$$\mathcal{D}_N(t; a, b) = \det\Big[\int_0^1 x^{j+k} x^b (1-x)^a e^{-t/x}\, dx\Big]_{j,k=0}^{N-1} \tag{6}$$

or equivalently

$$\mathcal{D}_N(t; a, b) = \frac{1}{N!}\int_{[0,1]^N} \prod_{1\leq j<k\leq N} (x_j - x_k)^2 \prod_{\ell=1}^{N} x_\ell^b (1-x_\ell)^a e^{-t/x_\ell}\, dx_\ell. \tag{7}$$

The equality of (6) and (7) follows from Heine's formula, as in Szegö's treatise on orthogonal polynomials[33]. The Hankel matrix associated with w_t is

$$\Big[\int_0^1 x^{j+k} w_t(x) dx\Big]_{j,k=0}^{\infty}$$

which gives a linear operator on ℓ^2; whereas the Hankel integral operator on $L^2(0,\infty)$ with scattering function $\varphi \in L^2(0,\infty)$ is $\Gamma_\varphi : f(x) \mapsto \int_0^\infty \varphi(x+y) f(y)\, dy$. In section two, we introduce the notion of a linear system and scattering function, and introduce a special linear system involving w_0 which realises $\varphi(x) = W_{\kappa,\mu}(x)/x^{\mu+1/2}$ as its scattering function. Then we relate the Hankel operator matrix with Hankel integral operators, and the linear system which gives rise to the Hankel integral operator. We are concerned with a family of weights w_t which deforms as $t \geq 0$ varies, so in section two, we introduce a family of linear systems $(-A, B_\varepsilon, C_\varepsilon)$ depending on $\varepsilon \geq 0$ which have scattering functions $\varphi(x + 2\varepsilon)$. In section three we show how the weight w_ε corresponds $(-A, B_\varepsilon, C_\varepsilon)$.

The fundamental eigenvalue distributions in random matrix theory are the bulk, soft edge and hard edge distributions, which specified by Fredholm determinants of kernels K known as the sine, Airy and Bessel kernels. Tracy and Widom[36] observed that such K can be expressed as products of Hankel operators $K = \Gamma_\phi^* \Gamma_\phi$, where ϕ satisfies second-order differential equations with rational coefficients. More generally they considered the matrix models which arise from rational second order differential equations, and the deformations which arise from the translation operation $\phi(x) \mapsto \phi(x + 2\varepsilon)$. The results in section two, three and four of this paper extend their program to the Whittaker kernel.

We recall fundamental facts about trace class operators and their Fredholm determinants. Let H and H_1 be separable Hilbert space and suppose that H has a complete orthonormal basis $(\phi_j)_{j=0}^\infty$. A linear operator $T : H \to H_1$

is said to be Hilbert–Schmidt if $\|T\|^2_{HS}=\sum_{j=0}^\infty \|T\phi_j\|^2_{H_1}$ is finite, and $\|T\|_{HS}$ is the Hilbert–Schmidt norm. A linear operator $R: H\rightarrow H$ is said to be trace class if there exist Hilbert–Schmidt operators $T: H\rightarrow H_1$ and $S: H_1\rightarrow H$ such that $R=ST$. If so, then the trace norm is $\|R\|_{c^1}=\inf\{\|S\|_{HS}\|T\|_{HS}: R=ST\}$. Such an R has eigenvalues $(\lambda_j)_{j=1}^\infty$ such that $\sum_{j=1}^\infty|\lambda_j|<\infty$, and the Fredholm determinant associated with R is $\det(I+R)=\prod_{j=1}^\infty(1+\lambda_j)$. A crucial observation is that $TS: H_1\rightarrow H_1$ is also trace class and

$$\det(I+ST)=\det(I+TS). \tag{8}$$

Given trace class $R: H\rightarrow H$, and finite rank orthogonal projections $P_N: H\rightarrow H$, the operators P_NRP_N are also trace class and we can compare the polynomial $\det(I-zP_NRP_N)$ with the entire function $\det(I-zR)$. In particular, we can choose P_N to be the orthogonal projection onto $\mathrm{span}\{\phi_j; j=0,\dots,N-1\}$, where P_NRP_n is represented by the matrix $[\langle R\phi_j,\phi_k\rangle_H]_{j,k=0}^{N-1}$. In Proposition 2.3 below, we choose $H=L^2(0,\infty)$, R to be a Hankel integral operator on H and $(\phi_n)_{n=0}^\infty=(\phi_n^{(\alpha)})_{n=0}^\infty$ to be a Laguerre basis of H; we also let $H_1=L^2(w_0(x)dx)$ and realise $\det[\langle R\phi_j,\phi_k\rangle_H]_{j,k=0}^{N-1}$ in terms of $\mathcal{D}_N(0;a,b)$ for $(a,b)=(2\alpha-2\mu+1,\mu-\kappa-1/2)$. This leads to the basic connection between Hankel integral operators and Hankel operators; in section three we refine this by deforming the Jacobi weight w_0 to w_t.

Painlevé considered how the solutions of second order rational differential equations behave when the coefficients are deformed; see Refs 18, 22, 27 for a modern treatment. In section three, we show how the determinants $\mathcal{D}_N(t;a,b)$ are related to Whittaker functions, and we observe that $\mathcal{D}_N(t;a,b)$ satisfies a version of the Painlevé transcendental differential equations V and VI as a function of the deformation parameter t. The Painlevé V equation has previously appeared in various ensembles in random matrix theory. Tracy and Widom[37] obtained PV from the Laguerre ensemble, and also from the Bessel ensemble from Ref 35 which is associated with hard edge distributions.

In section four, we introduce the Whittaker kernels in terms of our special linear system and associated Hankel integral operators. We obtain factorization theorems in the style of Tracy and Widom[34,35,36], which express the Whittaker kernel as products of Hankel integral operators of infinite rank on $L^2(0,\infty)$. Peller[29] describes in detail the connection between the eigenvalues of a Hankel operator Γ_ϕ, the spectral multiplicity and the smoothness of the symbol ϕ. In Corollary 4.3, we show that the eigenvalues of a variant of the Whittaker kernel are of rapid decay, by using methods from section 6 of Ref 4.

The case of Eqn (2) with $\kappa=a+1/2$ and $\mu=a$ is of particular interest, as in Ref 28, and we consider this in sections 5 and 6. Lisovyy considered how the determinant for the hypergeometric kernel degenerates to the determinant for the Whittaker kernel, and realised PV as a limiting case of PVI; see Ref 24, section 10.

A stochastic point process on $(0,\infty)$ is a probability measure on the space of point configurations of $(0,\infty)$. See section 5.4 of Ref 13 for the general definition of finite point processes. The process is said to be determinantal when the correlation functions are given by Fredholm determinants of operators on $L^2(0,\infty)$, as follows.

Definition (Determinantal point process) Let S be a continuous kernel on $(0,\infty)$ such that

(i) $S(x,y)=\overline{S(y,x)}$ for all $x,y\in(0,\infty)$;

(ii) the integral operator with kernel S satisfies the operator inequality $0\leq S\leq I$ as self-adjoint operators on $L^2(0,\infty)$;

(iii) for all $0<u<v<\infty$, the integral operator with kernel $\mathbf{I}_{[u,v]}(x)S(x,y)\mathbf{I}_{[u,v]}(y)$ is trace class, where $\mathbf{I}_{[u,v]}$ denotes the indicator function of $[u,v]$.

Then S gives rise to a determinantal point process on (u,v) in Soshnikov's sense[9,32]. Let T satisfy $I+T=(I-S)^{-1}$ as operators on $L^2(u,v)$. Then the probability that there are exactly N points in the realization, one in each subset dx_j for $j=1,\dots,N$ and none elsewhere is equal to

$$\det(I+T)^{-1}\det[T(x_j,x_k)]_{j,k=1}^N dx_1\dots dx_N, \tag{9}$$

where the $x_j\in[u,v]$.

Let $M_N^h(\mathbf{C})$ be the space of Hermitian complex matrices, and $U(N)$ the space of unitary $N\times N$ matrices which acts on $M_N^h(\mathbf{C})$ by $U: X\mapsto UXU^*$. A probability measure on $M_N^h(\mathbf{C})$ which is invariant under the action of $U(N)$ is called a unitary ensemble. The eigenvalues of X are real and are invariant under this action, and the eigenvalues $\lambda_1,\dots,\lambda$ may be regarded as random points on $\mathbf{R}$. Mehta[25] gives examples of unitary ensembles which produce determinantal point processes of the above form.

2. LINEAR SYSTEMS FOR THE WHITTAKER FUNCTIONS

Definition (Linear systems) Let H be a complex separable Hilbert space known as the state space, and H_0 a finite dimensional complex Hilbert space which serves as the input and output space. Usually, we take $H_0=\mathbf{C}$, although in section 5 we use $H_0=\mathbf{C}^2$. Let $L^2((0,\infty);H_0))$ be the space of strongly measurable functions $f:(0,\infty)\to H_0$ such that $\int_0^\infty \|f(t)\|^2_{H_0}dt<\infty$. Let $\mathcal{L}(H)$ be the space of bounded linear operators on H with the operator norm $\|T\|=\sup\{\|T\xi\|_H:\xi\in H;\|\xi\|_H\leq 1\}$. A linear system $(-A,B_0,C_0)$ consists of:

(i) $-A$, the generator of a strongly continuous semigroup $(e^{-tA})_{t\geq 0}$ of bounded linear operators on H such that $\|e^{-tA}\|\leq Me^{-\omega_0 t}$ for all $t\geq 0$ and some $M,\omega_0\geq 0$;

(ii) $B_0: H_0\to H$ a bounded linear operator;

(iii) $C_0: H\to H_0$ a bounded linear operator.

(In semigroup literature, a strongly continuous semigroup is described as being of class (C_0), but this should not be confused with our notation.) We define the scattering function by $\varphi(t)=\varphi_{(0)}(t)=C_0e^{-tA}B_0$, where $\varphi:[0,\infty)\to\mathcal{L}(H_0)$ is continuous by (i), (ii) and (iii) since H_0 is finite-dimensional. Then we define the Hankel operator with scattering function $\varphi:(0,\infty)\to\mathcal{L}(H_0)$ by

$$\Gamma_\varphi f(x)=\int_0^\infty \varphi(x+y)f(y)\,dy\qquad (f\in L^2((0,\infty);H_0)),\tag{10}$$

as in Ref 29. Note that if the integral $\int_0^\infty t\|\varphi(t)\|^2_{\mathcal{L}(H_0)}dt$ converges, then Γ_φ defines a Hilbert–Schmidt operator. In particular, this holds if $\omega_0>0$.

Let $B_\varepsilon=e^{-\varepsilon A}B_0$ and $C_\varepsilon=C_0e^{-\varepsilon A}$, and consider the linear system $(-A,B_\varepsilon,C_\varepsilon)$. For the moment, $\varepsilon>0$ may be viewed as a small parameter and $e^{-\varepsilon A}$ as a convergence factor in some of the subsequent formulas; in section three we show how this is related to deformations of w_ε. For $\varepsilon\geq 0$, we also introduce $R_\varepsilon: H\to H$ by

$$R_\varepsilon=\int_0^\infty e^{-tA}B_\varepsilon C_\varepsilon e^{-tA}\,dt,\tag{11}$$

and this integral plainly converges whenever $\omega_0>0$. See Ref 5 for some related results.

For Whittaker functions, the basic linear system is the following. Let $\varepsilon>0$, $H_0=\mathbf{C}$ and $H=L^2((1/2,\infty);\mathbf{C})$ and $\mathcal{D}(A)=\{f(s)\in H: sf(s)\in H\}$. Then

$$A: f(s)\mapsto sf(s),\qquad (f\in\mathcal{D}(A));$$

$$B_\varepsilon: b\mapsto e^{-\varepsilon s}(s+1/2)^{(\kappa+\mu-1/2)/2}(s-1/2)^{(-\kappa+\mu-1/2)/2}b,\qquad (b\in\mathbf{C});$$

$$C_\varepsilon: f(s)\mapsto \int_{1/2}^\infty e^{-\varepsilon s}\frac{(s+1/2)^{(\kappa+\mu-1/2)/2}(s-1/2)^{(-\kappa+\mu-1/2)/2}}{\Gamma(\mu-\kappa+1/2)}f(s)ds,\qquad (f\in\mathcal{D}(A)).\tag{12}$$

2.1 Lemma *(i) Suppose that $\Re\mu>\Re\kappa-1/2$. Then the scattering function of $(-A,B_\varepsilon,C_\varepsilon)$ is*

$$\varphi_{(\varepsilon)}(t)=\frac{W_{\kappa,\mu}(t+2\varepsilon)}{(t+2\varepsilon)^{\mu+1/2}}.\tag{13}$$

(ii) Suppose that $\kappa,\mu\in\mathbf{R}$ and $1/2>\mu>\kappa-1/2$. Then R_ε is self-adjoint and nonnegative, and $R_\varepsilon\to R_0$ in trace class norm as $\varepsilon\to 0+$.

(iii) Suppose that $\Re\mu > \Re\kappa - 1/2$, and either $\varepsilon > 0$, or $\Re\mu < 1/2$ and $\varepsilon = 0$. Then R_ε is trace class and

$$\det(I - \lambda R_\varepsilon) = \det(I - \lambda\Gamma_{\varphi_{(\varepsilon)}}) \qquad (\lambda \in \mathbf{C}). \tag{14}$$

Proof. (i) This is by direct computation of $\varphi_{(\varepsilon)}(t) = C_\varepsilon e^{-tA}B_\varepsilon$, which simplifies when one uses a representation formula

$$\varphi_{(\varepsilon)}(t) = \frac{W_{\kappa,\mu}(t+2\varepsilon)}{(t+2\varepsilon)^{\mu+1/2}} = \int_{1/2}^\infty e^{-s(t+2\varepsilon)}(s+1/2)^{(\kappa+\mu-1/2)}(s-1/2)^{(-\kappa+\mu-1/2)}\frac{ds}{\Gamma(-\kappa+\mu+1/2)} \qquad (t>0) \tag{15}$$

for the Whittaker function from Refs 15, 6.113(18); 17, (9.222), in which the integral is absolutely convergent. Clearly, replacing t by $t+2\varepsilon$ is equivalent to multiplying the integrand by $e^{-2\varepsilon s}$. Taking $\varepsilon \to 0+$, we obtain $\varphi_\varepsilon(t) \to \varphi(t) = W_{\kappa,\mu}(t)/t^{\mu+1/2}$.

(ii) The operator R_ε on $L^2(1/2,\infty)$ is represented by the kernel

$$\frac{e^{-\varepsilon s}(s+1/2)^{(\kappa+\mu-1/2)/2}}{(s-1/2)^{(\kappa-\mu+1/2)/2}}\frac{e^{-\varepsilon t}(t+1/2)^{(\kappa+\mu-1/2)/2}}{(t-1/2)^{(\kappa-\mu+1/2)/2}}\frac{1}{(s+t)\Gamma(-\kappa+\mu+1/2)}. \tag{16}$$

The first two factors are multiplication operators in s and t, while the final factor is the compression of Carleman's operator $\Gamma \in \mathcal{L}(L^2(0,\infty))$ to the bounded linear operator on $L^2(1/2,\infty)$ with kernel $1/(x+y)$, as discussed in Ref 29, p.440. Now Γ is non negative as an operator, hence R_ε is also non negative by Eqn (16). Since $R_\varepsilon \geq 0$, the trace class norm of R_ε equals trace(R_ε). Now R_0 has kernel

$$\frac{(s+1/2)^{(\kappa+\mu-1/2)/2}}{(s-1/2)^{(\kappa-\mu+1/2)/2}}\frac{(t+1/2)^{(\kappa+\mu-1/2)/2}}{(t-1/2)^{(\kappa-\mu+1/2)/2}}\frac{1}{(s+t)\Gamma(-\kappa+\mu+1/2)},$$

and

$$\mathrm{trace}(R_0) = \int_{1/2}^\infty \frac{(s+1/2)^{(\kappa+\mu-1/2)}}{(s-1/2)^{(\kappa-\mu+1/2)}}\frac{1}{2s\Gamma(-\kappa+\mu+1/2)}ds < \infty.$$

As $\varepsilon \to 0+$, we obtain trace$(R_\varepsilon) \to$ trace(R_0) by the dominated convergence theorem and $R_\varepsilon \to R_0$ in the weak operator topology. Hence $\|R_\varepsilon - R_0\|_{c^1} \to 0$ by Arazy's convergence theorem from Ref 1. See Ref 19 for more analysis of operators of this form.

(iii) One can introduce operators $\Xi_\varepsilon, \Theta_\varepsilon : L^2(0,\infty) \to L^2(1/2,\infty)$ by $\Theta_\varepsilon f = \int_0^\infty e^{-tA^\dagger}C_\varepsilon^\dagger f(t)\,dt$ so

$$\Theta_\varepsilon f(s) = \int_0^\infty \frac{e^{-ts-\varepsilon s}(s+1/2)^{(\kappa+\mu-1/2)/2}(s-1/2)^{(-\kappa+\mu-1/2)/2}}{\Gamma(\mu-\kappa-1/2)}f(t)\,dt \qquad (s>1/2)$$

and $\Xi_\varepsilon f = \int_0^\infty e^{-tA}B_\varepsilon f(t)\,dt$ such that $\Gamma_{\varphi_{(\varepsilon)}} = \Theta_\varepsilon^\dagger \Xi_\varepsilon$ and $R_\varepsilon = \Xi_\varepsilon\Theta_\varepsilon^\dagger$; see Ref 5 for more details. For $\varepsilon > 0$, it is straightforward to check that Θ_ε and likewise Ξ_ε are Hilbert–Schmidt. The kernel of Ξ_ε is

$$e^{-\varepsilon s - st}(s+1/2)^{(\kappa+\mu-1/2)/2}(s-1/2)^{(-\kappa+\mu-1/2)/2} \qquad (s>1/2, t>0), \tag{17}$$

which is Hilbert–Schmidt since

$$\int_{1/2}^\infty\int_0^\infty e^{-2\varepsilon s-2st}(s+1/2)^{(\Re\kappa+\Re\mu-1/2)}(s-1/2)^{(-\Re\kappa+\Re\mu-1/2)}dtds < \infty$$

for $-\Re\kappa + \Re\mu - 1/2 > -1$, since $e^{-\varepsilon s} \to 0$ rapidly as $s \to \infty$. Hence R_ε and likewise $\Gamma_{\varphi_{(\varepsilon)}}$ are trace class with

$$\det(I - \lambda R_\varepsilon) = \det(I - \lambda\Xi_\varepsilon\Theta_\varepsilon^\dagger) = \det(I - \lambda\Theta_\varepsilon^\dagger\Xi_\varepsilon) = \det(I - \lambda\Gamma_{\varphi_{(\varepsilon)}}). \tag{18}$$

For $\alpha \geq 0$, let $L_n^{(\alpha)}(x)$ be the generalized Laguerre polynomial of degree n, defined by

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{\Gamma(n+1)}\frac{e^x}{x^\alpha}\frac{d^n}{dx^n}\big(x^{n+\alpha}e^{-x}\big); \tag{19}$$

then the n^{th} generalized Laguerre function is $\phi_n^{(\alpha)}(x) = e^{-x/2}x^\alpha L_n^{(\alpha)}(x)$. See 8.970 of Ref 17.

2.2 Lemma *Let φ be the scattering function $\varphi(x) = W_{\kappa,\mu}(x)x^{-\mu-1/2}$, and let w_0 be the Jacobi weight*

$$w_0(\xi) = \xi^{\mu-\kappa-1/2}(1-\xi)^{2\alpha-2\mu+1} \qquad (0 < \xi < 1). \tag{20}$$

Then the operation of Γ_φ on the generalized Laguerre basis is represented by a matrix of moments for w_0, so

$$\big\langle \Gamma_\varphi \phi_\ell^{(\alpha)}, \phi_n^{(\alpha)}\big\rangle_{L^2(0,\infty)} = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}\frac{\Gamma(\ell+1+\alpha)}{\Gamma(\ell+1)}\int_0^1 \xi^{\ell+n}w_0(\xi)d\xi. \tag{21}$$

For $\alpha = 0$, this reduces to the Hankel matrix of w_0.

Proof. This is suggested by Ref 30. The Laplace transform of $\phi_n^{(\alpha)}$ satisfies

$$\hat{\phi}_n^{(\alpha)}(s) = \frac{1}{\Gamma(\alpha+n)}\frac{(s-1/2)^n}{(s+1/2)^{n+1+\alpha}}, \tag{22}$$

as one checks by repeatedly integrating by parts. Using the representation formula for φ as in Eqn (15), one can express $\langle \Gamma_\varphi \phi_n^{(\alpha)}, \phi_\ell^{(\alpha)}\rangle$ as an integral with respect to s over $(1/2, \infty)$. By changing variables to $\xi = (s-1/2)/(s+1/2)$, one obtains the integral of moments with respect to the weight w_0.

We now show how the leading minors of the Hankel operator Γ_φ are related to the Jacobi unitary ensemble. The joint probability density function of the Jacobi unitary ensemble on $[0,1]^N$ as in Refs 25, 36 is

$$\frac{1}{N!}\frac{1}{\Gamma(\kappa+\mu+1)^N}\prod_{j=0}^{N-1}\frac{\Gamma(j+1+\alpha)}{\Gamma(j+1)}\prod_{0\leq j<k\leq N-1}(x_j-x_k)^2\prod_{j=0}^{N-1}w_0(x_j). \tag{23}$$

Let $\Delta_N(t)$ be the multiple integral

$$\Delta_N(t) = \frac{1}{N!}\frac{1}{\Gamma(\kappa+\mu+1)^N}\prod_{j=0}^{N-1}\frac{\Gamma(j+1+\alpha)}{\Gamma(j+1)}\int_{[t,1]^N}\prod_{0\leq j<k\leq N-1}(x_j-x_k)^2\prod_{j=0}^{N-1}w_0(x_j)dx_j, \tag{24}$$

as in Chen and Zhang[12].

2.3 Proposition *(i) The leading minors of the determinant of Γ_φ satisfy*

$$\det\Big[\big\langle \Gamma_\varphi \phi_n^{(\alpha)}, \phi_\ell^{(\alpha)}\big\rangle\Big]_{\ell,n=0}^{N-1} = \Delta_N(0). \tag{25}$$

(ii) Let $x_0, \dots, x_{N-1}$ be a sample of N points from the Jacobi unitary ensemble. Then the probability of the event $[x_j \geq t$ for all $j]$ equals $\Delta_N(t)/\Delta_N(0)$.

Proof. (i) This identity follows from Lemma 2.2 and the Heine–Andreief identity from page 27 of Ref 33.

(ii) Let $P_n^{(a,b)}(x)$ be the monic Jacobi polynomial of degree n for the weight $w_0(x) = x^b(1-x)^a$ on $[0,1]$, where we choose $b = \mu+\kappa-1/2$ and $a = 2\alpha-2\mu+1$. See Ref 33, page 58. Then with the constants $\gamma_j = \int P_j^{(a,b)}(x)^2 w_0(x)dx$, the kernel

$$J_N(x,y) = \sum_{j=0}^{N-1}\frac{P_j^{(a,b)}(x)P_j^{(a,b)}(y)}{\gamma_j} \tag{26}$$

defines a self-adjoint operator on $L^2(w_0,[0,1])$ such that $0 \leq J_N \leq I$. Hence

$$\frac{\Delta_N(t)}{\Delta_N(0)} = \det\big(I - J_N \mathbf{I}_{(0,t)}\big). \tag{27}$$

2.4 Proposition *Let*

$$H_N(t) = t(1-t)\frac{d}{dt}\log\Delta_N(t). \tag{28}$$

Then $\sigma(t) = H_N(t) - d_1 - td_2$ *satisfies the* σ *form of Painlevé's transcendental differential equation* PVI*, so*

$$\sigma'\big[t(t-1)\sigma''\big]^2 + \big[2\sigma'(t\sigma'-\sigma) - (\sigma')^2 - \nu_1\nu_2\nu_3\nu_4\big]^2 = (\sigma'+\nu_1^2)(\sigma'+\nu_2^2)(\sigma'+\nu_3^2)(\sigma'+\nu_4^2) \tag{29}$$

where $\nu_1 = (a+b)/2$, $\nu_2 = (b-a)/2$, $\nu_3 = \nu_4 = (2N+a+b)/2$ *with initial conditions* $\sigma(0) = d_2$ *and* $\sigma'(0) = d_1$.

Proof. See Theorem 1 of of Ref 12, and Ref 22.

3. DETERMINANT FORMULAS FOR THE POLLACZEK–JACOBI TYPE WEIGHT

In this section, we consider the Pollaczek–Jacobi type weights, and show that translating the scattering function $\varphi(t) \mapsto \varphi(t+2\varepsilon)$ has the same effect as deforming the weight $w_0(x)$ through multiplication by $e^{-2\varepsilon/x}$. The m^{th} moment of the Pollaczek–Jacobi weight is defined to be

$$\mu_m(t;a,b) = \int_0^1 x^m x^b (1-x)^a e^{-t/x}\,dx \qquad (m = 0,1,\dots). \tag{30}$$

3.1 Proposition *(i) The moments satisfy*

$$\mu_m(t;a,b) = \Gamma(a+1)e^{-t/2}t^{(b+m)/2}W_{-(2a+b+m+2)/2,-(b+m+1)/2}(t). \tag{31}$$

(ii) The translation operation $\varphi(t) \mapsto \varphi_{(\varepsilon)}(t) = \varphi(t+2\varepsilon)$ *is equivalent to replacing* w_0 *by*

$$w_{(\varepsilon)}(x) = w_0(x)\exp\Big(-2\varepsilon\Big(\frac{1}{x}-\frac{1}{2}\Big)\Big). \tag{32}$$

Proof. (i) Making the change of variable $1/x = s+1/2$ in Eqn (30) we have

$$\mu_m(t;a,b) = e^{-t/2}\int_{1/2}^{\infty} e^{-st}(s-1/2)^{-\kappa+\mu-1/2}(s+1/2)^{\kappa+\mu-1/2}\,ds \tag{33}$$

with $\kappa = -(2a+b+m+2)/2$ and $\mu = -(b+m+1)/2$. See Ref 17, Eqn 9.222.

(ii) As in Eqn (15), we have

$$\varphi_{(\varepsilon)}(t) = \int_{1/2}^{\infty} e^{-s(t+2\varepsilon)}(s+1/2)^{(\kappa+\mu-1/2)}(s-1/2)^{(-\kappa+\mu-1/2)}\frac{ds}{\Gamma(-\kappa+\mu+1/2)}, \tag{34}$$

which involves an extra factor of $e^{-2\varepsilon s}$, and we recover $w_{(\varepsilon)}$ when change the variables from $s \in (1/2,\infty)$ back to $x \in (0,1)$.

For this problem, we shall be concerned with

$$D_N(\varepsilon) = \prod_{j=0}^{N-1}\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}[\Gamma(\kappa-\mu+(1/2))]^{-N}\frac{1}{N!}\int_{[0,1]^N}\prod_{0\le j<k\le N-1}(x_k-x_j)^2\prod_{\ell=1}^{N}w_{(\varepsilon)}(x_\ell)dx_\ell. \tag{35}$$

Hence a straightforward change of variables gives

$$D_N(\varepsilon) = C_N(\varepsilon)\mathcal{D}_N(2\varepsilon;a,b) \tag{36}$$

where $\mathcal{D}_N(t;a,b)$ is defined in Eqn (6) and

$$C_N(\varepsilon) = e^{\varepsilon N}[\Gamma(\kappa-\mu+1/2)]^{-N}\prod_{j=0}^{N-1}\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}. \tag{37}$$

As in Lemma 2.2 and Proposition 2.3, we have

$$D_N(\varepsilon) = \det\Big[\big\langle \Gamma_{\varphi(\varepsilon)}\phi_n^{(\alpha)}, \phi_\ell^{(\alpha)}\big\rangle\Big]_{\ell,n=0}^{N-1}. \tag{38}$$

Let the quantity $\tilde{H}_N(t)$ be defined as follows

$$\tilde{H}_N(t) = t\frac{d}{dt}\log \mathcal{D}_N(t) - N(N+b+a). \tag{39}$$

3.2 Theorem *Then $\tilde{H}_N(t)$ satisfies the Jimbo–Miwa–Okamoto σ form of Painlevé's PV for a special choice of parameters, so*

$$(t\tilde{H}_N'')^2 = -4t(\tilde{H}_N')^3 + (\tilde{H}_N')^2\big(4\tilde{H}_N + (b+2a+t)^2 + 4N(N+a+b) - 4a(b+a)\big)$$

$$+2\tilde{H}_N'\big(-(b+2a+t)\tilde{H}_N - 2Na(N+b+a)\big) + (\tilde{H}_N)^2. \tag{40}$$

Proof. This was found in Chen and Dai[11] page 2161.

3.3 Remarks (i) Chen and Dai[11] Theorem 6.1 also show that $(\tilde{H}_N)_{N=1}^{\infty}$ satisfies a second order nonlinear difference equation.

(ii) $\mathcal{D}_N(t;a,b)$ is the Wronskian determinant of

$$\{\mu_{2N-1}(t;a,b), \mu_{2N-1}'(t;a,b), \dots, \mu_{2N-1}^{(N-1)}(t;a,b)\}; \tag{41}$$

thus $\mu_{2N-1}(t;a,b)$ and its derivatives with respect to t determine $\mathcal{D}_N(t;a,b)$.

(iii) The modified Bessel function as in Eqn (83) satisfies $K_\mu(z) = (2z/\pi)^{-1/2}W_{0,\mu}(2z)$ as in Ref 31; so in view of these results, it is fitting that PV should emerge from the Whittaker kernel.

(iv) Tracy and Widom[37] considered the ensembles $U(N)$ of $N\times N$ complex unitary matrices U with Haar measure, and obtained PV from the distribution of trace(U). Remarkably, this is related to the uniform measure on the symmetric group S_N of permutations as $N\to\infty$. Borodin and Olshanski[8] page 98 considered the pseudo-Jacobi ensemble and obtained PV from a Fredholm determinant associated with the Whittaker functions $M_{\kappa,\mu}(1/x)$ with κ and μ real. Their results have applications to the infinite dimensional unitary group $U(\infty)$.

4. THE MATRIX WHITTAKER KERNEL

Borodin and Olshanski[7,8] have considered kernels of the form of Eqn (2). We can factorize the kernel in terms of products of Hankel operators, by analogy to the results of Refs 34, 37. In section 5, we see that the case $\kappa = a + 1/2$ and $\mu = a$ is of particular interest.

4.1 Proposition *The kernel satisfies*

$$\sqrt{zw}\frac{W_{\kappa,\mu}(z)W_{\kappa-1,\mu}(w) - W_{\kappa-1,\mu}(z)W_{\kappa,\mu}(w)}{(w-z)} = \int_1^\infty \Big(\sqrt{z}W_{\kappa,\mu}(sz)\sqrt{w}W_{\kappa-1,\mu}(sw) + \sqrt{z}W_{\kappa-1,\mu}(sz)\sqrt{w}W_{\kappa,\mu}(sw)\Big)\frac{ds}{2s} \tag{42}$$

Proof. Note that the left-hand side is a continuously differentiable function of $(z,w)\in(0,\infty)^2$ and that the left-hand side converges to zero as $z\to\infty$ or $w\to\infty$. In the following proof we use the matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{43}$$

Combining the differential equation (1) with the recurrence relation

$$z\frac{d}{dx}W_{\kappa,\mu}(z) = (\kappa - z/2)W_{\kappa,\mu}(z) - \big(\mu^2 - (\kappa-1/2)^2\big)W_{\kappa-1,\mu}(z), \tag{44}$$

we obtain the matrix differential equation

$$z\frac{d}{dz}W = \big((1/2)I + \Omega(z)\big)W, \tag{45}$$

$$W(z) = \begin{bmatrix} W_{\kappa,\mu}(z) \\ W_{\kappa-1,\mu}(z) \end{bmatrix}, \quad \Omega(z) = \begin{bmatrix} \kappa - 1/2 - z/2 & (\kappa - 1/2)^2 - \mu^2 \\ -1 & 1/2 - \kappa + z/2 \end{bmatrix} \tag{46}$$

in which $\mathrm{trace}(\Omega(z)) = 0$ for all z. The eigenvalues of $(1/2)I + \Omega(0)$ are $(1/2) \pm \mu$, so the eigenvalues differ by an integer if and only if 2μ is an integer; this characterizes the exceptional case for Birkhoff factorization into canonical form Ref 38.

The system in Eqns (45) and (46) resembles the system of matrix differential equation considered by Tracy and Widom[36], although in our paper the trace of the coefficient matrix is non-zero, so we use a variant on their methods as in Ref 3. We compute

$$\Big(z\frac{d}{dz} + w\frac{d}{dw}\Big)\frac{\langle JW(z), W(w)\rangle}{z-w} = \frac{\langle JzW'(z), W(w)\rangle}{z-w} + \frac{\langle JW(z), wW'(w)\rangle}{z-w} - \frac{\langle JW(z), W(w)\rangle}{z-w}$$

$$= \frac{\langle J\Omega(z)W(z), W(w)\rangle}{z-w} + \frac{\langle JW(z), \Omega(w)W(w)\rangle}{z-w}, \tag{47}$$

where $J\Omega(z) + \Omega(w)^t J = -(1/2)(z-w)\tilde{J}$, so

$$\Big(z\frac{d}{dz} + w\frac{d}{dw}\Big)\frac{\langle JW(z), W(w)\rangle}{z-w} = -(1/2)\langle \tilde{J}W(z), W(w)\rangle. \tag{48}$$

For comparison, we have

$$\Big(z\frac{d}{dz} + w\frac{d}{dw}\Big)\frac{1}{2}\int_1^\infty \langle \tilde{J}W(sz), W(sw)\rangle \frac{ds}{s} = \frac{1}{2}\int_1^\infty \Big(\langle \tilde{J}zW'(sz), W(sw)\rangle + \langle \tilde{J}W(sz), wW'(sw)\rangle\Big)ds$$

$$= \frac{1}{2}\int_1^\infty \frac{d}{ds}\langle \tilde{J}W(sz), W(sw)\rangle ds$$

$$= -\frac{1}{2}\langle \tilde{J}W(z), W(w)\rangle. \tag{49}$$

Hence the sum

$$\frac{\langle JW(z), W(w)\rangle}{z-w} - \frac{1}{2}\int_1^\infty \langle \tilde{J}W(sz), W(sw)\rangle \frac{ds}{s} \tag{50}$$

is a function of z/w, which converges to zero as $z \to \infty$ or $w \to 0$ in any way whatever, so this sum is zero. To obtain the stated result, we multiply by $\sqrt{zw}$ and rearrange the terms.

Let K be the integral operator on $L^2((a_1, a_2); dx)$ with kernel

$$K(x, y) = \frac{\langle JW(x), W(y)\rangle}{x-y}. \tag{51}$$

For suitable a_1 and a_2, we can suppose that $\|K\| < 1$ as an operator on $L^2((a_1, a_2); dx)$ and let $S = K(I-K)^{-1}$; so that $(I+S)(I-K) = I$ and $S : L^2((a_1, a_2); dx) \to L^2((a_1, a_2); dx)$ has a kernel

$$S(x, y) = \frac{Q(x)P(y) - Q(y)P(x)}{x-y}, \tag{52}$$

and we take $S(x, x) = Q'(x)P(x) - Q(x)P'(x)$ on the diagonal.

Let $D(a_1, a_2; K)$ be the Fredholm determinant of K, regarded as a function of the endpoints a_1 and a_2.

4.2 Proposition *(i) The restrictions of the kernels on $\{(x, y) \in (a_1, a_2)^2\}$ of the operators S and S^2 to the diagonal $\{(x, x) : x \in (a_1, a_2)\}$ satisfy*

$$x\frac{d}{dx}S(x, x) = -P(x)Q(x) + (S^2)(x, x). \tag{53}$$

(ii) The exterior derivative with respect to the endpoints satisfies

$$d \log D(a_1, a_2; K) = S(a_1, a_1)da_1 - S(a_2, a_2)da_2. \tag{54}$$

Proof. (i) Let M be the operator of multiplication by the independent variable x and D the operator of differentiation with respect to x. For a integral operator T let $\delta T = [MD, T] - T$, so that δT has kernel $(x\partial/\partial x + y\partial/\partial y)T(x, y)$; thus δ is a pointwise derivation on the kernels, while $T \mapsto [MD, T]$ is a derivation on operator composition. We use $\doteq$ to mean that an operator corresponds to a certain kernel. Then

$$[M, K] \doteq W_{\kappa,\mu}(x)W_{\kappa-1,\mu}(y) - W_{\kappa,\mu}(y)W_{\kappa-1,\mu}(x) \tag{55}$$

and Eqn (48) shows that

$$\delta K \doteq -2^{-1}\langle \tilde{J}W(x), W(y)\rangle. \tag{56}$$

Then $[M, S](I - K) = (I + S)[M, K]$, so the kernels have

$$[M, S] \doteq Q(x)P(y) - Q(y)P(x)$$

where $Q = (I-K)^{-1}W_{\kappa,\mu}$ and $P = (I-K)^{-1}W_{\kappa-1,\mu}$ are differentiable functions. Hence S is also an integral operator with kernel of the form of Eqn (52). Also, by some straightforward manipulations, we have

$$\delta S = (I - K)^{-1}(\delta K)(I - K)^{-1} + S^2, \tag{57}$$

where

$$(I - K)^{-1}(\delta K)(I - K)^{-1} \doteq -(1/2)(P(x)Q(y) + P(y)Q(x)) \tag{58}$$

and a short calculation shows that $(\delta S)(x, x) = x(d/dx)S(x, x)$. Hence the result.

(ii) This is a standard consequence of the results from Ref 21.

Let $\phi_{\kappa-1,\mu}(t) = W_{\kappa-1,\mu}(e^t)$ and $\phi_{\kappa,\mu}(t) = W_{\kappa,\mu}(e^t)$ and form the matrix

$$\Phi(t) = \begin{bmatrix} 0 & 0 & \phi_{\kappa,\mu}(t) & 0 \\ 0 & 0 & \phi_{\kappa-1,\mu}(t) & 0 \\ \phi_{\kappa-1,\mu}(t) & \phi_{\kappa,\mu}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{59}$$

(which is not symmetric) and the integral operator on $L^2((0, \infty); \mathbf{C})$ with kernel

$$T \doteq 2\frac{\phi_{\kappa,\mu}(t)\phi_{\kappa-1,\mu}(u) - \phi_{\kappa,\mu}(u)\phi_{\kappa-1,\mu}(t)}{e^t - e^u}. \tag{60}$$

Definition (i) Let K be a bounded linear operator on Hilbert space H. As in page 126 of Ref 29, we define the singular values of K to be $(s_j(K))_{j=0}^{\infty}$ where $s_j(K) = \inf\{\|K - R\|_{\mathcal{L}(H)} : R \in \mathcal{L}(H), \mathrm{rank}(R) \leq j\}$.

(ii) We say that a complex sequence $(z_j)_{j=0}^{\infty}$ decays rapidly if $j^p z_j \to 0$ as $j \to \infty$ for all $p \in \mathbf{N}$.

4.3 Corollary *Let $0 > \Re\mu > \Re\kappa - 1/2 > -1/2$. Let $(\lambda_j(T))_{j=0}^\infty$ be the eigenvalues of T, listed according to algebraic multiplicity in order of decreasing modulus, so $|\lambda_0(T)| \geq |\lambda_1(T)| \geq \dots$. Then*

(i) the Hankel operator with scattering function Φ operates on $L^2((0,\infty); \mathbf{C}^4)$ and satisfies

$$\det(I - T) = \det(I + \Gamma_\Phi); \tag{61}$$

(ii) $(s_j(T))_{j=0}^\infty$ decays rapidly, so that $j^p s_j(T) \to 0$ as $j \to \infty$ for all $p \in \mathbf{N}$;

(iii) $(\lambda_j(T))_{j=0}^\infty$ decays rapidly.

Proof. (i) By Proposition 4.1, with the change of variable $s = e^u$, we see that the integral operator satisfies

$$T = \Gamma_{\phi_{\kappa-1,\mu}}\Gamma_{\phi_{\kappa,\mu}} + \Gamma_{\phi_{\kappa,\mu}}\Gamma_{\phi_{\kappa-1,\mu}} \tag{62}$$

where Γ_ϕ is in the standard form of a Hankel operator on $L^2(0,\infty)$ as in Eqn (10). By some elementary determinant manipulations, the Hankel operator with matrix valued scattering function satisfies

$$\det(I + \Gamma_\Phi) = \det(I - \Gamma_{\phi_{\kappa-1,\mu}}\Gamma_{\phi_{\kappa,\mu}} - \Gamma_{\phi_{\kappa,\mu}}\Gamma_{\phi_{\kappa-1,\mu}}),$$

which produces the stated result.

(ii) By Eqn (62) and page 705 of Ref 29, we have the inequality

$$s_{2j}(T) \leq 2\|\Gamma_{\phi_{\kappa-1,\mu}}\|_{\mathcal{L}(L^2)} s_j(\Gamma_{\phi_{\kappa,\mu}})$$

which is an easy consequence of the definition of singular values. So it suffices to show that the singular values σ_j of $\Gamma_{\phi_{\kappa,\mu}}$ satisfy $j^p\sigma_j \to 0$ as $j \to \infty$ for all $p \in \mathbf{N}$. To do this, we replace the Hankel integral operator by a Hankel matrix.

Let $\psi_k(s) = e^{-s}L_k^{(0)}(2s)$ so that $(\psi_k)_{k=0}^\infty$ gives an orthonormal basis of $L^2(0,\infty)$, and let

$$\gamma_k = \int_{-\infty}^\infty \phi_{\kappa,\mu}(s)\psi_k(s)ds \tag{63}$$

so that $\Gamma_{\phi_{\kappa,\mu}}$ is represented by the Hankel matrix $G = [\gamma_{j+k-1}]_{j,k=1}^\infty$ with respect to $(\psi_j)_{j=0}^\infty$ as in Theorem 1.8.9 of Ref 29.

We now prove that γ_k decays rapidly as $k \to \infty$. By Plancherel's formula we have

$$\gamma_k = \frac{i}{2\pi}\int_{-\infty}^\infty \hat{\phi}_{\kappa,\mu}(\xi)\frac{(\xi - i)^k}{(\xi + i)^{k+1}}d\xi. \tag{64}$$

in which the Fourier transform of $\phi_{\kappa,\mu}(t) = W_{\kappa,\mu}(e^t)$ is

$$\hat{\phi}_{\kappa,\mu}(\xi) = \int_{-\infty}^\infty \phi_{\kappa,\mu}(t)e^{-i\xi t}\,dt = \int_0^\infty z^{-i\xi-1}W_{\kappa,\mu}(z)dz.$$

We recognise this as a Mellin transform of $W_{\kappa,\mu}(z)$, which is an absolutely convergent integral by Eqn (15) since $W_{\kappa,\mu}(z) = O(z^{\Re\mu+1/2})$ as $z \to 0+$; also, substituting Eqn (15) and changing the order of integration, this reduces to

$$\hat{\phi}_{\kappa,\mu}(\xi) = \frac{\Gamma(\mu + (1/2) - i\xi)}{\Gamma(\mu - \kappa + (1/2))}\int_0^\infty \frac{t^{\mu-\kappa-1/2}(1+t)^{\mu+\kappa-1/2}}{(1/2+t)^{\mu+1/2-i\xi}}dt. \tag{65}$$

The latest integral converges for $1/2 - \Re(i\xi) > \Re\mu > \Re\kappa - 1/2$, and is easy to bound from above for the stated parameters. From Stirling's asymptotic formula for the Gamma function in Eqn (65), we deduce that there exist $\delta_1, \delta_2 > 0$ such that $\hat{\phi}_{\kappa,\mu}$ is analytic on the horizontal strip $\{\xi \in \mathbf{C} : |\Im\xi| < \delta_1\}$ and $\hat{\phi}_{\kappa,\mu}(\xi) = O(e^{-\delta_2|\xi|})$ as $\Re\xi \to \pm\infty$ along the strip. As in Proposition 6.3 of Ref 4, we can estimate Eqn (64) by shifting the line of integration and thereby obtain convergence $k^p\gamma_k \to 0$ as $k \to \infty$ for all $p \in \mathbf{N}$.

By some simple matrix approximation arguments, this implies that σ_n also decays rapidly as $n \to \infty$. Indeed, the matrix $G_n = [\gamma_{j+k-1}\mathbf{I}_{j+k\leq n}(j,k)]_{j,k=1}^{\infty}$ is zero outside the top left corner and has rank less than or equal to n, so by definition of singular values, we deduce that

$$\sigma_n = s_n(G) \leq \|G - G_n\|_{\mathcal{L}(\ell^2)} \leq \|G - G_n\|_{HS} \leq \Big(\sum_{m=n}^{\infty}(m+1)\gamma_m^2\Big)^{1/2},$$

which decays rapidly as $n \to \infty$. Hence $s_n(T)$ decays rapidly as $n \to \infty$.

(iii) By Weyl's inequality, we have

$$|\lambda_n(T)|^{n+1} \leq \prod_{j=0}^{n}|\lambda_j(T)| \leq \prod_{j=0}^{n} s_j(T) \qquad (n = 1, 2, \ldots),$$

where in which $s_j(T) \leq M_p/(1+j)^p$ for some $M_p > 0$ and all j by (ii). By Stirling's formula, there exists $C_p > 0$ such that $|\lambda_n(T)| \leq C_p/n^p$ for all $n \in \mathbf{N}$.

4.4 Remark By Ref 17, 9.237, Whittaker's differential equation generalizes the differential equation that the associated Laguerre functions satisfy. The column vector $Y(z) = \sqrt{z}W(z)$ satisfies $z(d/dz)Y(z) = \Omega(z)Y(z)$, which resembles the differential equation for generalized Laguerre functions on Ref 36, page 60. In Remark 5.2 of Ref 4 we obtained a factorization theorem for certain Whittaker kernels which expressly excluded the case of generalized Laguerre functions.

5. A SPECIAL CASE OF THE WHITTAKER KERNEL

For $-1/2 < a < 1/2$, let R_a be the integral operator on $L^2(0,\infty)$ that has kernel

$$R_a \doteq \frac{(y/x)^a e^{-(x+y)/2}}{x+y} \tag{66}$$

and $\mathcal{R}^{(a)}$ be the operator on $L^2((0,\infty);\mathbf{C}^2)$ given by the block matrix

$$\mathcal{R}^{(a)} = \begin{bmatrix} 0 & R^{(a)} \\ -R^{(-a)} & 0 \end{bmatrix} \tag{67}$$

and let Ψ be the matrix-valued scattering function

$$\Psi(t) = \begin{bmatrix} 0 & \Gamma(1-2a)/(1+t)^{1-2a} \\ -\Gamma(1+2a)/(1+t)^{1+2a} & 0 \end{bmatrix} \qquad (t > 0). \tag{68}$$

5.1 Proposition *(i) The operators $R^{(\pm a)}$ are bounded on $L^2(0,\infty)$ for $|a| < 1/2$.*
(ii) For all $0 < a_1 < b_1 < \infty$, the operators $\mathcal{R}^{(a)}$ and Γ_Ψ are trace class on $L^2((a_1,b_1);\mathbf{C}^2)$ and satisfy

$$\det(I - \lambda\mathcal{R}^{(a)}) = \det(I - \lambda\Gamma_\Psi) \qquad (\lambda \in \mathbf{C}). \tag{69}$$

(iii) The operator $\mathcal{R}^{(a)}$ is skew self-adjoint, and $I + \mathcal{R}^{(a)}$ is invertible, and they satisfy in block matrix form

$$\mathcal{R}^{(a)}(I+\mathcal{R}^{(a)})^{-1} = \begin{bmatrix} R^{(a)}R^{(-a)}(I+R^{(a)}R^{(-a)})^{-1} & -R^{(a)}(I+R^{(-a)}R^{(a)})^{-1} \\ -R^{(-a)}(I+R^{(a)}R^{(-a)})^{-1} & R^{(-a)}R^{(a)}(I+R^{(-a)}R^{(a)})^{-1} \end{bmatrix}. \tag{70}$$

(iv) The integral operator $R^{(a)}R^{(-a)}$ is represented by the kernel

$$\Gamma(1-2a)\frac{W_{a+1/2,a}(x)W_{a-1/2,a}(y) - W_{a-1/2,a}(x)W_{a+1/2,a}(y)}{(x-y)\sqrt{xy}}. \tag{71}$$

Proof. (i) This was proved by Olshanski[28] using a Fourier argument. For completeness we give an equivalent proof involving Mellin transforms. First we check that the operators $R^{(\pm a)}$ are bounded on $L^2(0,\infty)$. The expression

$$\int_0^\infty \frac{(x/y)^a}{(x/y)+1}\frac{f(y)dy}{y} = \frac{1}{\sqrt{x}}\int_0^\infty \frac{(x/y)^a}{\sqrt{(x/y)}+\sqrt{(y/x)}}\frac{\sqrt{y}f(y)\,dy}{y} \tag{72}$$

is a Mellin convolution as in Ref 31, and one shows by a standard contour integration argument that

$$\int_0^\infty \frac{z^{a+i\sigma-1/2}dz}{z+1} = \pi\operatorname{sech}\pi(a+i\sigma) \tag{73}$$

is bounded for all $\sigma\in\mathbf{R}$ for all $|a|<1/2$.

(ii) This follows as in Lemma 2.1. Letting M_2 be the 2×2 complex matrices with Hilbert–Schmidt norm, we introduce the state space $H = L^2((\omega_0,\infty); M_2)$ and the input and output space $H_0=\mathbf{C}^2$, viewed as column vectors, then

$$\mathcal{D}(A)\to H:\quad f(x)\mapsto xf(x);$$

$$B_{\omega_0}: H_0\to H: b\mapsto \begin{bmatrix} 0 & e^{-x/2}x^{-a}\\ -e^{-x/2}x^a & 0\end{bmatrix} b;$$

$$C_{\omega_0}: H\to H_0:\quad g(y)\mapsto \int_{\omega_0}^\infty \begin{bmatrix} e^{-y/2}y^{-a} & 0\\ 0 & e^{-y/2}y^a\end{bmatrix} g(y)\,dy. \tag{74}$$

Then $\Vert e^{-tA}\Vert\leq e^{-t\omega_0}$ for all $t>0$, and the scattering function is

$$C_{\omega_0}e^{-tA}B_{\omega_0} = \int_{\omega_0}^\infty \begin{bmatrix} 0 & e^{-ty-y}y^{-2a}\\ -e^{-ty-y}y^{2a} & 0\end{bmatrix} dy; \tag{75}$$

and we note that

$$C_{\omega_0}e^{-tA}B_{\omega_0}\to\Psi(t) \tag{76}$$

as $\omega_0\to 0+$; also the 'corresponding R operator' for the linear system $(-A, B_{\omega_0}, C_{\omega_0})$ has 2×2 matrix kernel

$$\mathcal{R}^{(a)}_{\omega_0} = \int_0^\infty e^{-tA}B_{\omega_0}C_{\omega_0}e^{-tA}dt; \tag{77}$$

so as $\omega_0\to 0$, we obtain the operator from Eqn (67), namely

$$\mathcal{R}^{(a)} \doteq \int_0^\infty \begin{bmatrix} 0 & e^{-t(x+y)-(x+y)/2}(y/x)^a\\ -(x/y)^a e^{-t(x+y)-(x+y)/2} & 0\end{bmatrix} dt$$

$$\doteq \begin{bmatrix} 0 & (y/x)^a e^{-(x+y)/2}/(x+y)\\ -(x/y)^a e^{-(x+y)/2}/(x+y) & 0\end{bmatrix}. \tag{78}$$

(iii) Note that $R^{(a)*}=R^{(-a)}$ hence $\mathcal{R}^{(a)*}=-\mathcal{R}^{(a)}$ and $I+R^{(-a)}R^{(a)}$ is invertible. By elementary row operations one checks that $I+\mathcal{R}^{(a)}$ has inverse

$$\begin{bmatrix} I & R^{(a)}\\ -R^{(-a)} & I\end{bmatrix}^{-1} = \begin{bmatrix} (I+R^{(a)}R^{(-a)})^{-1} & -R^{(a)}(I+R^{(-a)}R^{(a)})^{-1}\\ R^{(-a)}(I+R^{(a)}R^{(-a)})^{-1} & (I+R^{(-a)}R^{(a)})^{-1}\end{bmatrix}, \tag{79}$$

in which we observe that the indices a and $-a$ alternate.

(iv) We have

$$R^{(a)}R^{(-a)} \doteq \int_0^\infty \frac{(x/y)^a e^{-(x+y)/2}}{x+y}\frac{(y/z)^{-a}e^{-(y+z)/2}}{y+z}dy$$

which we write as partial fractions

$$R^{(a)}R^{(-a)} \doteq \frac{(xz)^a e^{-(x+z)/2}}{z-x}\int_0^\infty \Bigl(\frac{1}{x+y}-\frac{1}{y+z}\Bigr) y^{-2a}e^{-y}\,dy, \tag{80}$$

where Refs 16, 14.2(17) and page 431 give the final integral in terms of an incomplete Gamma function which reduces to Whittaker's function

$$\int_0^\infty \frac{y^{-2a}e^{-y}}{x+y}dy = \Gamma(1-2a)x^{-a-1/2}e^{x/2}W_{a-1/2,a}(x). \tag{81}$$

We also have the identity from Ref 16, p 432

$$x^a e^{-x/2} = x^{-1/2}W_{a+1/2,a}(x). \tag{82}$$

The result follows on substituting Eqns (81) and (82) into (80).

6. DIAGONALIZING THE WHITTAKER KERNEL

The Whittaker's functions are related to Bessel's functions and Laguerre's functions. Let $K_\nu(x)$ be the modified Bessel function of the second kind given by

$$K_\nu(x) = \int_0^\infty \cosh(\nu t)e^{-x\cosh t}\,dt, \tag{83}$$

which is analytic on the open right half plane $\{x : \Re x > 0\}$ and decays rapidly as $x \to \infty$ through real values. When $\nu = im$ is purely imaginary, one refers to McDonald's function as in Ref 31.

Let $L^{(a)}$ be the differential operator

$$L^{(a)}f = -\frac{d}{dx}\Bigl(x^2\frac{df}{dx}\Bigr) + \bigl(x/2+a\bigr)^2 f(x). \tag{84}$$

Then $L^{(-\kappa)}$ has eigenfunctions

$$f_{\kappa,m}(x) = x^{-1}W_{\kappa,im}(x) \qquad (m \geq 0), \tag{85}$$

so that

$$L^{(-\kappa)}f_{\kappa,m}(x) = \bigl(1/4+m^2+\kappa^2\bigr)f_{\kappa,m}(x). \tag{86}$$

6.1 Proposition *(i) The differential operator $L^{(-a)}$ commutes with $R^{(a)}R^{(-a)}$ on $C_c^\infty((0,\infty);\mathbf{C})$, and*

$$L^{(-a)}R^{(a)} = R^{(a)}L^{(a)}. \tag{87}$$

(ii) $\mathcal{R}^{(a)}$ can be expressed as a diagonal operator with respect to $(f_{\pm a,m})_{m\geq 0}$ for $-1/2 < a < 1/2$.

Proof. (i) Suppose that f is smooth and has compact support inside $(0,\infty)$. Then we can repeatedly integrate by parts the integral

$$\int_0^\infty \frac{(y/x)^a e^{-(x+y)/2}}{(x+y)}\Bigl(-\frac{d}{dy}\Bigl(y^2\frac{df}{dy}\Bigr) + (y/2+a)^2 f(y)\Bigr)dy \tag{88}$$

without introducing boundary terms, to obtain

$$\int_0^\infty \frac{(x/y)^a e^{-(x+y)/2}}{x+y}\Bigl(\frac{y^2}{4}+a^2+\frac{y^2}{(x+y)^2}+ay+\frac{y^2}{x+y}+\frac{2ay}{x+y}-a-y+\frac{y^2}{(x+y)^2}-\frac{2y}{x+y}\Bigr)f(y)\,dy. \tag{89}$$

After a little reduction, one shows that this coincides with

$$\Big(-x^2\frac{d^2}{dx^2}-2x\frac{d}{dx}+(x/2-a)^2\Big)\int_0^\infty \frac{(y/x)^a e^{-(x+y)/2}}{x+y}f(y)\,dy. \tag{90}$$

Likewise, we have $L^{(a)}R^{(-a)}=R^{(-a)}L^{(-a)}$; so $L^{(-a)}$ commutes with $R^{(a)}R^{(-a)}$.

(ii) Erdélyi[16] in equations 14.3 (53) quotes the following formula for the Stieltjes transform

$$\int_0^\infty \frac{y^{-1-a}e^{-y/2}}{x+y}W_{-a,im}(y)dy=\Gamma((1/2)-a+im)\Gamma((1/2)-a-im)x^{-a-1}e^{x/2}W_{a,im}(x), \tag{91}$$

for $a<1/2$ so $f_{-a,m}(x)=x^{-1}W_{-a,im}(x)$ satisfies

$$R^{(a)}f_{-a,m}(x)=\Gamma((1/2)-a+im)\Gamma((1/2)-a-im)f_{a,m}(x). \tag{92}$$

It follows that $f_{-a,m}$ is an eigenvector for $R^{(-a)}R^{(a)}$. Wimp[40] considered a Fourier–Plancherel formula which decomposes $L^2((0,\infty);\mathbf{C})$ as a direct integral with respect to $f_{a,m}(x)$, where $f_{a,m}(x)$ are eigenfunctions of $L^{(-a)}$, so there is a transform pair $h\leftrightarrow f$

$$h(m)=\Gamma(1/2-\kappa-im)\Gamma(1/2-\kappa+im)\int_0^\infty f(t)f_{\kappa,m}(t)\,dt,$$

$$f(x)=\frac{1}{\pi^2}\int_0^\infty m\sinh(2\pi m)f_{\kappa,m}(x)h(m)\,dm. \tag{93}$$

Hence we can take $(f_{a,m})_{m\geq 0}$ to be a basis of $L^2((0,\infty);\mathbf{C})$, and $(f_{-a,m})_{m\geq 0}$ to be a basis of another copy of $L^2((0,\infty);\mathbf{C})$ and diagonalize $\mathcal{R}^{(a)}$ with respect to the combined basis $(f_{\pm a,m})_{m\geq 0}$ of $L^2((0,\infty);\mathbf{C}^2)$.

6.2 Corollary *For $-1/2<a<1/2$, the kernel*

$$K\doteq\frac{\Gamma(1-2a)\cos^2\pi a}{\pi^2}\,\frac{W_{a+1/2,a}(x)W_{a-1/2,a}(y)-W_{a-1/2,a}(x)W_{a+1/2,a}(y)}{\sqrt{xy}(x-y)} \tag{94}$$

gives a determinantal point process on $[0,\infty)$ such that random points λ_j, ordered by $0\leq\lambda_1\leq\lambda_2\leq\dots$, satisfy

$$\Pr[\lambda_1\geq s]=\det(I-K\mathbf{I}_{[0,s]}). \tag{95}$$

Proof. Note that

$$\Gamma(a+1/2+im)\Gamma(a+1/2-im)\Gamma(-a+1/2+im)\Gamma(-a+1/2-im)=2\pi^2/(\cos 2\pi a+\cosh 2\pi m)$$

so

$$\|R^{(a)}R^{(-a)}\|_{\mathcal{L}(L^2(0,\infty))}=\pi^2\sec^2\pi a. \tag{96}$$

This cancels with $\pi^{-2}\cos^2\pi a$; hence we have $0\leq K\leq I$ as operators on $L^2(0,\infty)$, and $K(x,y)$ has a continuous kernel. Hence by Mercer's theorem, K restricts to a trace class operator on $L^2(0,b)$ for all $0<b<\infty$ with

$$\operatorname{trace}(K)=\int_0^b K(x,x)\,dx. \tag{97}$$

Note that the right-hand side is finite for all $b>0$, but diverges to ∞ as $b\to\infty$, hence $\det(I-K\mathbf{I}_{[0,b]})\to 0$ as $b\to\infty$.

By Soshnikov's theorem[32], there is a point process on $[0,b]$ with K as the generating kernel. The point process distributes random x in $[0,b]$ such that only finitely many x can lie in each Borel subset of $[0,b]$, and the joint distribution of the points is specified as follows. Let B_j $(j=1,\dots,m)$ be disjoint Borel subsets of $[0,b]$ and $n_j=$

$\sharp\{x\in B_j\}$ the number of random points that lie in B_j. Then the joint probability generating function of the random variables n_j is

$$\mathbf{E}\prod_{j=1}^m z_j^{n_j} = \det\Bigl(I+\mathbf{I}_{[0,b]}\sum_{j=1}^m (z_j-1)K\mathbf{I}_{B_j}\Bigr)\qquad (z_j\in\mathbf{C}). \tag{98}$$

Equivalently, as in Ref 9, p. 597 we compress K to $\mathbf{I}_{[0,b]}K\mathbf{I}_{[0,b]}$ on $L^2([0,b])$ and introduce
$T_{[0,b]}=\mathbf{I}_{[0,b]}K\mathbf{I}_{[0,b]}(I-\mathbf{I}_{[0,b]}K\mathbf{I}_{[0,b]})^{-1}$ so that

$$\det(I+T_{[0,b]})=\bigl(\det(I-\mathbf{I}_{[0,b]}K\mathbf{I}_{[0,b]})\bigr)^{-1}. \tag{99}$$

For each integer $\ell\geq 0$, the ℓ-point correlation function is defined to be the positive symmetric function

$$\rho_\ell(x_1,\dots,x_\ell)=\det(I+T_{[0,b]})^{-1}\det\bigl[T_B(x_j,x_k)\bigr]_{j,k=1}^{\ell}\qquad (x_j\in[0,b]), \tag{100}$$

such that

$$\Pr(\text{there are exactly}\quad\ell\quad\text{particles in}\quad B)=\frac{1}{\ell!}\int_{B^\ell}\rho_\ell(x_1,\dots,x_\ell)dx_1\dots dx_\ell \tag{101}$$

for all Borel subsets B of $(0,b)$.

Remarks 6.3 (i) In particular as in Ref 31, (7.4.25) we can write

$$f_{0,m}(x)=\frac{1}{\sqrt{\pi x}}K_{im}(x/2)=\frac{1}{\sqrt{2}}\int_1^\infty P_{-1/2+im}(s)e^{-sx/2}ds \tag{102}$$

in terms of the MacDonald and associated Legendre functions. The function $W_{0,im}$ also occurs in the spectral decomposition of the Laplace operator over the fundamental domain that arises from the action of $SL(2,\mathbf{Z})$ on the upper half plane; see Ref 23, page 318 for a discussion of Maass cusp forms. In Ref 4, we considered the Hankel operators that commute with second order differential operators, and found the case L_0 and R_0 as Q(vii).

(ii) Another case of Q(vii) from Ref 4 is $L^{(-\kappa)}$ commuting with the Hankel operator with scattering function $x^{-1}W_{\kappa,1/2}(x)$ on $C_c^\infty(0,\infty)$. If $a\neq 0$, then $R^{(a)}$ is not a Hankel operator but an operator of Howland's type[19], and $L^{(a)}$ does not commute with $R^{(-a)}$.

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