A Spectral Analysis Argument to prove the Riemann Hypothesis

Dr. Klaus Braun
79106 Freiburg i. Br.
www.riemann-hypothesis.de

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Abstract

The Hilbert transform [24] of Jacobi's Theta function is applied in combination with Müntz' formula [37] to verify the Berry conjecture [2]. The basic idea behind this approach is applying the concept of hyper functions to get a weak formulation of the Riemann duality equation, enabling the application of spectral theory in the framework of Distribution and Pseudo Differential Operator theory [21], [24], [35].

The Riemann duality equation [11], valid for all complex $s \in C$,

$$\xi(s) := \zeta(s)\Omega(s) = \xi(1-s) \quad \text{with} \quad \Omega(s) := (s-1)\int_0^\infty x^s (xf'(x))d\log x$$

is equivalent to the Jacobi's $\vartheta -$ relation [17]

$$1 + \psi(x^2) := G(x) := \sum_{-\infty}^{\infty} f(nx) = \frac{1}{x}G(\frac{1}{x}),$$

which is the Poisson summation formula [24] for the Gauss-Weierstrass density function $f(x) := e^{-\pi x^2}$. The regularity of the $\vartheta -$ function relates to the regularity of f(x), which is analytical, which isn't the case for f(1/x). This jeopardizes convergent (Mellin transform) integrals in the critical stripe, when trying to apply Müntz formula [37] to $G(x)$ and $G(1/x)/x$ in this domain. Transforming the Theta function by the Hilbert transform and managing the corresponding Mellin integrals in a distribution theory framework overcome this issue. Nevertheless, spectral theory can be applied in the distribution theory framework [36], which verifies the Berry conjecture.

In a strong sense there is an only formally valid representation of Riemann's duality equation as transform of an integral operator in the form ([11], 10.3):

$$\zeta(s)\int_0^\infty x^s (xf'(x))d\log x = \frac{\xi(s)}{s-1} = \frac{s}{2}\int_0^\infty x^{1-s} G(x)\frac{dx}{x}.$$

This operator has no transform at all, as the integral does not converge for any s. The integral would converge at infinity if the constant term $f(0) = \hat{f}(0) = 1$ is absent. Hyper functions (23], [24], are distributions allowing to treat "functions" by Fourier transform, which can transmit unexpected (non-analytical!) signals, represented by a Laurent-series description with vanishing constant Fourier term. Applying the Hilbert transform to f gives a Cauchy principle-valued function with vanishing constant Fourier terms, Applying Müntz formula then defines a distribution valued holomorphic Zeta fake function $\xi^*(s)$ fulfilling the relation

$$(\xi_s^*, \varphi) = (\xi_{1-s}^*, \varphi) \quad \text{for each } \varphi \in C_c^\infty$$

with

$$\xi^*(s) := \zeta(s)\Omega^*(s) = \int_0^\infty x^s G^*(x)d\log x = \int_0^\infty x^{1-s} G^*(x)d\log x, \quad 0 < \mathrm{Re}(s) < 1,$$

and

$$\Omega^*(s) := \int_0^\infty x^s (Hf)(x)d\log x \quad \text{and} \quad G^*(x) = \frac{1}{x}G^*(\frac{1}{x}) \quad \text{in the distribution sense.}$$

As a consequence ξ_s^* is the weak form of the Zeta function $\xi(s)$, now represented as transform of a Pseudo Differential (integral) operator in the critical stripe, where the spectral theory can be applied to. The several equivalent criteria to the Riemann Hypothesis can be investigated, using the "dual" structure of $\psi^*(x^2)$ as an enabler to overcome current singularity handicaps, when trying to verify those criteria in a non distribution sense.

0. Introduction

Our terminology follows those of [11] H.M. Edwards, [24] B. E. Petersen and [36] R.S. Strichartz.

Specific properties of the Gauss-Weierstrass density function

(0.1) $$f(x) := e^{-\pi x^2}$$

with its non-analytical "dual" counterpart e^{-1/x^2} and the Mellin transform in the form

(0.2) $$\tilde{\Pi}(s) := \Gamma(1+s/2)\pi^{-s/2} = \int_0^\infty x^s f'(x)dx$$

enables Jacobi's $\vartheta-$relation resp. Riemann's duality equation in the form ([11] H.M. Edwards, 1.6ff)

$$\vartheta(x^2) := G(x) := \sum_{-\infty}^{\infty} f(nx) = G(1/x)/x$$

$$\xi(s) := \zeta(s)(s-1)\tilde{\Pi}(s) = \xi(1-s)\,,\ s \in C \quad .$$

[11] H.M. Edwards, chapter 10): *Let V be the vector space of all complex-valued functions on R^+ with the inner product*

$$(u,v) := \int_0^\infty u(x)v(x)dx \,.$$

By $I : v(x) \mapsto \int_0^\infty v(ux)F(u)du$ *an integral operator* $\mathrm{I} : V \to V$ *is defined. An operator is said to be invariant if it commutes with all translation operators* $T_u : v(x) \mapsto v(ux)$*. The transform of an invariant operator is the function whose domain is the set of complex numbers* s *such that the function* $v(x) := x^{-s}$ *lies in the domain of the operator and whose value for such an s is the factor by which the operator multiplies* $v(x) := x^{-s}$ *. Thus e.g. the Zeta function* $\zeta(s)$ *for* $\mathrm{Re}(s) > 1$ *is the transform of the summation operator*

$$v(x) \mapsto \sum_1^\infty v(nx) \,.$$

When defining the adjoint of an invariant operator on V the inner product is defined on a rather small subset of V, whenever both side of $(Lu,v) = (u,L^*v)$ *are defined.*

There is an only formally valid representation of Riemann's duality equation as transform of an integral operator

(0.3a) $$I : v(x) \mapsto \int_0^\infty v(ux)G(u)du$$

in the form ([11] H.M. Edwards, 10.3, (2.8) below)

(0.3b)

$$\int_0^\infty x^{1-s}G(x)\frac{dx}{x}=\frac{2\xi(s)}{s(s-1)} \quad .$$

But the operator (0.3a) has no transform at all, as the integral does not converge for any s. The integral would converge at ∞ if the constant term $f(0)=\hat{f}(0)=1$ is absent.

If one would find an integral operator in the from (0.3a) satisfying the same functional equation than G does and if

(0.3c) $\int_0^\infty x^{1-s}G(x)\frac{dx}{x}$ converges and $\int_0^\infty x^{1-s}G(x)\frac{dx}{x}=\int_0^\infty x^{s}G(x)\frac{dx}{x}$

Then this operator would be self-adjoint in the sense of above ([11] H.M. Edwards, 10.2, 10.3).

Hyperfunctions are distributions, allowing to treat "functions" by Fourier transform, which can transmit unexpected (non-analytic!) signals, represented by a Laurent-series description with vanishing constant "Fourier term" ([23] R. Penrose, 9.2). In the one-dimensional case hyper-functions are the distributions of the dual space $C^{-\omega}$ of the real-analytical functions of a real variable C^{ω}, defined on some connected segment $\subset R$ ([23] R. Penrose, 9.7, [24] B. E. Petersen, 1.16) and appendix). This gives the link of our approach to Penrose's thoughts and ideas moving forward "the road to reality". In the one-dimensional case the concept of hyperfunctions enables a link between distributions and a holomorphic, i.e. a complex-analytical function, as any distribution f on R can be realized as the "jump" of the corresponding in $C-R$ holomorphic Cauchy integral function

$$F(x):=\frac{1}{2\pi i}\oint\frac{f(t)dt}{t-x}$$

across the real axis, given by

$$(f,\varphi)=\lim\int_{-\infty}^{\infty}F(x+iy)-F(x-iy))\varphi(x)dx \quad \text{for } y\to 0^+ \quad .$$

The Hilbert transform H (see (1.1) below) gives a Cauchy principle-valued function with Fourier terms

(0.4)
$$(Hu)_\nu=-i\,\mathrm{sgn}(\nu)u_\nu \quad .$$

The study of the Hilbert transform and the study of operational calculus for non-commuting operators in quantum mechanics (e.g. the Weyl operator) contain some of the basic ingredients of the theory of pseudo differential operators ([24] B. E. Petersen, 3.1). Freeding the Hilbert transform from its too intimate link connection with complex variables techniques Calderon and Zygmund introduced the algebra of singular integral operators (modulo compact operators) based on salient features of the Hilbert transform ([24] B. E. Petersen, 2.9). This also stimulated the study of the algebra generated by singular differential operators and partial differential operators ([24] B. E. Petersen, 4.1ff), which all leads into the concept of pseudo differential operator.

The Hilbert transform (1.1), which is a classical Pseudo differential operator, transforms the Gauss-Weierstrass density function into a “P.v. distribution” ([24] B. E. Petersen, 1.7,) in the form

(0.5) $$Hf(x) = f(x) * \frac{1}{\pi x} = \hat{f}(x)\left[\frac{1}{\pi x}\right]^{\wedge} \quad \text{with} \quad \hat{f}(x) = e^{-\pi x^2}$$

resp. $$FHF^{-1} = 2\pi F\left(\frac{1}{\pi x}\right) = -i\,\mathrm{sgn}(x) \cdot$$

With reference to (0.4) we mention Euler’s famous formula ([22] N. Nielsen, chapter IX, §51)

$$sign(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin(tx)dt}{t} \cdot$$

As link to a well known Zeta function constant we mention ([24] B. E. Peterson, 1.15, [5] R. P. Brent)

$$\left[P.f.\frac{1}{|x|}\right]^{\wedge} = -2\gamma - 2\log|\xi| \quad ,$$

whereby P.f. denotes Hadamard’s “partie finie” or “finite part”. The P.v. distribution (0.5) can be calculated, which we state in

Lemma 0.1: The Hilbert transform of the Gauss-Weierstrass density function (0.1) and its related Fourier transform are given by

i) $$[H(f)](x) = 4\pi\int_0^{\infty} f(\xi)\sin(2\pi\xi x)d\xi \quad ,$$

ii) $$[H(f)]^{\wedge}(\omega) = 2\pi i\int_0^{\infty} f(\xi)[\delta(\omega - 2\pi\xi) - \delta(\omega + 2\pi\xi)]d\xi \quad \cdot$$

Proof of lemma 0.1: is given in the appendix.

With respect to lemma 0.1 ii) we refer to the definition of hyperfunctions (appendix B). With respect to lemma 0.1iii) we refer to [18] J.M. Hill.

[24] B. E. Petersen, chapter 1, §15: *Let* $z \to g_z$ *be a function defined on a open subset* $U \subset C$ *with values in the distribution space. Then* g_z *is called a holomorphic in* $U \subset C$ *(or* $g(z) := g_z$ *is called holomorphic in* $U \subset C$ *in the distribution sense), if for each* $\varphi \in C_c^{\infty}$ *the function* $z \to (g_s, \varphi)$ *is holomorphic in* $U \subset C$ *in the usual sense*.

The constant Fourier term of $(Hf)(x)$, $(Hf')(x)$ and $(Hxf')(x)$ vanish. As f' is odd it holds

$$H(f')(x)=\frac{1}{x}(H(xf'))(x) \quad \text{whereby} \quad 0=\int H(f')(x)dx=\frac{1}{x}\int H(tf'))(t)dt \ .$$

This enables the definition of the distribution complex-valued function

(0.6)
$$\Omega^*(s):=(s-1)\tilde{\Pi}_s^*:=(s-1)\tilde{\Pi}^*(s):=\int_0^\infty x^s(Hxf')(x)dx \ .$$

The Müntz formula (lemma 2.1) builds representations of the Zeta in the form

$$\zeta(s)\int_0^\infty x^s\frac{\omega(x)dx}{x}=\int_0^\infty x^s\left[\sum_1^\infty\omega(nx)-\frac{1}{x}\int_0^\infty\omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0<\mathrm{Re}(s)<1 \ ,$$

when $\omega(x)$ satisfies certain conditions. We apply this formula to the function

(0.7)
$$G^*(x):=2\psi^*(x^2):=2\sum_1^\infty\omega(nx)\in L_2 \quad \text{with } \omega(x):=H[\circ f'(\circ)](x) \ ,$$

to build a distribution valued holomorphic function

$$\xi_s^*=\zeta(s)\Omega^*(s) \quad \text{for } 0<\mathrm{Re}(s)<1.$$

Jacobi's $\vartheta-$relation and the vanishing constant Fourier term of $H[xf'(x)]$ then imply

(0.8)
$$\psi^*(x^2)=\frac{1}{x}\psi^*(\frac{1}{x^2}) \qquad \text{in the distribution sense.}$$

The distribution framework ensures the convergence of the Mellin transform integrals, when applying the variable transform $x\to\frac{1}{y}$ resp. $\frac{dx}{x}=\frac{dy}{y}$. This then leads to the relation

(0.9)
$$(\xi_s^*,\varphi)=(\xi_{1-s}^*,\varphi) \qquad \text{for each } \varphi\in C_c^\infty \ .$$

Thus, in the sense of (0.3), the corresponding integral operator

$$I: v(x)\mapsto\int_0^\infty v(ux)\psi^*(u^2)du$$

is self-adjoint and spectral theory can be applied. (0.8), (0.9) is the main result of this manuscript, which we summarize in

Proposition 0.3: The distribution valued holomorphic Zeta fake function ξ_s^* fulfills the Riemann duality equation in the critical stripe, i.e. ξ_s^* is the weak representation of $\xi(s)$ represented as Mellin transform of a self-adjoint integral operator.

For the rest of this section we give the link to an appropriate Hilbert space environment.

The weighted Hermite polynomials (e.g. [36] R.S. Strichartz, 7.6)

(0.10) $$\varphi_n(x) := \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \quad \text{with} \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \ , \quad H_0(x) = 1 \, , \quad H_1(x) = x \, ,$$

form a set of orthonormal functions in $L_2(-\infty,\infty)$, i.e. the Hermite polynomials have only real zeros. The relation to (0.1) is given by

$$f(x) = \pi^{1/4} \varphi_0(\sqrt{2\pi} x) \ .$$

The Hilbert transforms of Hermite polynomials are given in the appendix. Lemma 1.3 indicates the orthogonality relations to the original weighted Hermite polynomials $\varphi_n \in L_2$ leading to $H\varphi_n \in L_2$ and

(0.11) $$L_2 := H := span[\varphi_n(x)] = span[H(\varphi_n(x))] \ .$$

In [6] D. Bump, et. al. it's shown that all zeros of the Mellin transforms of the weighted Hermite polynomials lie on the critical line using the recursion formula of the Hermite polynomials in combination with an argument from Polya ([27]), which he developed to analyze the zeros of $q(s) := K_s(x)$ (see [37] E. C. Titchmarsh, 10.23 and appendix part B (5.1)):

From [40] 13.14 we recall

Lemma 0.4: Let f be an even Schwartz test function with

$$\int_0^\infty f(x)dx = 0 \ , \ f(0) = 0 \ .$$

Then it holds

$$\sum_{n=1}^\infty f(nx) = \sum_{n=1}^\infty \hat{f}(\frac{n}{x}) \ ,$$

whereby $\hat{f}$ denotes the Fourier transform of f .

According to lemma 0.1 and lemma 1.3 below it then follows

Corollary 0.5: Let

$$f^H(x) := Hf(x) = 4\pi\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi$$

be the Hilbert transform of $f(x) = e^{-\pi x^2}$. Then it holds:

i) $f^H(0) = 0$

ii) $f^H(x)$ is odd, i.e. $H(xf^H(x)) = x(Hf^H)(x)$

iii) $\left[f^H\right]_{\nu=0} = \int_{-\infty}^{\infty} f^H(x)dx = 0$.

Lemma 0.6: If $-\infty < c < \infty$ and $K(z)$ is an entire function of genus 0 or 1 that assumes real values for real z , has only real zeros and has at least one real zero, then the function

$$K(z-ic)+K(z+ic)$$

also has only real zeros.

We note the relations

$$\sin(z-ic)+\sin(z+ic)=2\sin x\cosh c$$

$$\cos(z-ic)+\cos(z+ic)=2\cos x\cosh c .$$

Applying the arguments from [6] D. Bump et.al., the Mellin transforms of the functions $H\left(\varphi_n(x)\right)$ have their zeros on the critical line.

By (0.9) a Hermitian operator is defined. From spectral theory it then follows that its spectrum is real, i.e. it holds

Corollary 0.7: The zeros of the complex-valued distribution Mellin transform of $\psi^*(x^2)$, which are the zeros of ξ_s^* , lie all on the critical line. This proves the RH in the distribution sense. Applying standard density arguments this then proves the RH itself.

In [7] D.A. Cardon an analysis is given to study convolution operators and the zeros of corresponding entire function. For a special class of probability (!) distribution (in the sense of probability theory!!) functions F there is a generalization of Polya's lemma such that the convolution ([7,8] D. A. Cardon),

(0.12) $$(K*dF)(z):=\int_{-\infty}^{\infty}K(z-iu)dF(u)$$

has only real zeros. Being especially $F(x)$ the normal distribution ([7] D. A. Cardon)

(0.13) $$F(x):=\int_{-\infty}^{x}e^{-\pi u^2}du=\int_{-\infty}^{x}f(u)du \quad \text{and} \quad K(z):=z^n$$

the corresponding convolution (0.9) enables a representation of the Hermite polynomials in the form

(0.14) $$h_n(z):=(K_n*dF)(z)=\int_{-\infty}^{\infty}(z-iu)^n dF(u)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(z-iu)^n e^{-u^2}du=2^{-n}H_n(z) \quad .$$

We refer to

Lemma 0.8 ([7] D.A.Cardon): If the function $\Xi(t):=\xi(1/2+it)$ with

$$\xi(s):=\zeta(s)(s-1)\tilde{\Pi}(s)=\xi(1-s)$$

can be realized as a convolution $\Xi(t)=(K*dF)(t)$ where $K(t)\in LP^*$, i.e. is a entire function from the Laguerre-Polya class of order <2, i.e. $\Phi(z)=c^z me^{\alpha z-\beta z^2}\prod_k(1-z/\alpha_k)e^{z/\alpha_k}$, where c,α,α_k are real, $\beta\geq 0$ and m is a nonnegative integer, this would prove the RH.

Hypothesis: On the critical line the representation (0.7) fulfills the assumptions of lemma 0.6 below in the distribution sense due to the convolution structure of the Hilbert transform (see also appendix (B.1)).

1. Hyperfunctions and Pseudo Differential Operators,

Mellin-, Fourier-, Wavelet-, Hilbert-Transforms

The Fourier transform plays a key role in the theory of signal processing. For real functions $u(t)$ the positive real frequency axis already contains all information about the waveform in the time domain. In the one-dimension case the Riesz operator is identical with Hilbert transform ([24] B. E. Petersen, 2.9), that is a Cauchy principle-valued function, expressed in the form

(1.1) $$(R_1u)(x) := (Hu)(x) := \lim \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{u(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy \qquad \text{for } \varepsilon \to 0$$

fulfilling $$(Hu)_\nu = -i\,\mathrm{sgn}(\nu) u_\nu \ .$$

The Hilbert transform is a classical pseudo-differential operator ([24] B. E. Petersen, 3.6) with symbol $i\,\mathrm{sgn}(s)$. The principle value $P.v.(1/x)$ of the not locally integrable function $1/x$ is the distribution g defined by ([24] B. E. Petersen, 1.7)

$$(g,\varphi) := \lim \int_{x \geq \varepsilon} \varphi(x) \frac{dx}{x} = \int_{-\infty}^{\infty} \log|x| \varphi'(x) dx \qquad \text{for each } \varphi \in C_c^\infty \ .$$

The relation of this specific principle value to the Fourier and Hilbert transform (1.1) is given by ([24] B. E. Petersen, 2.9)

(1.2) $$\left[P.v.(\frac{1}{x})\right]^\wedge = -i\pi\,\mathrm{sgn}(s) \quad \text{and} \quad \left[P.v.(\frac{1}{x})\right]^{\wedge\wedge} = -2\pi P.v.(\frac{1}{x}) \cdot$$

For example the function $\sin(\omega t)$ is the Hilbert transform of $\cos(\omega t)$. This gives a $\pm\pi/2-$ phase-shift operator, which is a basic property of the Hilbert transform. It can be used to remove the not needed negative frequency axis.

To make a rigorous presentation of the Hilbert transform theory one have to apply distribution theory. We state some main properties of the Hilbert transform in

Lemma 1.1: For the Hilbert transform it holds

i) $\|H\| = 1$, $H^* = -H$, $H^2 = -I$, $H^{-1} = H^3$,

ii) $H(f * g) = f * Hg = Hf * g$, $f * g = -Hf * Hg$

iii) If $(\varphi_n)_{n \in N}$ is an orthogonal system, so it is for the system $(H(\varphi_n))_{n \in N}$, i.e.

$$(H\varphi_n, H\varphi_n) = -(\varphi_n, H^2\varphi_n) = (\varphi_n, \varphi_n).$$

iv) $\|Hu\|^2 = \|u\|^2$, i.e. if $u \in L_2$ then $Hu \in L_2$.

For other properties related e.g. to rotations we refer to [35] E.M. Stein.

A complex function is called Hermitian if its real part is even and its imaginary part is odd. If $g(t)$ is a real function, then e.g. $\hat{g}(\xi)$ is Hermitian and therefore $|\hat{g}(\xi)|^2$ is even.

A complex signal u is called a strong analytical signal if it fulfills $Hu = iu$. For strong analytical signals u it holds that $H(\mathrm{Re}(u)) = \mathrm{Im}(u(x))$, i.e.

$$z(t) = u(t) + iH(u(t))$$

is a strong analytical signal.

The specific properties of the operator (1.1), which are essential for our arguments, we summaries in

Lemma 1.3: The Hilbert Operator (1.1) fulfills

i) The Fourier term for $\nu = 0$ is $(Hu)_0 = 0$

ii) $H(xu(x)) = xH(u(x)) - \frac{1}{\pi}\int_{-\infty}^{\infty} u(y)dy$

iii) for odd functions it hold $H(xu(x)) = x(Hu)(x)$

iv) $Hu(x) = u(x) * \frac{1}{\pi x}$, $\frac{1}{\pi x} = \lim \frac{x}{\pi(x^2+\rho^2)}$ $\rho \to 0$

v) If $u, Hu \in L_2$ then u and Hu are orthogonal, i.e. $\int_{-\infty}^{\infty} u(y)(Hu)(y)dy = 0$.

Proof of lemma 1.3 is given in the appendix.

2. A Distribution valued Zeta Function as Transform of a Pseudo Differential Operator

The Gauss-Weierstrass density function

(2.1) $$f(x) := e^{-\pi x^2}$$

with its Mellin transform in the form

(2.2) $$\tilde{\Pi}(s) := \Gamma(1+\frac{s}{2})\pi^{-s/2} = \int_0^\infty x^s (xf'(x))\frac{dx}{x}$$

enables Jacobi's $\vartheta-$relation (see e.g. [17] H. Hamburger)

(2.3) $$1+2\psi(x^2) := G(x) := \sum_{-\infty}^{\infty} f(nx) = \frac{1}{x}G(\frac{1}{x})$$

and Riemann's duality equation

(2.4) $$\xi(s) := \zeta(s)(s-1)\tilde{\Pi}(s) = \xi(1-s)$$ valid for all complex $s \in C$.

The $\vartheta-$relation (2.3) is a direct consequence of Poisson summation formula ([24] B. E. Petersen, 2.11)

(2.5) $$\sum_{n=-\infty}^{\infty} \hat{\varphi}(2\pi n) = \sum_{n=-\infty}^{\infty} \varphi(n)$$

and the Fourier transform of the Gauss-Weierstrass kernel ([24] B. E. Petersen, 2.3)

(2.6) $$\frac{1}{2\pi}\left[e^{-\varepsilon|x|^2}\right]^{\wedge} = \frac{1}{\sqrt{4\pi\varepsilon}} e^{-|\xi|^2/(4\varepsilon)} \ .$$

The constant Fourier term $(\xi = 0)$ doesn't vanish. Therefore it (unfortunately) doesn't hold

$$\psi(x^2) \neq \frac{1}{x}\psi(\frac{1}{x^2}) \ .$$

There is an only formally valid representation ([11] H.M. Edwards, 10.3) of (1.7) as transform of an integral operator in the form

(2.7) $$\int_0^\infty x^{1-s} G(x)\frac{dx}{x} = \frac{2\xi(s)}{s(s-1)} \ .$$

The operator (2.7) has no transform at all, as the integral does not converge for any s. The integral would converge at ∞ if the constant term $f(0)=\hat{f}(0)=1$, $(n=0)$ is absent. Roughly speaking solves the measure $xf'(x)dx$ the convergence issue of (2.7).

The function ([11] H.M. Edwards, 10.3)

(2.8) $$H(x):=\frac{d}{dx}\left[x^2\frac{d}{dx}G(x)\right]=\frac{d}{dx}\left[x^2\frac{d}{dx}\left[G(x)-1\right]\right]=2\sum_1^\infty(2\pi^2n^4x^4-3\pi n^2x^2)e^{-\pi n^2x^2}>0$$

overcomes the convergence issue of (2.7) by "differentiation" to get ride off the "jeopardizing" non-vanishing constant Fourier term, fulfilling both (self-adjoint) conditions, i.e. it holds

$$H(x)=\frac{1}{x}H(\frac{1}{x})$$ and $$\int_0^\infty x^{1-s}H(x)\frac{dx}{x}=\int_0^\infty y^{-(-s)}\frac{1}{y}H(\frac{1}{y})\frac{dy}{y}=\int_0^\infty x^sH(x)\frac{dx}{x}$$ converge for any s,

defining an entire function, which is invariant concerning $s\leftrightarrow 1-s$, i.e.

(2.9) $$\int_0^\infty x^{1-s}H(x)\frac{dx}{x}=s(s-1)\int_0^\infty 2x^{1-s}\psi(x^2)\frac{dx}{x}=2\xi(1-s)\cdot$$

The operator (2.8) is a "differentiation" operator, balancing with respect to (2.9) the move to a higher regularly Hilbert scale by combining it with a multiplication operator, but the prize to be paid for that "move" is a corresponding change of underlying zeros resp. eigenvalues behavior.

On the other side looking at the Hilbert transform of (2.1) and its related Fourier transform there is an only formally valid representation of the corresponding Poisson summation formula in the strong sense. But as the Fourier transform exists in the distribution sense the weak $\vartheta-$relation

(2.10) $$G^*(x):=\sum_{-\infty}^\infty H\left[f(nx)\right]=\frac{1}{x}G(\frac{1}{x})\quad ,$$

fulfills as well

(2.11) $$\psi^*(x^2):=\sum_1^\infty H\left[f(nx)\right]=\frac{1}{x}\psi^*(\frac{1}{x^2})$$ in the distribution sense.

Combining lemma 0.1 with (2.11) gives

(2.12) $$\psi^*(x^2)=4\pi\sum_1^\infty\int_0^\infty f(\xi)\sin(2\pi n\xi x)d\xi$$ in the distribution sense.

The fact, that in opposite to (2.6) the Fourier transform of the Hilbert transform of (2.1) vanishes at $x=0$ (see (0.8)) suggests to replace

(2.13) $$xf'(x) \to Hf(x) = 4\pi\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi \;.$$

The adjoint of the differential operator $v(x) \mapsto xv'(x)$, found by integration by parts, is

$u(x) \mapsto \frac{d}{dx}[xu(x)]$ ([11] H.M. Edwards, 10.32). In terms of transforms this operation is related to the substitution $s \mapsto (1-s)$.

A vanishing constant Fourier term plays also a key role in the wavelet theory. A wavelet transform is similar as a Fourier transform. A Fourier transform delivers the frequency spectrum of a timely signal $\varphi(t)$ without any loss of information, although it gives the frequencies without any information about the points in time, when the frequencies occur. The wavelet transform delivers this sort of information in a better distinguishing form: one gets both the frequency analysis and the points in time, when those frequencies happen. In this sense the wavelet transform describes the music of an orchestra on a 2-dimensional instead of a 1-dimensional paper. This reminds to [32] M. du Sautoy's statement (p. 120ff), that "the primes have music in them", where the behavior of the zeros of the Zeta function gives the melody and the loudness of its music.

We recall Müntz' formula, which gives a representation of the Zeta function in the critical stripe:

Lemma 2.1 (**Müntz' formula**) For $\omega(x), \omega'(x)$ continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ and $\omega(x) = o(x^{-\beta})$ for $x \to \infty$ and $\alpha, \beta > 1$ it holds

(2.14) $$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s \left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0 < \operatorname{Re}(s) < 1.$$

Proof: is given in the appendix. ●

Applying Müntz formula with $\omega(x) := \psi^*(x^2)$ (2.11) it follows the

Corollary 2.2: in the critical stripe there exists a holomorphic function $\xi_s^* := \xi^*(s)$ in the distribution sense represented by

(2.15) $$\xi^*(s) := \zeta(s)\int_0^\infty x^s Hf(x)\frac{dx}{x} = \int_0^\infty x^s \psi^*(x^2)\frac{dx}{x} \qquad \text{for } 0 < \operatorname{Re}(s) < 1,$$

which is dual in the distribution sense, i.e. it fulfills

(2.16) $$(\xi_s^*, \varphi) = (\xi_{1-s}^*, \varphi) \qquad \text{for each } \varphi \in C_c^\infty \;.$$

Applying standard density arguments to corollary 2.2 leads to

Corollary 2.3: The holomorphic function $\xi_s^* := \xi^*(s)$ is the weak distribution representation of the entire function $\xi(s)$ and the standard Zeta function theory can be applied to $\xi_s^* := \xi^*(s)$ accordingly e.g. it holds ([11] H.M. Edwards, 1.8)

(2.17) $$\xi^*(s) = \int_0^\infty x^s \psi^*(x^2)\frac{dx}{x} = \frac{1}{2}\int_1^\infty \sqrt{x}\cosh\left[(s-\frac{1}{2})\log x\right]\psi^*(x^2)\frac{dx}{x}$$ in the distribution sense,

i.e. there is a series expansion in the form $\xi^*(s) = \sum_1^\infty a_{2n}(s-\frac{1}{2})^{2n}$ with

$$a_{2n} := \frac{1}{2}\int_1^\infty \sqrt{x}\cosh\left[(s-\frac{1}{2})\log x\right]\psi^*(x^2)\frac{(\log x)^{2n}}{(2n)!}\frac{dx}{x}$$

$$a_{2n} := 2\pi\sum_{k=1}^\infty \int_0^\infty\int_1^\infty x^{-1/2}\cosh\left[(s-\frac{1}{2})\log x\right]\frac{(\log x)^{2n}}{(2n)!}f(\xi)\sin(2\pi\xi kx)dxd\xi \ .$$

For the abbreviation

$$g(x) := \frac{1}{2}x^{1/2}\psi^*(x^2) \quad \text{and} \quad G(u) := g(e^u)$$

it holds with (2.11)

$$g(\frac{1}{x}) := \frac{1}{2}x^{1/2}\frac{1}{x}\psi^*(\frac{1}{x^2}) = \frac{1}{2}x^{1/2}\psi^*(x^2) = g(x) \ .$$

Substituting the variable $u = \log x$ (2.17) resp. $x = 1/y$ (2.17) can be re-written in the form

Corollary 2.4: The complex-valued distribution holomorphic function $\Xi^*(t) := \xi^*_{1/2+it} := \xi^*(1/2+it)$ can represented in the form

(2.18) $$\Xi^*(t) := \int_1^\infty g(x)\cos(t\log x)\frac{dx}{x} = \int_1^\infty G(u)\cos(tu)du$$

and fulfills

$$\Xi^*(t) = \int_1^\infty g(x)\cos(t\log x)\frac{dx}{x} = \int_0^1 g(y)\cos(t\log y)\frac{dy}{y} = \frac{1}{2}\int_0^\infty g(x)\cos(t\log x)\frac{dx}{x} = \frac{1}{2}\int_0^\infty G(u)\cos(tu)du \ .$$

Riemann's Hypothesis is saying, that all zeros of (2.18) are real, which is the fulfilled, if the underlying integral operator of (2.18) is self-adjoint ([11] H.M. Edwards, 10.3), as a hermitian operator has a real spectrum only.

Appendix

Proof of lemma 0.1

i) The Fourier transform of $\varphi_0(t) := \pi^{-1/4} e^{-t^2/2}$ is given by $\hat{\varphi}_0(t) := \sqrt{2}\pi^{1/4} e^{-\omega^2/2}$. With (0.4) we get

$$[H(\varphi_0)]^{\wedge}(\omega) = -i\,\mathrm{sgn}(\omega)\hat{\varphi}_0(\omega) \ .$$

Applying the inverse Fourier transform then gives

$$[H(\varphi_0)](t) = \sqrt{2}\pi^{1/4} \int_{-\infty}^{\infty} (-i\,\mathrm{sgn}(\omega)) e^{-\omega^2/2} e^{-i\omega t} d\omega \ .$$

Since $\mathrm{sgn}(\omega) e^{-\omega^2/2}$ is odd we have

$$[H(\varphi_0)](t) = 2\sqrt{2}\pi^{1/4} \int_0^{\infty} e^{-\omega^2/2} \sin(\omega t) d\omega \ .$$

With $f(x) = \pi^{1/4}\varphi_0(\sqrt{2\pi}x)$ it follows

$$\pi^{1/4}[H(\varphi_0)](\sqrt{2\pi}x) = 2\sqrt{2\pi} \int_0^{\infty} e^{-\omega^2/2} \sin(\sqrt{2\pi}\omega x) d\omega \ .$$

Substituting the variables $\omega = \sqrt{2\pi}\xi$ then leads to

$$[H(f)](x) = 4\pi \int_0^{\infty} e^{-\pi\xi^2} \sin(2\pi\xi x) d\xi \ .$$

ii) We recall the Fourier transforms

$$g_1(x) := \sin(2\pi a x) \qquad \hat{g}_1(\omega) = \frac{i}{2}\left[\delta(\omega - 2\pi a) - \delta(\omega + 2\pi a)\right]$$

$$g_2(x) := \begin{cases} \frac{i}{2}\pi sign(x) & |x| \le 2a \\ 0 & |x| > 2a \end{cases} \qquad \hat{g}_2(\omega) = \frac{\sin 2(a\omega)}{\omega} \ ,$$

which leads to ii)

●

Lemma (**Müntz' formula**) For $\omega(x), \omega'(x)$ continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ and $\omega(x) = o(x^{-\beta})$ for $x \to \infty$ and $\alpha, \beta > 1$ it holds

$$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s \left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0 < \operatorname{Re}(s) < 1.$$

Proof: recalled from [37] E. C. Titchmarsh:

i) because $\omega(x)$ is continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ it holds

$$\sum_1^\infty \frac{1}{n^s}\left|\int_0^\infty x^{s-1}\omega(x)\right|dx \quad \text{exists for} \quad 1 < \sigma < \alpha \ ,$$

i.e. the inversion leading to the left hand side of (4.3) is justified.

ii)
$$\sum_1^\infty \omega(nx) - \int_0^\infty \omega(xt)dt = x\int_0^\infty \omega'(t)(t-[t])dt = x\int_0^{1/x} O(1)dt + x\int_{1/x}^\infty O((xt)^{-\beta})dt = O(1)$$

The first summand is justified, because $\omega(x)$ is continuous and bounded in any finite interval the second summand is justified, because $\omega(x) = o(x^{-\alpha})$, i.e. it holds

$$\sum_1^\infty \omega(nx) = O(1) + \frac{c}{x} \ \text{ with } \ c = \int_0^\infty \omega(t)dt \ .$$

Hence

$$\int_0^\infty x^s \sum_1^\infty \omega(nx) + \frac{dx}{x} = \int_0^1 x^s \left[\sum_1^\infty \omega(nx) - \frac{c}{x}\right]\frac{dx}{x} + \int_1^\infty x^s \sum_1^\infty \omega(nx)\frac{dx}{x} + \frac{c}{s-1}$$

for $\sigma > 0$ except $s = 1$. Also

$$-c\int_1^\infty x^{s-2}dx = \frac{c}{s-1} \qquad \text{for } \sigma < 1$$

and therefore the result for $0 < \sigma = \operatorname{Re}(s) < 1$ ●

Proof of lemma 1.3

i) is given in [24] B. E. Petersen, 2.9

ii) Consider the Hilbert transform of $xu(x)$

$$H(xu(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yu(y)}{x-y}dy \; .$$

The insertion of a new variable $z = x - y$ yields

$$H(xu(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x-z)u(x-z)}{z}dz = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(xu(x-z)}{z}dz - \frac{1}{\pi}\int_{-\infty}^{\infty}u(x-z)dz = xH(u(x)) - \frac{1}{\pi}\int_{-\infty}^{\infty}u(y)dy$$

iii) follows directly from i) and ii)

iv) is given in [24] B. E. Petersen, 2.9

v) $\int_{-\infty}^{\infty}u(y)(Hu)(y)dy = \frac{i}{2\pi}\int_{-\infty}^{\infty}sign((\omega)|\hat{u}(\omega)|^2 d\omega$ with $|\hat{u}(\omega)|^2$ is even gives the result ●

2. Hermite Polynomials

The Hermite polynomials $H_n(x)$ fulfill the recursion formula

(2.9) $$H_n(\sqrt{2\pi}x) = 2xH_{n-1}(\sqrt{2\pi}x) - (n-1)b_n\varphi_{n-2}(x) - 2(n-1)H_{n-2}(\sqrt{2\pi}x) \; .$$

Using the abbreviation

$$a_n := \sqrt{\frac{2(n-1)!}{n!}} \qquad b_n := \sqrt{\frac{(n-2)!}{n!}}$$

this gives the recursion formula

(2.10) $$\varphi_n(x) := a_n x\varphi_{n-1}(x) - (n-1)b_n\varphi_{n-2}(x), \quad \varphi_0(x) := \pi^{-1/4}e^{-\frac{x^2}{2}}, \quad \varphi_1(x) := 2^{-1/2}\pi^{-1/4}xe^{-\frac{x^2}{2}} \quad ,$$

from which the recursion formula for the corresponding Hilbert transforms can be calculated

(2.11) $$\hat{\varphi}_n(x) := a_n\left[x\hat{\varphi}_{n-1}(x) - \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_{n-1}(y)dy\right] - (n-1)b_n\hat{\varphi}_{n-2}(x)$$

$$\hat{\varphi}_0(x) = \pi^{1/4}\int_{-\infty}^{\infty}e^{-\frac{\omega^2}{2}}\sin(\omega x)d\omega \; .$$

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