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A CLOSURE PROBLEM RELATED TO THE RIEMANN ZETA-FUNCTION

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It is rather obvious that any property of the Riemann zeta-function may be expressed in terms of some other property of the function $\rho(x)$ defined as the fractional part of the real number x, i.e., $x = \rho(x) \bmod 1$. This note will deal with a duality of the indicated kind which may be of some interest due to its simplicity in statement and proof. In the sequel, C will denote the linear manifold of functions

$$f(x) = \sum_1^n c_\nu \rho\left(\frac{\theta_\nu}{x}\right), \; 0 < \theta_\nu \leq 1, \qquad n = 1, 2, \ldots,$$

where the c_ν are constants such that $f(1 + 0) = \sum_1^n c_\nu \theta_\nu = 0$.

THEOREM. *The Riemann zeta-function is free from zeros in the half-plane $\sigma > 1/p$, $1 < p < \infty$, if and only if C is dense in the space $L^p(0, 1)$.*

Let C^p denote the closure of C in the space $L^p = L^p(0, 1)$, and let T_a, $0 < a \leq 1$, be the operator which takes a function $f(x)$ defined over $(0, 1)$ into the function which is equal to $f(x/a)$ for $0 \leq x \leq a$ and equal to 0 for $a < x \leq 1$. This semi-group of operators has the following properties which will be important for our problem: Each T_a carries C into itself and is norm-diminishing in each space L^p. From this we easily conclude that *C is dense in L^p if and only if C^p contains the function k which is equal to* 1 *over the unit interval.* For, if k belongs to C^p, the same must be true of the characteristic function of any subinterval (a, b) of $(0, 1)$, this function being equal to $T_b k - T_a k$.

We next point out that, for $\sigma > 0$,

$$\int_0^1 \rho\left(\frac{\theta}{x}\right) x^{s-1}\, dx = \frac{\theta}{s-1} - \frac{\theta^s \zeta(s)}{s}. \tag{1}$$

For $f \in C$ we will have

$$\int_0^1 f(x) x^{s-1}\, dx = -\frac{\zeta(s) \sum_1^n c_\nu \theta_\nu^s}{s}, \qquad \sigma > 0. \tag{2}$$

Assume first that $C^p = L^p$. We can then find an $f \in C$ such that $||1 + f||_p < \epsilon$, where ϵ is a given positive number. By equation (2),

$$\int_0^1 (1 + f(x))\, x^{s-1}\, dx = \frac{1}{s}\left(1 - \zeta(s) \sum_1^n c_\nu \theta_\nu^s\right), \qquad \sigma > 0. \quad (3)$$

If $\sigma > 1/p$, Hölder's inequality yields the following majorant of equation (3):

$$||1 + f||_p|| x^{s-1}||_q < \epsilon\left\{\frac{1}{q(\sigma - 1/p)}\right\}^{1/q}.$$

Consequently,

$$\left|1 - \zeta(s) \sum_1^n c_\nu \theta_\nu^s\right|^q < \frac{\epsilon^q |s|^q}{q(\sigma - 1/p)}, \qquad \sigma > 1/p. \quad (4)$$

This proves that $\zeta \neq 0$ in the region $D_{p,\,\epsilon}$ where the right-hand member of relation (4) is < 1, i.e., for

$$\sigma > \frac{1}{p} + \frac{\epsilon^q\ |s|^q}{q}.$$

As $\epsilon \downarrow 0$, $D_{p,\,\epsilon}$ increases and exhausts the half-plane $\sigma > 1/p$. We should also observe at this instance that $D_{p,\,\epsilon}$ is convex and that its boundary intersects the line $\sigma = 1$ at the two points where $|s| = 1/\epsilon$.

The proof of the necessity of the condition $C^p = L^p$ is less trivial. If C is not dense in L^p, we know by a classical theorem of F. Riesz and Banach that the dual space L^q must contain a nontrivial element g which is orthogonal to C in the sense that

$$0 = \int_0^1 g(x) f(x)\, dx, \qquad f \epsilon C. \quad (5)$$

Since T_a takes C into itself, it follows that

$$0 = \int_0^1 g(x) T_a f(x)\, dx = a \int_0^1 g(ax) f(x)\, dx, \qquad f \epsilon C, \qquad 0 < a \leq 1. \quad (6)$$

Let E_g^r, $1 \leq r \leq q$, be the closed linear subset of L^r spanned in the topology of that space by the set $\{g(ax), 0 < a \leq 1\}$. From equation (6) and the fact that C consists of bounded functions, we conclude that each $f \epsilon C$ is orthogonal to each function belonging to any of the sets E_g^r. We now recall that the positive reals ≤ 1 form a semigroup S under multiplication and that each continuous (and normalized) character of S has the form $\varphi = x^\lambda$, where λ is an arbitrary complex or real number. Clearly $\varphi \epsilon L^q$ if and only if $\Re_e(\lambda) > -1/q$. The problem of whether or not a set of the kind E_g^p contains a character is of considerable complexity. It has been studied earlier by the author[1, 3] and by Nyman.[2] However, the following result[3] can be proved by elementary function theoretic means: *Let $g(x)$ belong to a space $L^q(0, 1)$, $1 < q < \infty$, and have the property*

$$\int_0^x |g(x)|\, dx > 0, \qquad x > 0. \quad (7)$$

Then there exists at least one character x^λ, $\Re_e(\lambda) > -1/q$, which is contained in each set E_g^r for $1 \leq r < q$ (but not necessarily in E_g^q).

In order to apply this theorem, we have first to show that condition (7) is satisfied by our function g. For this pupose assume that g vanishes a.e. on some interval $(0, a)$, $a > 0$. Choose b such that $a < b < \min(1, 2a)$, and $f(x) = b\rho(a/x) -$

$a\rho(b/x)$. This f belongs to C; it vanishes for $x > b$ and takes the value a for $a < x < b$. Therefore,

$$0 = \int_0^1 f(x)g(x)\,dx = a\int_a^b g(x)\,dx, \qquad a < b < \min(1, 2a), \tag{8}$$

and it follows that $g = 0$ a.e. for $x < \min(1, 2a)$. On repeating the same argument a finite number of times, we find that $g = 0$ a.e. over $(0, 1)$, which is a contradiction. The cited theorem may thus be applied to g and yields the existence of a number λ, $\mathcal{R}_e(\lambda) > -1/q$, such that x^λ is orthogonal to C. On defining $s_0 = 1 + \lambda$, we will have, in particular

$$0 = \int_0^1 \left(\rho\left(\frac{1}{x}\right) - \left(\frac{1}{\theta}\right)\rho\left(\frac{\theta}{x}\right)\right) x^{s_0 - 1}\,dx = \frac{(\theta^{s_0-1} - 1)}{s_0}\,\zeta(s_0),\ 0 < \theta < 1. \tag{9}$$

Obviously, $s_0 \neq 1$. Since θ can be chosen such that, at any given point $\neq 1$, $\theta^{s-1} - 1 \neq 0$, it follows that s_0 is a zero of ζ. We also have $\mathcal{R}_e\,(s_0) > 1 - 1/q = 1/p$, and this concludes the proof.

We finally point out that the problem of how well $k = 1$ can be approached by functions $\epsilon\ C$ has a direct bearing on the distribution of the primes even in case ζ does have zeros arbitrary close to the line $\sigma = 1$.

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INTEGRABLE AND SQUARE-INTEGRABLE REPRESENTATIONS OF A SEMISIMPLE LIE GROUP

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Let G be a connected semisimple Lie group. We shall suppose for simplicity that the center of G is finite. Let π be an irreducible unitary representation of G on a Hilbert space $\mathfrak{H}$. We say that π is integrable (square-integrable) if there exists an element $\psi \neq 0$ in $\mathfrak{H}$ such that the function $(\psi, \pi(x)\psi)$ $(x \epsilon G)$ is integrable (square-integrable) on G, with respect to the Haar measure. Assuming that the Haar measure dx has been normalized in some way once for all and that π is square-integrable, we denote by d_π the positive constant given by the relation[1]

$$\int_G |(\psi, \pi(x)\psi)|^2\,dx = \frac{1}{d_\pi},$$

where ψ is any unit vector in $\mathfrak{H}$. Let $C_c^\infty(G)$ denote the set of all complex-valued functions on G which are everywhere indefinitely differentiable and which vanish outside a compact set. Then the following result is an easy consequence of the Schur orthogonality[1] relations.