The Zeta Function as Transform of a Self-adjoint Singular Integral Operator and its relation to the Helmholtz Free Energy

Dr. Klaus Braun
79106 Freiburg i. Br.
www.riemann-hypothesis.de
January 25, 2012

Abstract

1. There is an only formally valid representation of the Riemann's duality equation ([ETi] 2.1)

$$\xi(s):=\zeta(s)\Omega(s)=\xi(1-s) \quad , \quad \zeta(s)=\chi(s)\zeta(1-s)$$

with

$$\Omega(s)=\Gamma(\frac{s}{2})\frac{1-s}{2}\pi^{-s/2}=\int_0^\infty x^s\frac{d}{dx}\left[e^{-\pi x^2}\right]dx=:\int_0^\infty x^s f'(x)dx \quad , \quad \chi(s)\pi^{-\frac{1-s}{2}}\Gamma(\frac{s}{2}):=\pi^{\frac{s}{2}}\Gamma(\frac{1-s}{2})$$

as transform of a self-adjoint integral operator in the form ([HEd] 10.3)

$$\int_0^\infty x^{1-s}G(x)\frac{dx}{x}=\frac{2\xi(s)}{s(s-1)}=\zeta(s)\Gamma(\frac{s}{2})\pi^{-s/2} .$$

In [KBr1], [KBr2] the Riemann Hypothesis has been proven by building representations of both above duality equations as singular Mellin integral transformations for a complex-valued distribution Zeta function ξ_s^*. In [KBr1] this is built by the orthogonal Hilbert transform (which is a wavelet function)

$$f_H(x)=4\pi\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi=4\pi x f(x)_1F_1(\frac{1}{2};\frac{3}{2};\pi x^2)=2\sqrt{\pi}x\int_0^1 e^{-\pi x^2(1-t)}t^{-1/2}dt$$

of the Gauss-Weierstrass distribution function $f(x)$. Its Mellin transform for $\mathrm{Re}(s)<1$ is given by

$$M\left[f_H\right](s)=\pi^{(1-s)/2}\tan(\pi\frac{s}{2})\Gamma(\frac{s}{2})=\pi^{(1-s)/2}\cot(\pi\frac{1-s}{2})\Gamma(\frac{s}{2})=2\sqrt{\pi}\tan(\pi\frac{s}{2})M\left[f\right](s) .$$

A modified Li-function can be defined by

$$Li_H(x):=Ei_H(\log x):=-\int_{-\log x}^\infty f_H(\sqrt{\frac{x}{\pi}})\frac{dx}{x} .$$

The Müntz formula in a weak L_2-form ([RDu], [ETi] 2.11), which is valid in the critical stripe, is applied to build the complex-valued distribution Zeta function

$$\xi^*(s):=\zeta^*(s)\pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2})=\int_0^\infty x^s G_H(x)\frac{dx}{x}:=\int_0^\infty x^s\left[\sum_{n=1}^\infty f_H(nx)\right]\frac{dx}{x}=\xi^*(1-s) .$$

Then spectral theory can be applied to the corresponding dual singular integral equation representation on the critical line $s=1/2+it$ with $t\in R$ to prove the Riemann Hypothesis in a (variational) weak sense ([DCa]). By standard density arguments then the RH follows in a strong sense.

2. The Hilbert transform of the $2\pi-$ periodic sawtooth function $\varphi(x)$ is given by

$$\varphi_H(x)=\sum_1^\infty\frac{\cos\nu x}{\nu}=\log 2\sin\frac{x}{2} .$$

In the critical stripe by

$$2\int_0^\infty x^{s-1}\varphi_H(2\pi x)dx=2\oint x^{s-1}\log(2\sin\pi x)dx=\zeta^{**}(s)\chi(1-s)$$

a holomorphic function in the distribution sense is defined, built on the convolution integral function $\varphi_H(x)$ [KBr2]. A duality equation is valid in the form

$$(\xi_s^{**},\chi)_{-1/2}=(\xi_{1-s}^{**},\chi)_{-1/2} \quad \text{for all } \chi\in H_{-1/2} \text{ and } 0<\mathrm{Re}(s)<1 ,$$

Consequently the zeros of the Zeta function ξ_s^{**} have to lie on the critical line ([DCa]).

3. There is a relation between the Gauss-Weierstrass density function and the Zeta function, when describing the total (Planck black body) radiation and its spectral density at the same time by

$$\frac{x^{-4}}{e^{1/x}-1}\frac{dx}{x}=\frac{x^4}{e^x-1}\frac{dx}{x} .$$

Therefore the alternative kernel distribution function $f_H(x)$ provides an alternative model for the Planck radiation formula. Corresponding options are sketched for quantum theory modeling of free energy, vacuum energy of electromagnetic fields, the density matrix for a one-dimensional harmonic oscillator.

§ 1 The Riemann Duality Equation

Concerning the notations we refer to [HEd], [SGr]. The Gauss-Weierstrass density function

$$f(x) := e^{-\pi x^2} = \hat{f}(x)$$

In combination with

$$\frac{1}{2\pi}\left[e^{-\varepsilon|x|^2}\right]^{\wedge} = \frac{1}{\sqrt{4\pi\varepsilon}} e^{-|\xi|^2/(4\varepsilon)}$$

and

$$\hat{f}_H(x\omega) = \frac{1}{x}\hat{f}_H(\frac{\omega}{x}) \quad , \quad x \in R^+$$

gives the Jacobi's $\vartheta-$relation formulas in the forms ([HEd] 1.6ff., [HHa])

i) $\vartheta(x^2) := G(x) := \sum_{-\infty}^{\infty} f(nx) = G(1/x)/x =: 1 + 2\psi(x^2)$

ii) $\frac{1}{\pi}\frac{d}{dx}\log(\sinh(\pi x) = \coth(\pi x) = i\cot i\pi x = \frac{i\cos(ix)}{\sin(ix)} = 1 + 2\sum_{1}^{\infty} e^{-2\pi n x} = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{1}^{\infty}\frac{1}{x^2+n^2}$.

Both formulas are equivalent to the Riemann's duality equation ([HHa])

$$\xi(s) := \zeta(s)(s-1)\Pi(s) = \xi(1-s)$$

whereby

$$\Pi(s) := \Gamma(1+s/2)\pi^{-s/2} = \int_0^{\infty} x^s f'(x)dx \ .$$

The $\vartheta-$relation i) is a consequence of Poisson summation formula ([BPe] 2.11)

$$\sum_{n=-\infty}^{\infty} \hat{f}(2\pi n) = \sum_{n=-\infty}^{\infty} f(n)$$

and the Fourier transform of the Gauss-Weierstrass kernel ([BPe] 2.3)

$$\hat{f}_{\alpha=x}(n) = \frac{1}{\sqrt{x}}\hat{f}_{\alpha=x}(\frac{n}{2\pi x}) \ .$$

Riemann's extention of Euler's product formula to the Zeta function counting primes ended up in his hypothesis ([HEd] 1.11, appendix, [JHa] 16): "*The non-trivial zeros of the Zeta function all have real part one-half".*

The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self adjoint operator.

There is an only formally valid representation of the Riemann's duality equation as transform of an self-adjoint integral operator in the form ([HEd] 10.3)

$$(*)\qquad \frac{1}{2}\int_0^\infty x^{1-s}G(x)\frac{dx}{x}=\frac{\xi(s)}{s(s-1)}=\frac{1}{2}\zeta(s)\Gamma(\frac{s}{2})\pi^{-s/2}=\frac{\zeta(s)}{s}\int_0^\infty x^s\left[xf'(x)\right]\frac{dx}{x}\ ,$$

whereby it holds ([ETi] 2.4, ([HEd] 6.3 (4))

$$-\frac{1}{s(1-s)}\approx-\frac{\Gamma(1-s)}{s}=\Gamma(-s)\quad ,\quad \frac{d}{ds}\log(\Gamma(s)\approx-\log\frac{1}{1-s}\ .$$

The integral (*) is divergent for any $s\in C$. The valid representations

i) $\int_0^\infty x^{1-s}\psi(x^2)\frac{dx}{x}=\frac{1}{2}\zeta(s)\Gamma(\frac{s}{2})\pi^{-s/2}$ for $1<\mathrm{Re}(s)$

i) $\int_0^\infty x^{1-s}\psi(x^2)\frac{dx}{x}=\frac{1}{2}\zeta(1-s)\Gamma(\frac{1-s}{2})\pi^{-(1-s)/2}$ for $1<\mathrm{Re}(1-s)$ i.e. $\mathrm{Re}(s)<0$

show disjunctive support, i.e. it's either $1<\mathrm{Re}(s)$ or $\mathrm{Re}(s)<0$.

The integral would converge at ∞ if the constant term $f(0)=\hat{f}(0)=1$ is absent. If such an integral operator would exist, the following two conditions ([HEd] 10.2, 10.3) would be fulfilled:

i) the Jacobi's $\vartheta-$relation:

$$\frac{1}{x}G^*(\frac{1}{x})=G^*(x)$$

ii) the convergence of the integrals:

$$\int_0^\infty x^{1-s}G^*(x)\frac{dx}{x}=\int_0^\infty\left[\frac{1}{x}\right]^{1-s}G^*(\frac{1}{x})\frac{dx}{x}=\int_0^\infty x^s\frac{1}{x}G^*(\frac{1}{x})\frac{dx}{x}=\int_0^\infty x^sG^*(x)\frac{dx}{x}\ .$$

A modified integral operator representation in the form

$$(**)\qquad \frac{2\xi(s)}{s(s-1)}=\int_0^\infty x^{1-s}\left[G(x)-1-\frac{1}{x}\right]\frac{dx}{x}=\int_0^\infty x^{(1-s)/2}\left[\psi(x)-\frac{1}{2\sqrt{x}}\right]\frac{dx}{x}$$

is used to prove the *Hardy theorem* ([HEd] 11.1), i.e. that *there are infinitely many roots of* $\xi(s)=0$ *on the line* $\mathrm{Re}(s)=1/2$. The idea of Hardy's proof is to use information about

$$G(x)-1-\frac{1}{x}=\frac{1}{2\pi i}\int_{1/2-i}^{1/2+i\infty}\frac{2\xi(s)}{s(s-1)}x^{s-1}ds \quad \text{resp.} \quad H(x):=x\frac{d^2}{dx}\left[xG(x)\right]=\frac{1}{2\pi i}\int_{1/2-i}^{1/2+i\infty}2\xi(s)x^{s-1}ds\ ,$$

specifically that $G(x)$ and all its derivatives approach zero as x approaches $i^{1/2}=e^{i\pi/4}$, while the right hand side of the equation is an integral involving the function $\xi(1/2+it)$, which is real on the critical line (see below) and therefore provides information about its zeros resp. the behavior of changes the sign along the critical line.

Therefore, based on the representation

$$\sqrt{x}H(x) = \sum_0^\infty c_{2n}(i\log x)^{2n}$$

with

$$c_{2n} := \frac{1}{\pi(2n)!}\int_{1/2-i}^{1/2+i\infty}\xi(1/2+it)t^{2n}dt$$

the behavior of sign changes for both sides of the integral equation is analyzed and put into relation to each other to verify the Hardy theorem.

We further note that the Riemann-Siegel formula ([HEd] 7.9) is based on a representation in the form (see remark “the 1st Mellin inverse problem” below)

$$\frac{2\xi(s)}{s(s-1)} = F(s) + \overline{F(1-\bar{s})} \ .$$

If the underlying integral operator of (**) would be self-adjoint, spectral theory could be applied.

Regarding the integral kernel function of (**) we note below

- the Müntz formula ([ETi] 2.11)
- the reciprocality of the function $g^*(x)$ below ([ETi] 2.7).

Lemma (Müntz formula): For $\omega(x), \omega'(x)$ continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ and $\omega(x) = o(x^{-\beta})$ for $x \to \infty$ and $\alpha, \beta > 1$ it holds

$$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s\left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0 < \mathrm{Re}(s) < 1.$$

Proof: is given in the appendix.

Lemma: The function

$$g^*(\sqrt{2\pi}x) := g(x) := \frac{1}{e^x - 1} - \frac{1}{x}$$

is self-reciprocal, i.e. it holds

$$g^*(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(y)\sin(xy)dy \ .$$

Its relation to the Zeta function is given by

$$\zeta(s)\Gamma(s) = \int_0^\infty g^*(x)x^{s-1}dx \ .$$

Remark: We note Hadamard's formula ([HEd] 3.8, [JHa] 16.10)

$$\xi(s)=-e^{-As}\prod_{\zeta(\rho)=0}(1-\frac{s}{\rho})e^{\frac{s}{\rho}}$$

Whereby

$$-A:=1+\frac{\gamma}{2}-\log(2\sqrt{\pi})=\sum_{\operatorname{Im}(\rho)\geq 0}\left[\frac{1}{\rho}+\frac{1}{1-\rho}\right] .$$

and the Laurent expansion ([JHa] 12.2, 12.5, [CBe], 7 entry 13, 8 entry 18(i), [ETi] 2.1)

$$\zeta(s)=\frac{1}{s-1}+\gamma+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\gamma_m(s-1)^m=\frac{1}{2}\int_1^{\infty}\left[x^{1-s}+x^{s-1}\right]\frac{dx}{x}+\int_1^{\infty}\frac{[x]-x}{x}\frac{dx}{x}+1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\gamma_m(s-1)^m$$

with the Stieltjes constants

$$\gamma_m:=\lim_{n\to\infty}\sum_{r=1}^{n}\left[\frac{\log^m r}{r}-\frac{\log^{m+1}n}{m+1}\right] .$$

Within the rectangle $0\leq\sigma:=\operatorname{Re}(s)\leq 1,\ 0\leq t\leq T$ the zeros of the Zeta function can be localized arbitrarily exact by the formula ([ETi] 4.14)

$$\zeta(s)=\sum_{n\leq x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}+O(x^{-\sigma}) \quad \text{whereby} \quad \sum_{n>x}\frac{1}{n^s}=-\frac{1}{2i}\int_{x-i\infty}^{x+i\infty}\frac{\cot(\pi z)}{z^s}dz \quad \text{for } \sigma>1 .$$

From this formula it follows for $T\geq 2$ ([HEd] 9.7)

$$\int_0^T\left|\zeta(\frac{1}{2}+it)\right|^2 dt=O(T\log^2 T) .$$

Its proof uses Hardy and Littlewood's estimate of the integral

$$-\int_N^{\infty}g(x)x^{-s}\frac{dx}{x}$$

With

$$g(x):=[x]-x+\frac{1}{2}=\sum_{n=0}^{\infty}\frac{\sin(2n\pi x)}{n\pi},$$

enabled by the fact that $g(x)$ is boundedly convergent, which justifies termwise integration by the Lebesgue dominated convergence theorem.

It holds ([ETi] 2.1)

i) $\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+s\int_1^{\infty}g(x)x^{-s}\frac{dx}{x}$ for $\operatorname{Re}(s)>1$

ii) $\zeta(s)=s\int_1^{\infty}(g(x)-\frac{1}{2})x^{-s}\frac{dx}{x}=-\int_1^{\infty}(g(x)-\frac{1}{2})\frac{d}{dx}(x^{-s})dx$ for $0<\operatorname{Re}(s)<1$

ii) $\zeta(s)=s\int_0^{\infty}g(x)x^{-s}\frac{dx}{x}$ for $-1<\operatorname{Re}(s)<0$.

Its counterpart related to the Gauss-Weierstrass function is given by ([HEd] 10.5)

i) $$\frac{\xi(s)}{s/2}=(s-1)\int_0^\infty (G(x)-\frac{1}{x})x^{1-s}\frac{dx}{x}=\frac{1}{2}(s-1)\pi^{(1-s)/2}\int_0^\infty (G(\sqrt{\pi y})-\frac{1}{\sqrt{\pi y}})y^{(1-s)/2}\frac{dy}{y}$$ for $\mathrm{Re}(s)>1$

ii) $$\frac{\xi(s)}{s/2}=(s-1)\int_0^\infty (G(x)-1-\frac{1}{x})x^{1-s}\frac{dx}{x}=\frac{1}{2}(s-1)\pi^{(1-s)/2}\int_0^\infty (G(\sqrt{\pi y})-1-\frac{1}{\sqrt{\pi y}})y^{(1-s)/2}\frac{dy}{y}$$ for $0<\mathrm{Re}(s)<1$

ii) $$\frac{\xi(s)}{s/2}=(s-1)\int_0^\infty (G(x)-1)x^{1-s}\frac{dx}{x}=\frac{1}{2}(s-1)\pi^{(1-s)/2}\int_0^\infty (G(\sqrt{\pi y})-1)y^{(1-s)/2}\frac{dy}{y}$$ for $\mathrm{Re}(s)<0$

whereby

$$\frac{\xi(s)}{s/2}:=\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) \quad .$$

From [HEd] we recall the

Lemma: The Zeta function $\xi(s)$ is analytical, it is real for real $s\in R$ and it holds

$$\xi(s)=\xi(1-s) \quad .$$

Then the Schwarz reflection formula can be applied, i.e. it holds for $t\in R$

$$\bar{\xi}(1/2+it)=\xi(1/2+it) \quad ,$$

i.e. $\xi(s)$ is real on the critical line.

So locating roots on the critical line reduces to locating sign changes of $\xi(1/2+it)$. This fact in combination with the duality equation is used to deduce the Riemann-Siegel approximation ([HEd] 7) to

$$Z(t):=e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$$

with

$$\xi(s)=\xi(\frac{1}{2}+it)=\Gamma(\frac{s}{2})\frac{s(s-1)}{2}\pi^{-s/2}\zeta(s)=\left[-\frac{t^2+1/4}{4\pi^{1/4}}e^{\mathrm{Re}(\log(\Gamma(\frac{1}{2}(\frac{1}{2}+it))))}\right]*Z(t)$$

and

$$\vartheta(t):=\mathrm{Im}(\log(\Gamma(\frac{1}{2}(\frac{1}{2}+it))))-i\frac{t}{2}\log(\pi) \quad .$$

The alternative weak duality equations below provide alternative functions $Z^*(t)$ with corresponding alternative (weak) Riemann-Siegel approximations.

To overcome the convergence issues above a distribution framework is proposed, building complex-valued generalized transforms with domain being the critical stripe. This enables convergent integrals (by an appropriate inner product of a Hilbert space) in the critical stripe

$$0 < \mathrm{Re}(s), \mathrm{Re}(1-s) < 1 \ .$$

The key idea to build the coresponding kernel function is to replace the Gauss-Weierstrass function $f(x)$ by its Hilbert transform

$$f_H(x) = 4\pi \int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi \ ,$$

while keeping the Theta relation property, building a convolution integral representation, but loosing the analyticality of $f(x)$ and the purely Fourier integral representation).

The same idea is applied to the Hilbert transform $\varphi_H(x)$ of the $2\pi-$periodic sawtooth function $\varphi(x)$, now with corresponding Mellin transform with respect to the inner product of Hilbert space $H^*_{-1/2}(0,2\pi)$.

The key properties are given by

i) $f, f_H \in H_0$ and both functions are orthogonal, i.e. $(f, f_H)_0 = 0$

ii) $\varphi, \varphi_H \in H_{-1/2}$ and both functions are orthogonal, i.e. $(\varphi, \varphi_H)_{-1/2} = 0$.

Our main result is

Proposition: In the critical stripe by

$$\xi^*(s) := \zeta(s)\pi^{(1-s)/2}\cot(\frac{\pi}{2}(1-s)\Gamma(\frac{s}{2}) = \int_0^\infty x^s G_H(x)\frac{dx}{x} := \int_0^\infty x^s\left[\sum_{n=1}^\infty f_H(nx)\right]\frac{dx}{x}$$

$$\xi^{**}(s) := \zeta(s)\chi(1-s) = 2\int_0^\infty x^s\varphi_H(2\pi x)\frac{dx}{x} = 2\oint x^s \log 2\sin(\pi x)\frac{dx}{x}$$

holomorphic functions are defined in the distribution H_0 – resp. $H_{-1/2}$ – sense, fulfilling the two duality equations

$(\xi^*_s, \chi)_0 = (\xi^*_{1-s}, \chi)_0$ for all $\chi \in H$,

$(\xi^{**}_s, \chi)_{-1/2} = (\xi^{**}_{1-s}, \chi)_{-1/2}$ for all $\chi \in H_{-1/2}$.

which are representations as transforms of a self-adjoint (singular, convolution) integral operator on the critical line.

Remark: From [GSz] 5.7 we recall that for $\alpha > -1$ the systems

$$e^{-x/2}x^{\alpha/2}x^n \ , \ e^{-x^2/2}x^n \quad , \qquad n = 0,1,2,3......$$

are closed in

$$L_2(0,\infty) \, , \quad L_2(-\infty,\infty) \ .$$

Remark (invariant additive vs. multiplicative measure, [HEd] 10.2**)**: With the related "measures"

$$\mu_f(x) := -\int_x^\infty f(t)\frac{dt}{t} \quad , \quad \mu_{f_H}(x) := -\int_x^\infty f_H(t)\frac{dt}{t}$$

the corresponding Mellin transforms are given by (see below)

i) $$\sqrt{\pi}\int_0^\infty x^s f(x)\frac{dx}{x} = \sqrt{\pi}\int_0^\infty x^s d\mu_f(x) = \frac{1}{2}\pi^{(1-s)/2}\Gamma(\frac{s}{2})$$

ii) $$\int_0^\infty x^s f_H(x)\frac{dx}{x} = \int_0^\infty x^s d\mu_{f_H}(x) = \pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2}) \ .$$

It holds

$$\int_0^\infty d\mu_f(x) = \infty \quad \text{but} \quad \int_0^\infty d\mu_{f_H}(x) < \infty \ .$$

Remark (alternative Li-function): From [SGr] 8.211, 8.240 we recall the formulas

$$Li(x) := Ei(\log x)$$

with

$$Ei(x) := \begin{cases} -\int_{-x}^\infty e^{-t}\frac{dt}{t} & for \quad x < 0 \\ \lim_{\varepsilon\to 0}\left[\int_{-x}^{-\varepsilon} e^{-t}\frac{dt}{t} + \int_\varepsilon^\infty e^{-t}\frac{dt}{t}\right] & for \quad x > 0 \end{cases} \ .$$

It holds for $\mathrm{Re}(s) > 0$ and $x > 1$ ([HEd])

$$Li(x^s) = \int_0^x t^s \frac{dt}{t\log t} = -\int_{-\log x}^\infty e^{-st}\frac{dt}{t} \ .$$

The above definition of the measure $d\mu_{f_H}$ in combination with the following formulas ([SGr] 8.232, 3.721, 3.952) (see also ([GWa] 13-3, for its relation to the Bessel functions)

i) $$si(x) := -\int_x^\infty \sin(t)\frac{dt}{t} = -\frac{\pi}{2} + \sum_{k=1}^\infty \frac{(-1)^{k-1}x^{2k-1}}{(2k-1)(2k-1)!},$$

ii) $$\int_0^\infty \sin(at)\frac{dt}{t} = \frac{\pi}{2}sign(a)$$ (Euler's formula)

iii) $$\int_0^\infty e^{-\eta}\sqrt{\eta}\sin(b\sqrt{\eta})\frac{d\eta}{\eta} = 2\int_0^\infty e^{-x^2}\sin(bx)dx = \frac{b}{2}e^{-b^2/4}{}_1F_1(\frac{1}{2};\frac{3}{2};\frac{b^2}{4})$$

leads to the following modified definition and related representations of the Li-Function:

Definition & remark: By $f_H(x)$ a modified Li-function can be defined by

$$Li_H(x) := Ei_H(\log x) := -\int_{-\log x}^\infty f_H(\sqrt{\frac{t}{\pi}})\frac{dt}{t}.$$

It fulfills the relations

$$Li_H(x) = -2\sqrt{\pi}\int_{-\log x}^\infty \left[\int_0^\infty \sqrt{\eta}e^{-\eta}\sin(2t\sqrt{\frac{\eta}{\pi}})\frac{d\eta}{\eta}\right]\frac{dt}{t}$$

$$Li_H(x) = -2\int_{-\log x}^\infty e^{-t^2/\pi}{}_1F_1(\frac{1}{2};\frac{3}{2};\frac{4t^2}{\pi})dt.$$

With respect to the notation of §3 we refer to [RFe]:

In §3 we suggest an alternative Hilbert space than current standard $L_2 \cong l_2$ – Hilbert space to model quantum physical states, based on the function $\varphi^H(x)$ fulfiing

$$\frac{d}{dx}[\varphi_H(x)] = \frac{1}{2}\cot\frac{x}{2} \quad , \quad \frac{1}{2\pi}\int_0^{2\pi}\varphi_H(y)dy = 0 \quad , \quad \frac{1}{2\pi}\int_0^{2\pi}\varphi_H^2(y)dy = c \, .$$

This is being linked appropriately to the Helmholtz free energy formula and the Planck radiation body formula, given by

$$\frac{dR(\lambda,T)}{d\lambda} = \frac{c_1}{\lambda^5}\frac{1}{e^{c_2/\lambda T}-1} = \frac{c_1}{\lambda^5}\sum_1^\infty e^{-nc_2/\lambda T}$$

with $c_1 = 2\pi hc^2$ and $c_2 = hc/k$. The relation to the Zeta function $\zeta(s)\Gamma(s)$ is given by

$$\frac{\pi^4}{90} = \zeta(4)\Gamma(4) = \int_0^{+\infty} x^4 \frac{1}{e^x-1}\frac{dx}{x} = \int_0^{+\infty} x^{-4}\frac{1}{e^{\frac{1}{x}}-1}\frac{dx}{x} \, .$$

This duality formula describes the total radiation and its spectral density at the same time, i.e.

$$\frac{x^{-4}}{e^{1/x}-1}\frac{dx}{x} = \frac{x^4}{e^x-1}\frac{dx}{x} \, ,$$

whereby

$$\psi(\frac{x}{\pi}) = \sum_1^\infty f(\sqrt{\frac{nx}{\pi}}) = \sum_1^\infty e^{-nx} = \frac{1}{e^x-1} \, .$$

The exact value of the free energy F of a linear system of harmonic oscillators is given by

$$\beta F := \sum_{k=1}^\infty L(\beta\lambda_k) \quad \text{with} \quad \frac{1}{\beta} := k_B T \quad \text{and} \quad \lambda_k := \frac{\hbar\omega_k}{2}$$

with the related probability values in the form

$$a_k = e^{-\beta(\lambda_k - F)} \, .$$

Due to convergence issues in order to calculate a normalization factor Z the ground state zero term $\beta\lambda_0$ is (just!) omitted and F is replaced by

$$\beta F^* = \sum_{k=1}^\infty \log(1-e^{-2\beta\lambda_k}) = -\sum_{k=1}^\infty K(2\beta\lambda_k)$$

leading to

$$a_k^* = e^{-\beta(\lambda_k - F^*)} = \frac{1}{Z^*}e^{-\beta\lambda_k} \quad \text{and} \quad \varphi^* := \sum_{k=1}^\infty a_k^*\varphi_k \in H_0 \, .$$

§ 2 The Riemann Duality Equation as Transform of a Self-adjoint Integral Operator

Concerning the notation we refer to [HEd] and [GWa]. The transform

$$g(s):=\int_0^\infty K(s,x)h(x)dx$$

is called the integral transform and $K(s,x)$ is called the kernel of the transform. If $h:R^+\to C$ is a function such that $x^{s-1}h(x)\in L_1(0,\infty)$ for some $s\in C$ then the Mellin transform is defined by

$$M[h](s):=\int_0^\infty x^{s-1}h(x)dx=\int_0^\infty x^s d\sigma(x) \quad \text{with} \quad d\sigma(x):=h(x)\frac{dx}{x}.$$

If the integral is bounded then the transform exists, but converse is not necessarily true. The most popular Mellin transform example is the Gamma function $g(s)=\Gamma(s)$ in case $h(x):=e^{-x}$.

We summaries some well known properties of the Mellin transform in

Lemma: i) $M[h'](s)=(1-s)M[h](s-1)$

ii) $M[xh'](s)=-sM[h](s)$

iii) $M[(xh)'](s)=(1-s)M[h](s)$

iv) $M[h''](s)=(s-1)(s-2)M[h](s-2)$

v) $(h*M[g])(s)=\int_0^\infty h(\frac{x}{u})g(u)\frac{du}{u}=\int_0^\infty u^{-s}h(\frac{x}{u})g(u)u^s\frac{du}{u}$.

The property iii) is used to build a structure related to the entire Zeta function

$$\xi(s)=\zeta(s)(s-1)\Omega(s)$$

based on appropriately defined Mellin transforms.

From [GEs], [ESh] we recall the following relationship between the Hilbert and the Mellin transform of $g\in L_2(0,\infty)$:

$$H[g](x)=p.v.\frac{1}{\pi}\int_0^\infty \frac{g(y)}{x-y})dy=\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} x^{-s}(i\frac{1+e^{2\pi is}}{1-e^{2\pi is}})M[g](s)ds.$$

The operator is bounded on L_p, $1<p<\infty$. The function $\vartheta(2\pi x)$ with

$$\vartheta(x)=\frac{1}{1-e^{iz}}=\frac{1}{2}\left[1+\frac{1-e^{iz}}{1-e^{iz}}\right]$$

is a Mellin multiplier on L_p. We mention the similarity to the Cayley transformation of a hermitean operator resp. its spectrum.

Following the concept of [KBr1], [KBr2] the objective is to represent a complex-valued distribution Zeta function ξ_s^* as transform of a self-adjoint (singular) integral operator in an appropriate Hilbert space. Its building is enabled by the Hilbert transform, which builds the basic concept of the theory of pseudo-differential operators.

The Hilbert transform

$$\omega_H(x) := [H\omega](x) := \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\omega(y)}{x-y} dy$$

has the following properties [BPe]:

Lemma: i) $H^2 = -I$, $(Hu, v) = -(u, Hv)$

ii) $H[\omega'](x) = (H[\omega])'(x)$

iii) $\hat{\omega}_H(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \omega_H(y) dy = 0$

iv) $[xH - Hx](\omega(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \omega(y) dy$

i.e., if $\omega(x)$ is odd, then $H(x\omega(x)) = x(H\omega)(x)$.

The property iii) above provides the key to overcome the current issues to build the Zeta function as transform of a self-adjoint integral operator (§1). The property iv) is mentioned in the context of §3.

Holomorphic functions in the distribution sense are defined in the following way ([BPe] I.15):

Definition : Let $z \to g_z$ be a function defined on a open subset $U \subset C$ with values in the distribution space. Then g_z is called a holomorphic function in $U \subset C$ (or $g(z) := g_z$ is called holomorphic in $U \subset C$ in the distribution sense), if for each $\varphi \in C_c^\infty$ the function $z \to (g_s, \varphi)$ is holomorphic in $U \subset C$ in the usual sense.

The Gauss-Weierstrass density function

For later usage we note the formulas

i) $$\sqrt{\pi}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}=2^{1-s}\cos(\frac{\pi}{2}s)\Gamma(s)$$

ii) $$\Gamma(\frac{s}{2})\Gamma(\frac{1+s}{2})=2^{1-s}\sqrt{\pi}\Gamma(s)\ .$$

With respect to the function $\chi(s)$, defined in [ETi] 2.1 we note

Corollary: It holds

$$\chi(1-s):=\pi^{1/2-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}=\frac{1}{\pi}(2\pi)^{1-s}\cos(\frac{\pi}{2}s)\Gamma(s)$$

Remark: With respect to §3 below we note the following Fourier transforms

$$\frac{\pi}{\cosh(\pi\xi)}=\int_{-\infty}^{\infty}e^{-i\xi x}\frac{1}{2\cosh\frac{x}{2}}dx\quad,\quad\frac{\pi\xi}{\sinh(\pi\xi)}=\int_{-\infty}^{\infty}e^{-i\xi x}\frac{1}{4\cosh^{2}\frac{x}{2}}dx\ .$$

Remark: From [SGr] 3.952 we recall the representations

i) $$f_H(x)=4\pi\int_0^{\infty}f(\xi)\sin(2\pi\xi x)d\xi$$

ii) $$f_H(x)=4\pi xf(x)_1F_1(\frac{1}{2};\frac{3}{2};\pi x^2)$$

iii) $$f_H(x)=2\sqrt{\pi}xf(x)\int_0^1 e^{\pi x^2 t}t^{-1/2}dt$$

iv) $$f_H(x)=-\frac{1}{2\pi}\int_0^1 t^{-1/2}\frac{d}{dx}\left[e^{-\pi x^2(1-t)}\right]\frac{dt}{1-t}$$

whereby the confluent hypergeometric function (series) (see also [NNi] §65) is given by

i) $${}_1F_1(\alpha;\beta;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{\binom{-\alpha}{n}}{\binom{-\beta}{n}}\frac{z^n}{n!}\quad\text{with}\quad(\alpha)_n=\alpha(\alpha+1)...(\alpha+n-1);(\alpha)_0:=1$$

ii) $${}_1F_1(\alpha;\beta;z)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\Gamma(\alpha)}\int_0^1 e^{zt}t^{\alpha-1}(1-t)^{\beta-\alpha-1}dt\ .$$

Remark: The confluent hypergoemetric function is a solution of the confluent (or Kummer-schen) differential equation. Special examples are the Bessel functions, Laguerre and Hermite polynomials.

Lemma: It holds

i) $$f_H(x)=\frac{2}{\sqrt{\pi}x}\int_0^{\pi/2}\frac{1}{\sin\tau}\frac{d}{d\tau}\left[e^{-\pi x^2\cos^2\tau}\right]\tau d\tau$$

ii) $$f_H(x)=-2\sqrt{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{\pi}\cos\tau}\frac{d}{dx}\left[e^{-\pi x^2\cos^2\tau}\right]\tau d\tau$$

Proof: By variable substitutions $t=y^2$ and $y=\sin\tau$ it follows

$$f_H(x)=2\sqrt{\pi}x\int_0^1 e^{-\pi x^2(1-t)}t^{1/2}\frac{dt}{t}=2\sqrt{\pi}x\int_0^1 2e^{-\pi x^2(1-y^2)}y\frac{dy}{y}=2\sqrt{\pi}x\int_0^{\pi/2}2e^{-\pi x^2\cos^2\tau}\cos\tau d\tau$$

$$=2\sqrt{\pi}x\int_0^{\pi/2}\frac{1}{\pi x^2\sin\tau}\frac{d}{d\tau}\left[e^{-\pi x^2\cos^2\tau}\right]\tau d\tau=2\int_0^{\pi/2}\frac{1}{\sqrt{\pi}x\sin\tau}\frac{d}{d\tau}\left[e^{-\pi x^2\cos^2\tau}\right]\tau d\tau\,.$$ **q.e.d.**

Remark: From [SGr] 3.952 we recall

i) $$\int_0^\infty e^{-\pi x^2}\sin(\xi x)dx=\frac{1}{2\pi}\xi e^{-\frac{\xi^2}{4\pi}}{}_1F_1(\frac{1}{2};\frac{3}{2};\frac{\xi^2}{4\pi})$$

ii) $$\int_0^\infty e^{-\pi x^2}\sin(\xi x)\frac{dx}{x}=\frac{1}{2}\xi e^{-\frac{\xi^2}{4\pi}}{}_1F_1(1;\frac{3}{2};\frac{\xi^2}{4\pi})\,.$$

Lemma: With respect to the Polya theorem ([GPo2]) we note

i) $$f'_H(\frac{\sqrt{x}}{2\pi})=8\pi^2\int_0^\infty \xi f(\xi)\cos(x\xi)d\xi=4\pi\left[1-\frac{1}{2}\sum_{k=1}^\infty \pi^{1/2-k}\frac{(-1)^{k-1}(k-1)!}{(2k-1)!}x^k\right]$$

ii) $$\alpha\le -x\frac{f'_H(x)}{f_H(x)}=1-\frac{2\pi\int_0^\infty \xi^2 f(\xi)\sin(2\pi\xi x)d\xi}{\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi}=1+\frac{\int_0^\infty \xi f(\xi)\frac{d}{dx}\cos(2\pi\xi x)d\xi}{\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi}\le\beta$$

Proof: i) follows with ([SGr] 3.952)

ii) From ([SGr] 3.952.3) it follows

$$-xf'_H(x)=-4\pi x\int_0^\infty 2\pi\xi f(\xi)\cos(2\pi\xi x)d\xi=-4\pi\int_0^\infty \xi f(\xi)\frac{d}{d\xi}\sin(2\pi\xi x)d\xi$$

$$=4\pi\int_0^\infty\left[f(\xi)+\xi f'(\xi)\right]\sin(2\pi\xi x)d\xi=f_H(x)-8\pi^2\int_0^\infty \xi^2 f(\xi)\sin(2\pi\xi x)d\xi$$

$$=f_H(x)-2\pi\left[2x+\frac{1-2\pi x^2}{\sqrt{\pi}}\sum_0^\infty\frac{(-1)^k k!}{(2k+1)!}(2\sqrt{\pi}x)^{2k+1}\right].$$

Lemma: The Fourier transform of the odd function $f_H(x)$ is given by

$$\hat{f}_H(x) = -\frac{i}{2\pi}\int_0^1 t^{-1/2}\frac{x}{\sqrt{1-t}}e^{-\frac{x^2}{4\pi(1-t)}}\frac{dt}{1-t} = i\int_0^1 t^{-1/2}\left[\frac{d}{dx}e^{-\frac{x^2}{4\pi(1-t)}}\right]\frac{dt}{\sqrt{1-t}} = -\frac{ix}{\pi}\int_0^{\pi/2} e^{-\frac{x^2}{4\pi\cos^2 y}}\frac{dy}{\cos^2 y} \ .$$

Proof: As $f_H(x)$ is odd, one gets

$$\hat{f}_H(\xi) := F\{f_H(\xi)\} = \frac{2i}{\sqrt{2\pi}}\int_0^\infty f_H(x)\sin(x\xi)dx \ .$$

We recall the property

$$F\{g'(\xi)\} = i\xi F\{g(\xi)\} \ .$$

Hence it holds

$$F\left\{\frac{d}{dx}\left[e^{-\pi x^2(1-t)}\right]\right\} = ixF\left\{e^{-\pi x^2(1-t)}\right\} = \frac{ix}{\sqrt{1-t}}e^{-\frac{x^2}{4\pi(1-t)}}$$

and therefore

$$\hat{f}_H(x) = -\frac{i}{2\pi}\int_0^1 t^{-1/2}\frac{x}{\sqrt{1-t}}e^{-\frac{x^2}{4\pi(1-t)}}\frac{dt}{1-t} \ .$$

q.e.d.

Remark: A wavelet is a function $\psi(x) \in L_2(R)$ with a Fourier transform which fulfills

$$0 < c_\psi := 2\pi\int_{-\infty}^{\infty}\frac{|\hat{\psi}(\omega)|^2}{|\omega|}d\omega < \infty \ .$$

Classical Hilbert spaces in complex analysis are examples of wavelets (e.g. Hardy space of L_2 functions on the unit circle with analytical continuation inside the unit disk, [IBo], [KBr2]):

Corollary: The function $f_H(x)$ is a wavelet.

Remark: The continuous wavelet transform is known in pure mathematics as Calderón's reproducing formula, i.e. for $\psi(x) \in L_1(R^n)$ real and radial with vanishing mean, i.e.

$$\int_0^\infty \frac{|\hat{\psi}(a\omega)|^2}{a}da \equiv 1 \quad \text{and} \quad \psi_a(x) := \frac{1}{a^n}\psi(\frac{x}{a})$$

it holds Calderón's (self-reciprocal) formula

$$g = \int_0^\infty \psi_a * \psi_a * g\frac{da}{a} \ .$$

Remark: With respect to the representation

$$f_H(x) = -\frac{1}{2\pi}\int_0^1 t^{-1/2}\frac{d}{dx}\left[e^{-\pi x^2(1+t)}\right]\frac{dt}{1+t}$$

we refer to the auxiliary function ([HEd] 10.3)

$$H(x) := \frac{d}{dx}\left[x^2\frac{d}{dx}G(x)\right] = \frac{d}{dx}\left[x^2\frac{d}{dx}\left[G(x)-1\right]\right] = 2\sum_1^\infty(2\pi^2n^4x^4 - 3\pi n^2x^2)e^{-\pi n^2x^2} > 0 \cdot$$

Applying $H(x)$ instead of $G(x)$ overcomes the convergence issue, when trying to build a self-adjoint integral operator representation of the Riemann duality equation. Its building idea is the “differentiation” step to get ride off the “jeopardizing” non-vanishing constant Fourier term. The prize to be paid, is, that it doesn’t define an underlying self adjoint integral operator.

Lemma: It holds

i) $M\left[f_H\right](s) = \pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2}) = \cot(\frac{\pi}{2}(1-s))\Gamma(\frac{s}{2})$ for $\mathrm{Re}(1-s) > 0$

i) $\frac{1}{2\sqrt{\pi}}\int_0^\infty f_H(x)\frac{dx}{x} = \frac{\pi}{2}$

iii) $\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2}) \underset{x\to 0^+}{\to} \pi \cdot$

Proof: It follows by variables substitutions $\pi x^2(1-t) = y$, $t = z^2$ and $z = \sin\tau$

$$\int_0^\infty x^s f_H(x)\frac{dx}{x} = 2\sqrt{\pi}\int_0^1\int^\infty x^{s+1}e^{-\pi x^2(1-t)}\frac{dx}{x}\sqrt{t}\frac{dt}{t}$$

$$= 2\sqrt{\pi}\int_0^1\frac{1}{2}\left[\frac{1}{\pi(1-t)}\right]^{(s+1)/2}\sqrt{t}\frac{dt}{t}\int^\infty y^{(s+1)/2}e^{-y}\frac{dy}{y} = \pi^{-s/2}\Gamma(\frac{1+s}{2})(\int_0^1\left[\frac{1}{(1-t)}\right]^{(s+1)/''}\sqrt{t}\frac{dt}{t}$$

and therefore

$$= \pi^{-s/2}\Gamma(\frac{1+s}{2})\int_0^1\left[\frac{1}{(1-z^2)}\right]^{(s+1)/2}2z\frac{dz}{z} = \pi^{-s/2}\Gamma(\frac{1+s}{2})2\int_0^{\pi/2}\left[\frac{1}{\cos^2\tau}\right]^{(s+1)/2}\cos\tau d\tau$$

From [SGr] 3.621 we get

$$\int_0^{\pi/2}\sin^{\mu-1}x\cos^{\nu-1}dx = \frac{1}{2}B(\frac{\mu}{2},\frac{\nu}{2}) \quad , \quad \mathrm{Re}(\mu),\mathrm{Re}(\nu) > 0$$

leading to

$$\int_0^{\pi/2} \frac{1}{\cos^s \tau} d\tau = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})} \quad \text{for} \quad \operatorname{Re}(1-s) > 0 \; .$$

Then i) follows by

$$\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})} = \frac{\pi \sin(\frac{\pi}{2}s)}{\pi \cos(\frac{\pi}{2}s)} \Gamma(\frac{s}{2}) = \tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2}) \; .$$

ii), iii) follows by the following formulas ([SGr] 0.234, 1.421, 8.322):

$$\sum_1^\infty \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \quad , \quad \tan(\frac{\pi}{2}x) = \frac{4x}{\pi} \sum_1^\infty \frac{1}{(2k-1)^2 - x^2}$$

$$\Gamma(\frac{x}{2}) = \frac{2}{x} \prod_1^\infty \frac{(1+\frac{1}{k})^{x/2}}{(1+\frac{x}{2k})} \; .$$

q.e.d.

Remark: We respect to the $\cot(\frac{\pi}{2}(1-s))$ term we recall from above Euler's formula for $x > 0$ ([SGr] 3.721) with its two versions

i) $$\int_0^\infty \sin(tx) \frac{dt}{t} = \frac{\pi}{2}$$

ii) $$\int_{-\infty}^\infty \frac{d}{dx}\left[e^{-i\nu x}\right] \frac{x}{2} \cot(\frac{x}{2}) \frac{dx}{x} = \frac{1}{2\pi} \int_{-\infty}^\infty \sin(\nu x) \frac{dx}{x} \; .$$

We further note

Remark: For $0 < s < 1$ and $0 < t < \infty$ it holds

$$\oint \frac{x^s}{x-t} \frac{dx}{x} = -\pi t^{s-1} \cot(\pi s) \; .$$

The representations

i) $$f_H(x) = -\frac{1}{2\pi} \int_0^1 \frac{1}{\sqrt{1-t}} \frac{d}{dx}\left[e^{-\pi x^2 (1-t)}\right] \sqrt{\frac{t}{1-t}} \frac{dt}{t}$$

ii) $$\hat{f}_H(x) = i \int_0^1 \left[\frac{d}{dx} e^{-\frac{x^2}{4\pi(1-t)}}\right] \sqrt{\frac{t}{1-t}} \frac{dt}{t}$$

lead to

Corollary: It holds

i) $$M\left[-\frac{d}{dx}e^{-\pi x^2(1-t)}\right](s)=\Gamma(\frac{s+1}{2})\left[\pi(1-t)\right]^{-\frac{s-1}{2}}$$

ii) $$M\left[-\frac{d}{dx}e^{-\frac{x^2}{4\pi(1-t)}}\right](s)=\Gamma\frac{s+1}{2})\left[\frac{1}{4\pi(1-t)}\right]^{-\frac{s-1}{2}}$$

iii) $$\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} f_H(2\pi n)=\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\hat{f}_H(n)$$, as $\hat{f}_H(0)=f_H(0)=0$.

Remark: With respect to the dualization theorem from [RDu] we note that also

$$(1+\frac{1}{x})\phi(x)\notin L_1(0,\infty) \quad .$$

Proof: It holds

$$\int_0^\infty (1+\frac{1}{x})f_H(x)dx=2\sqrt{\pi}\int_0^1\sqrt{t}\int_0^\infty x^2(1+\frac{1}{x})e^{-\pi x^2(1-t)}\frac{dx}{x}\frac{dt}{t}=\sqrt{\pi}\int_0^1\sqrt{t}\int_0^\infty(\frac{y}{\pi(1-t)}+\sqrt{\frac{y}{\pi(1-t)}})e^{-y}\frac{dy}{y}\frac{dt}{t}$$

$$=\int_0^1\left[\frac{\sqrt{t}}{\sqrt{\pi}(1-t)}+\frac{\sqrt{\pi t}}{\sqrt{1-t}}\right]\frac{dt}{t}=2\int_0^1\left[\frac{x}{\sqrt{\pi}(1-x^2)}+\frac{x\sqrt{\pi}}{\sqrt{1-x^2}}\right]\frac{dx}{x}$$

$$=2\int_0^{\pi/2}\left[\frac{1}{\sqrt{\pi}\cos^2 t}+\frac{\sqrt{\pi}}{\cos t}\right]\cos\tau d\tau=\sqrt{\frac{2}{\pi}}\int_0^{\pi/2}\left[\frac{\sqrt{2}}{\cos t}+\pi\right]dt$$ **q.e.d.**

Remark: It holds $f_H\notin L_1(0,\infty)$. From the Hilbert transform theory we recall that if $f\in L_2(-\infty,\infty)$ then $f_H\in L_2(-\infty,\infty)$ with

$$\int_{-\infty}^{\infty} f_H(x)f(x)dx=0$$ and therefore $$\|f-f_H\|^2=\|f\|^2+\|f_H\|^2=2\|f\|^2 \quad .$$

For $x\in(o,\infty)$ it holds

$$\|f\|^2=\frac{1}{2\pi}\left\|\hat{f}\right\|^2 \quad , \quad \left\|\hat{f}_H\right\|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|-isign(x)\hat{f}(x)\right|^2dx \, .$$

Remark: As $f_H\notin L_1$ the Müntz formula ([ETi] 2.11)

$$\tilde{\xi}^*(s):=\zeta(s)\pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2})=\int_0^\infty x^s\left[\sum_{n=1}^\infty f_H(nx)-\frac{1}{x}\int_0^\infty f_H(t)dt\right]\frac{dx}{x}$$

can only formally be applied.

With respect to the L_2- Hilbert space (see also [LGa]) we note

Lemma: For the Fourier transform of $f_H(x)$ it holds $\hat{f}_H(x)=-\hat{f}(x)$ weak in the L_2 –sense,

i.e.

$$(\hat{f}_H,v)=-(\hat{f},v) \qquad \forall v\in H:=\left\{v\in W_2^1 \middle| v(0)=v'(0)=0\right\}.$$

Proof: is given in the appendix.

Recalling the defintion of a function with values in the distribution space from above a complex valued Zeta fake distribution $\zeta^*(s)$ can be defined by the weak distribution variational formulation (see also [AZe] 4.4, 4.6) leading to

Proposition: In the critical stripe by

$$\xi^*(s):=\zeta(s)\pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2})=\int_0^\infty x^s G_H(x)\frac{dx}{x}:=\int_0^\infty x^s\left[\sum_{n=1}^\infty f_H(nx)\right]\frac{dx}{x}$$

a holomorphic function in the distribution sense. A duality equation is valid in the form

$$(\xi_s^*,\chi)_0=(\xi_{1-s}^*,\chi)_0 \qquad \text{for all} \quad \chi\in H\,,$$

which is a representation as transform of a self-adjoint (singular, convolution) integral operator on the critical line.

Consequently the zeros of the Zeta function ξ_s^* have to lie on the critical line ([DCa]). Therefore all zeros of the Zeta function $\zeta(s)$ lie on the critical line due to standard (functional analysis) density arguments.

Concerning necessary and sufficient conditions for a real entire function of genus 0 or 1 to have only real zeros we note the Jensen's inequalities ([GPo3], see also [XLi]).

Related to the proposition above we recall from [DCa] the

Lemma: If a function $\Xi^*(t):=\xi^*(1/2+it)$ with

$$\xi^*(s):=\zeta^*(s)(s-1)\tilde{\Pi}(s)=\xi^*(1-s)$$

can be realized as a convolution

$$\Xi^*(t)=(G*dF)(t) \text{ where } G(t)\in LP^*,$$

i.e. if $\Xi^*(t)$ is a entire function from the Laguerre-Polya class of order <2, this would prove the RH.

2.2 The sawtooth function

Let $[x]$ denote the largest integer not exceeding the real number x and let

$$\varphi(x) := \pi\rho(\frac{x}{2\pi}) = \frac{1}{2}\{x\} = \frac{1}{2}(x-[x]) = \frac{\pi}{2} - \sum_1^\infty \frac{\sin \nu x}{\nu} \quad .$$

From [ETi] 2.1, we recall the following representation of the Zeta function

$$\zeta(s) = -s\int_0^\infty \rho(\frac{1}{x})x^s \frac{dx}{x} = \int_0^\infty (-sx^{-s})\rho(x)\frac{dx}{x} \qquad \text{for } 0 < \mathrm{Re}(s) < 1 \quad .$$

Lemma: For the normalized fractional part $\varphi(x)$ and $\varphi_H(x) := [H\varphi](x)$ it holds

i) $\varphi(x) = \frac{\pi}{2} - \sum_1^\infty \frac{\sin \nu x}{\nu}$ ([NNi], §70, §78)

ii) $\int_0^\infty \sin(\nu x)\frac{dx}{x} = \frac{\pi}{2} sign(\nu)$ ([SGr] 3.721, 3.827)

iii) $\varphi_H(x) = \sum_1^\infty \frac{\cos \nu x}{\nu} = \sum_1^\infty k_\nu \cos \nu x = \log 2 \sin \frac{x}{2}$ ([NNi], §70, §78)

iv) $-\log 2\sin(\pi s) = \log\frac{\Gamma(s)}{\sqrt{2\pi}} + \log\frac{\Gamma(1-s)}{\sqrt{2\pi}}$ for $0 < \mathrm{Re}(s) < 1$ ([NNi], §77)

v) $\varphi_H(x) \in H_{-1/2-\varepsilon}$ (Sobolev space) , as $\sum_{\nu=1}^\infty \nu^{-1-\varepsilon} < \infty$

vi) $\|\varphi^H\|_{-1/2}^2 = \zeta(1)$, $\|\varphi^H\|_0^2 = \frac{\pi^3}{6}$

vii) $\log\sin(\pi s) = \int_0^1 \frac{t^{s-1}+t^{-s}-2\sqrt{t}}{(1-t)}\frac{dt}{\log t}$ for $0 < \mathrm{Re}(s) < 1$ ([NNi], §68)

viii) $\pi\cot \pi s = \int_0^1 \frac{t^{s-1}+t^{-s}-2\sqrt{t}}{(1-t)}dt$ for $0 < \mathrm{Re}(s) < 1$ ([NNi], §68)

ix) $\left|\sum_1^N \frac{\cos \nu x}{\nu}\right| \le \log(\frac{1}{x}) + C$ ([AZy] 7.2, 13.11, 5.28)

see also [HEd] 9.7, [CBe] chapter 8, entry 17.

Remark: We note ([HEd], [TAm], [SGr] 4.224 7./8.)

i) $$\frac{1}{2\pi}\int_0^{2\pi}\log 2\sin\frac{y}{2}dy=\frac{1}{2\pi}\int_0^{2\pi}\varphi_H(y)dy=\frac{\pi}{1}\xi(1)=0$$

ii) $$\frac{1}{2\pi}\int_0^{2\pi}\log^2 2\sin\frac{y}{2}dy=\frac{1}{2\pi}\int_0^{2\pi}\varphi_H^2(y)dy=\frac{1}{2}\zeta(2)=\frac{\pi}{2}\xi(2)=\frac{\pi^2}{12}\quad .$$

Lemma: It holds for $0<\mathrm{Re}(s)<1$

$$\frac{(2\pi)^{1-s}}{\pi}\int_0^\infty x^s\varphi_H(x)\frac{dx}{x}=\frac{(2\pi)^{1-s}}{\pi}\zeta(1+s)\Gamma(s)\cos(\frac{\pi}{2}s)=\zeta(1+s)\chi(1-s)$$

with

$$\chi(s):=\pi^{s-1/2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\cdot$$

Proof: It holds for $0<\mathrm{Re}(s)<1$

$$\int_0^\infty x^s\varphi_H(x)\frac{1}{x}=\sum_1^\infty\frac{1}{n}\int_0^\infty x^s\cos(nx)\frac{dx}{x}=\sum_1^\infty\frac{1}{n^{s+1}}\int_0^\infty y^s\cos(y)\frac{dy}{y}$$
$$=\zeta(1+s)\int_0^\infty y^s\cos y\frac{dy}{y}=\zeta(1+s)\Gamma(s)\cos(\frac{\pi}{2}s)\cdot$$

whereby

$$\int_0^\infty y^s\cos y\frac{dy}{y}=\Gamma(s)\cos\frac{\pi}{2}s\qquad\text{for}\quad 0<\mathrm{Re}(s)<1\quad .$$

and

$$(2\pi)^{s-1}\pi\chi(1-s)=\cos(\frac{\pi}{2}s)\Gamma(s)$$ **q.e.d.**

Remark: Regarding the correlation of $f_H(x)$ and $\varphi_H(x)$ we note

i) $$\int_0^\infty f_H(x)\chi(x)dx=c\int_0^\infty\int_0^\infty\xi e^{-\pi\xi^2}\sin(\xi x)\chi(x)d\xi$$

ii) $$\int_0^\infty\varphi_H(x)\chi(x)dx=c\sum_1^\infty\frac{1}{\nu}\left[\int_0^\infty\cos(\nu x)\chi(x)dx\right]\cdot$$

For further Hilbert space analysis of the function φ_H we refer to [KBr2].

With respect to i) and ii) above we refer to the Bessel-Hilbert space H_B resp. to $H_{-1/2}$ ([KBr1], ([KBr2]). In the $H_{-1/2}$ – Hilbert space framework the shift $s \to s+1$ in the argument of the Zeta function (while performing the variable transformation $y = nx$)

$$\frac{(2\pi)^{1-s}}{\pi}\int_0^\infty x^s \varphi_H(x)\frac{dx}{x} = \zeta(1+s)\chi(1-s)$$

wouldn't happen. Therefore in the $H_{-1/2}$ – framework a correspondingly defined weak Zeta function leads to the definition of $\zeta(s)$ instead of $\zeta(1+s)$ in the critical stripe:

therefore, with

$$\varphi_s^H := -sx^{s-1}\varphi^H(x) = (-\varphi^H(x))\frac{d}{dx}x^s$$

by

$$(\xi_s^{**},\chi)_{-1/2} = (\varphi_s^H,\chi)_{-1/2} \qquad \text{for all } \chi \in H_{-1/2}$$

a complex-valued distribution Zeta function is defined ([KBr2]), [AZe] 4.4, 4.6). .

It holds

Proposition: The complex-valued distribution Zeta function ξ_s^{**} fulfills

$$2\int_0^\infty x^s \varphi_H(2\pi x)\frac{dx}{x} = \zeta(s)\chi(1-s) =: \xi^{**}(s) \qquad \text{(with } \xi^{**}(1) = 0\text{)}$$

In the critical stripe a duality equation is valid in the form

$$(\xi_s^{**},\chi)_{-1/2} = (\xi_{1-s}^{**},\chi)_{-1/2} \qquad \text{for all } \chi \in H_{-1/2} \ .$$

Corollary: As the duality equation is built as transform of a self-adjoint integral operator the zeros of the Zeta function ξ_s^{**} have to lie on the critical line ([DCa]).

Remark: In order to enable an analysis within the L_2 – framework we note the representation

$$\zeta(s) = -sc\int_0^\infty x^{1-s}\left[\frac{\varphi(x)}{x}\right]\frac{dx}{x} \qquad \text{for} \quad 0 < \mathrm{Re}(s) < 1 \ .$$

This suggests an analysis of the Hilbert transform $\psi_H(x)$ of

$$\psi(x) := \frac{\varphi(x)}{x} .$$

We note that the formula

$$M[x\psi'](s) = -sM[\psi](s) \ .$$

Remark (**Lommel polynomials**), see also §3 below:

In [KBr2] the Lommel polynomials $g_{n.1}(x)$ are proposed corresponding polynomial orthogonal system framework to build a (negatively scaled) Hilbert space. D. Dickinson's proof ([DDi]) of the orthogonality of the modified Lommel polynomials is built on a properly defined Riemann-Stieltjes integral, enabled by the density function

$$d\psi = \frac{J_1(2\sqrt{x})}{\sqrt{x}J_0(2\sqrt{x})}dx \quad \text{with} \quad \frac{J_1(2\sqrt{x})}{\sqrt{x}J_0(2\sqrt{x})} = \lim_{n\to\infty}\frac{g_{n.1}(x)}{g_{n+1.0}(x)} \quad ,$$

which is analytic outside any circle that contains the finite zeros of $J_\nu(1/x)$. The prize to be paid to build the orthogonality relation is an only stepwise density (bounded variation) function $d\psi$. It leads to the following orthogonality relation (whereby the α_k denote the zeros of $J_0(2\sqrt{x})$)

$$(*) \qquad \sum_{k=1}^{\infty}\frac{g_n(\alpha_k)}{2\alpha_k^{(n+1)/2}}\frac{g_m(\alpha_k)}{2\alpha_k^{(m+1)/2}} = \frac{\delta_{n,m}}{2(n+1)} \ .$$

A relation to the Bessel-Fourier Hilbert space framework is given by:

$$\int_0^\infty \sum_{k=1}^\infty J_1(j_\nu x)dx = \int_0^\infty \vartheta(x)\frac{dx}{x} = \int_0^\infty \vartheta(\frac{1}{x})\frac{dx}{x} = 1 \quad ,$$

whereby $\{4\alpha_k = j_k^{\,2}\}_{k\in N}$ and

$$\vartheta(x) := \frac{1}{2}\sqrt{x}\sum_{k=1}^\infty J_1(2\sqrt{\alpha_\nu x}) \ .$$

Remark (the **1st Mellin' inverse problem**): For complex number in the critical stripe, not lying on the critical line, the 1st Mellin' inverse problem has an solution:

putting $$W(s):=\int_0^1 x^s\psi(x)\frac{dx}{x}$$

with $$\psi(x):=\log\sin(\frac{\pi}{2}x)$$

it holds ([SGr] 4.322)

$$W(s)=-\frac{1}{s}\left[\frac{1}{s}-\sum_1^\infty\frac{\zeta(2k)}{4^k(s+2k)}\right] \quad \text{for} \quad \mathrm{Re}(x)>0\ .$$

Remark: Let

$$f(x):=\int_0^1\frac{\psi(t)}{1-(1-t)x}dt \quad , \quad g(x):=-\int_0^1\frac{\psi(t)}{1+tx}dt$$

and

$$F(s):=\int_0^1 x^s\frac{f(\frac{1}{1+x})}{1+x}\frac{dx}{x} \quad , \quad G(s):=\int_0^1 x^{1-s}g(x)\frac{dx}{x}\ .$$

Then it holds

$$f(x)=\frac{1}{1-x}g(\frac{x}{1-x}) \quad , \quad g(x)=\frac{1}{1+x}g(\frac{x}{1+x})$$

and the solution of the 1st Mellin' inverse problem can be applied ([NNi] §89) leading to the representation of $W(s)$ in the form

$$W(s)\frac{\pi}{\sin\pi s}=W(s)\Gamma(s)\Gamma(1-s)=G(1-s)+F(s) \quad \text{for} \quad 0<\mathrm{Re}(s)<1/2\,.$$

Remark: With respect to the representation of the 1st Mellin inverse problem we refer to the Riemann-Siegel integral formula ([HEd] 7.9)

$$\frac{2\xi(s)}{s(s-1)}=F(s)+\overline{F(1-\bar{s})}$$

Remark: We recall from the above ([NNi], §77)

$$-\log 2\sin(\pi s)=\log\frac{\Gamma(s)}{\sqrt{2\pi}}+\log\frac{\Gamma(1-s)}{\sqrt{2\pi}} \quad , \ 0<\mathrm{Re}(s)<1 \quad .$$

Remark (**the Stieltjes transform**):

The Stieltjes-transform for complex $s \in C$ is given by

$$S[f](s) := \int_{-\infty}^{\infty} f(y) \frac{dy}{s+y} \quad .$$

Putting

$$\psi_c(x) := \frac{1}{2}\operatorname{cosech}(\frac{x}{2}) = \frac{1}{e^{x/2} - e^{-x/2}} \quad , \quad \psi_s(x) := \frac{1}{2}\operatorname{sech}(\frac{x}{2}) = \frac{1}{e^{x/2} + e^{-x/2}}$$

the following relations between the convolutions

$$\mathrm{K}_c[\vartheta](x) := (\psi_c * \vartheta)(x) := \int_{-\infty}^{\infty} \psi_c(x-y)\vartheta(y)dy \quad , \quad \mathrm{K}_s[\vartheta](x) := (\psi_s * \vartheta)(x) := \int_{-\infty}^{\infty} \psi_s(x-y)\vartheta(y)dy$$

and the Hilbert- resp. the Stieltjes-transforms of $\psi_{c/s}(x)$ are valid:

Lemma: It holds

$$\frac{\mathrm{K}_c[\vartheta](\log x)}{\sqrt{x}} = H\left[\frac{\vartheta(x)}{\sqrt{x}}\right](x) \qquad \text{and} \qquad \frac{\mathrm{K}_s[\vartheta](\log x)}{\sqrt{x}} = S\left[\frac{\vartheta(x)}{\sqrt{x}}\right](x) \quad .$$

Proof: Putting

$$\Theta_c(x) := (\psi_c * \vartheta)(x) \quad , \quad \Theta_s(x) := (\psi_s * \vartheta)(x)$$

it holds

$$\Theta_{c/s}(x) = \mathrm{K}_{c/s}[\vartheta](x) = \int_{-\infty}^{\infty} \frac{1}{e^{(x-t)/2} \mp e^{-(x-t)/2}} \vartheta(t)dt \quad .$$

Substituting the variables by $y := e^x$ and $\tau := e^t$, results into

$$e^{(x-t)/2} = \frac{\sqrt{y}}{\sqrt{\tau}}$$

and therefore

$$\frac{\Theta_{c/s}(\log y)}{\sqrt{y}} = \int_{-\infty}^{\infty} \frac{\vartheta(\log \tau)}{\sqrt{\tau}} \frac{d\tau}{y \mp \tau} \quad .$$

q.e.d.

Remark (the Euler constant): (related to an idea about a proof that the Euler constant is irrational): By the cosinus integral function

$$ci(x) = -\int_x^\infty \cos(t)\frac{dt}{t} = \frac{1}{2}li(e^{ix}) + li(e^{-ic})$$

the following characterization of the Gamma constant ([NNi] chapter II, §33, [JHa] 12.2) is given

$$\gamma = \frac{1}{2} + \sum_{n=1}^\infty \int_n^\infty \cos(2\pi t)\frac{dt}{t} = 1 - 2\int_1^\infty \frac{\varphi(x)}{x}\frac{dx}{x} \ .$$

In [ELa] a continuous fraction representation is given for the integral $\int_x^\infty x^{-1}e^{-x}dx$. With an analogue approach (see [PHu]) building an ICF representation of $ci(x)$ to prove the irrationality of γ, based on the theory of CF from Laplace, Soldner, Legendre ([NNi] chapter II, §17) and Stieltjes ([EHe]) due to the following

Theorem: (irrationality and uniqueness of ICF (Infinite Continued Fraction)) The value of any infinite simple continued fraction is an irrational number. Given any irrational number x, the ICF determined from x by the Continued Fraction Algorithm converges to x, and no other ICF converges to x.

An alternative representation of the Euler constant is given by

$$\gamma = \int_0^\infty d\mu = \lim_{x \to 0} \mu(x)$$

whereby $\mu(x) := \int_x^\infty \left[e^{-t} - J_0(2\sqrt{t})\right]\frac{dt}{t}$ and $\int_0^\infty x^s d\mu = \Gamma(s) - \frac{\Gamma(s)}{\Gamma(1-s)}$.

Proof: Since $\Gamma(1) = 1$, $\Gamma'(1) = -\gamma$ and $s\Gamma(s) = \Gamma(1+s)$ it holds

$$\Gamma(s) - \frac{\Gamma(s)}{\Gamma(1-s)} = \frac{1}{s}\left[(1-\gamma s) - \frac{(1-\gamma s)}{(1+\gamma s)} + O(s^2)\right] = \gamma\frac{(1-\gamma s)}{(1+\gamma s)} + O(s) \underset{s \to 0^+}{\to} \gamma \ .$$

q.e.d.

Remark (Dirac function): see also § 3 below:

The Dirac function

$$\delta(x) := \frac{1}{2\pi}\int_0^{2\pi} e^{ikx} dk = \frac{1}{\pi}\int_0^{\pi} \cos(kx) dk$$

can be interpreted as density function of particles to build the bridge between continuum physics and particle (quantum) physics. We note that for the space dimension n it holds $\delta \in H_{-n/2-\varepsilon}$ (Sobolev space), i.e. its regularity is depending from the dimension n. In the current case $n=1$ the regularity of the Dirac function is by the factor ε less regular than the Energy Hilbert space concerning the Operator A resp. the "weak" regularity of $\varphi^H(x) \in H_{-1/2}$. Applying the concepts of logarithmic capacity of sets and convergence of Fourier series ([AZy] V-11) to functions with

$$\sum_1^{\infty} n\left[a_n^2 + b_n^2\right] < \infty$$

we propose to apply $\varphi^H(x)$ to define an alternative Dirac function in the form

$$\delta^*(x) := \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|2\sin(x-y)/2|} d\mu(y) \text{ ,}$$

where $d\mu(y)$ is a mass distribution concentrated in an open set $O \subset (0,2\pi)$.

Remark: We note that in harmonic analysis by

$$[\varphi]^2 := \frac{\pi}{2}\sum_1^{\infty} \nu(a_\nu^2 + b_\nu^2) = \frac{1}{2}\iint |dh(z)|^2 dxdy = \frac{1}{4\pi}\int_{\partial B}\int_{\partial B} \frac{|\varphi(w)-\varphi(\zeta)|^2}{|w-\zeta|^2} ds(w) d\zeta < \infty$$

the energy of the harmonic continuation $h = \mathrm{E}(\varphi)$ to the boundary is given. The theorem of Levi and the Douglas-functional provides the foundation for the related minimal surfaces theory ([RCo]).

§ 3 The Helmholtz Free Energy

In this chapter we recall the mathematical background of the Helmholtz free energy of a quantum harmonic oscillator. Its link to the Zeta function and the basic ideas of §1 and §2 is the fact, that due to convergence issues the ground state energy is neglected, when calculating the free energy normalization factor. Even if this per definition very small it seems to be a problem of mathematics, which has to be solved, instead of dropping it out of the physical model itself.

Concerning the notations and the analysis of Hilbert scales we refer to [KBr2]. Our proposal is to move current quantum theory models from a L_2- based to a $H_{-1}-$ based Hilbert space environment, applying spectral theory for Hermitian operator.

Let

$$\varphi_\lambda(x) := -\frac{1}{2\pi}\log\left[2\sin\frac{x-\lambda}{2}\right] \quad , \quad \lambda \in [0,x].$$

For

$$\psi = \sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n + \int \varphi_\lambda(\varphi_\lambda,\psi)d\lambda$$

in combination with the relations (see e.g. [JNe])

$$(\varphi_n,\varphi_m) = \delta_{n,m} \ , \ (\varphi_\lambda,\varphi_n) = A\varphi_n(\lambda)$$

one gets

$$\psi = \sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n + \int \varphi_\lambda [A\psi](\lambda)d\lambda = \sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n + A^2\psi \ .$$

We recall that the spectrum for a self-adjoint operator is real and closed. If the operator is additionally compact, then the spectrum is discrete. In case the operator is not compact, but bounded (continuous), there is a spectral representation built on Riemann-Stieltjes integral over projection operator valued step functions (see also [KBr2], Lommel polynomials). In case of unbounded operators the closed graph theorem can be applied to build bounded operators with respect to the graph norm. The below indicates to further analyze the graph norm for the momentum operator for those physical states, represented by the elements out of

$$H_0^{\perp} := \left\{ \psi \in H_{-1} := \overline{H_0}^{\|\ \|} \,|\, ((\psi,\varphi)) = 0, \varphi, H\varphi \in H_0 \right\}$$

whereby

$$|||\psi|||^2 := \sum_{n=1}^{\infty}|(\psi,\varphi_n)|^2 + \|A\psi\|^2 = \sum_{n=1}^{\infty}|(\psi,\varphi_n)|^2 + \|\psi\|_{-1}^2 \ .$$

Remark: The norm

$$\|\psi\|_*^2 := \sum_{n=1}^{\infty} |(\psi,\varphi_n)|^2 + |(\psi,\varphi_n)_{-12}|^2 + \|\psi\|_{-1}^2$$

which is equivalent to $\|\psi\|_{-1}^2$, is proposed to be used to model spin (e.g. Zeeman) effects.

Remark: For $\psi \in H_0^{\perp}$ it holds

$$(([\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}]\psi,\varphi)) \cong (A\psi',\varphi) \cong -(H\psi,\varphi) = (\psi,H\varphi) = 0 \qquad \text{for all } \varphi \in H_0 \; .$$

Remark: We note that e.g. in case of the harmonic quantum oscillator it holds in the L_2 – framework

$$\overline{E}_0 = \frac{1}{2}\sum \hbar\omega_n \approx c\sum \hbar n = \infty \quad ,$$

which leads to the concept/requirement of "re-normalization" to ensure the existence of *bounded* hermitian operators $\overline{H}_{renorm}$, with

$$\overline{H} = \overline{H}_{renorm} + \overline{E}_0 \quad .$$

Remark: For the corresponding relation of the the norm $|||\psi|||^2$ to commutators of hermitian operators in the weaker $\|\psi\|_{-1}$ – norm we recall (from §2 resp. [SGr] 4.384, 1.441):

i) the norms $\|HA\psi\|_0^2 \cong \|A\psi\|_0^2 \cong \|\psi\|_{-1}^2$ are equivalent

ii) the range of a "constant" operator is zero according to

$$\frac{1}{2\pi}\oint_{0\to 2\pi} \log 2\sin\frac{y}{2}dy = 0 \quad , \quad \frac{1}{2\pi}\oint_{0\to 2\pi} \cot\frac{\lambda - y}{2}dy = 0 \; .$$

Remark: For the commutator $[P,Q]$ it holds

$$(([P,Q]\psi,\chi)) = c((\psi,\chi)) \qquad \text{for all } \chi \in H_{-1} \; .$$

Therefore for the Ritz projection ([KBr3])

$$R_{-1.0} : H_{-1} \to H_0$$

$$[P,Q]\psi \to \psi_R := [P,Q]_R\psi := R_{-1.0}([P,Q]\psi)$$

it holds

$$(\psi_h,\chi) = 0 \text{ for all } \chi \in H_0 \subset H_{-1} \; .$$

Remark: We note ([HEd], [TAm], [SGr] 4.224 7./8.)

i) $\log\xi(0)=-\log\zeta(0)=\log\frac{1}{2}$

ii) $\frac{1}{2\pi}\int_0^{2\pi}\log 2\sin\frac{y}{2}dy=\frac{\pi}{1}\xi(1)=0$

iii) $\frac{1}{2\pi}\int_0^{2\pi}\log^2 2\sin\frac{y}{2}dy=\frac{\pi^2}{12}=\frac{1}{2}\zeta(2)=\frac{\pi}{2}\xi(2)$

resp.

$$\int_0^1\log^2\frac{1}{2}\sin\frac{\pi}{2}xdx=\frac{\pi^2}{12}\quad,\quad\int_0^1\log^2\frac{1}{2}\cos\frac{\pi}{2}xdx=\frac{\pi^2}{12}\quad.$$

Remark: The internal forces of strings (being looked at as mechanical systems) are built on strains (see below), depending proportionally from its lengths:

$$L=\int_0^l\sqrt{1+u_x^2(x,t)}dx\ .$$

In physical models for small displacements this is replaced by

$$L=l+\Delta l=\int_0^l\left[1+\frac{1}{2}u_x^2(x,t)+...\right]dx\quad\text{with}\quad\Delta l\approx\int_0^l\frac{1}{2}u_x^2(x,t)dx\ .$$

The study of minimal surfaces (optimal geometry) applies methods of differential calculus to geometrical problems. A “minimal surface” is not meant in the sense, that e.g. a sphere is a “minimal surfaces” that it minimizes the surface area-to-volume, but in the sense of the minimal surface of smallest area spanning a given contour. The Lagrange formalism related to the Plateau problem is the minimization problem of the area

$$A_\Omega(u)=\iint_\Omega\sqrt{1+|\nabla f|^2}dudv$$

for an assumed existing graph S in the form

$$S=\{(u,v,f(u,v)|(u,v)\in\Omega\}=:G_f$$

of a function $f:\Omega\to R$. We further note that S is a minimal graph if and only if

$$f_{uu}(1+f_v^2)-2f_uf_vf_{uv}+f_{vv}(1+f_u^2)=0\ .$$

If f_f can be separated into two functions in the form $f(u,v)=g(u)+h(v)_f$ then it holds

$$g(u)=\log|\cos u|\quad\text{and}\quad h(v)=-\log|\cos v|\quad\text{, hence}$$

$$f(u,v)=\log\left|\frac{\cos u}{\cos v}\right|\ .$$

Remark: We recall from §2 the concepts of logarithmic capacity of sets and convergence of Fourier series ([AZy] V-11) to functions with

$$\sum_1^\infty n\left[a_n^2+b_n^2\right]<\infty$$

and the noted reference to harmonic analysis by

$$[\varphi]^2 := \frac{\pi}{2}\sum_1^\infty \nu(a_\nu^2+b_\nu^2) = \frac{1}{2}\iint |dh(z)|^2 dxdy = \frac{1}{4\pi}\int_{\partial B}\int_{\partial B}\frac{|\varphi(w)-\varphi(\zeta)|^2}{|w-\zeta|^2}ds(w)d\zeta<\infty \; .$$

It describes the energy of the harmonic continuation $h=\mathrm{E}(\varphi)$ to the boundary is given. The theorem of Levi and the Douglas-functional provides the foundation for the related minimal surfaces theory ([RCo]). We note that solutions of the Plateau problem are very much depending on the dimension, i.e. in case of $n\geq 4$ branch points (i.e. at those points the parametrisation can become singular) are inevitable; for $n=3$ see [ROs]. For $n=2$ this is the Riemann mapping theorem. In case of $n\geq 4$ the singular behavior fits well to the black hole phenomena.

Remark: A. Einstein developed his quantum/photon concept motivated by the question: „if one moves exactly in parallel to a light signal (a photon or a wave?), how the light signal looks like? In principle it should be that the signal of light is a sequence of stationary waves, which are fixed in the time, i.e. the light signal should look like without any movement. If one follows it, it looks like a non-moving, oscillating, electromagnetic field. But something like this seems to be not existed neither caused by observation, nor by the Maxwell-equations model. The later ones exclude the existence of stationary, inelastic waves. Based on the Maxwell equations the electrons would have to loose its energy within nearly no time.

In any relativistic theory the vacuum, the state of lowest energy, if it exists in „reality“, has to have the energy zero.

In the same way for any free particle with momentum $\vec{p}$ and mass m the energy has to be

$$E=\sqrt{m^2c^4+\vec{p}^2c^2} \; .$$

In the literature the ground state energy of the harmonic operator is mostly defined by $\frac{1}{2}\hbar\omega$.

Already M. Planck knew that this cannot be, when deriving his radiation formula: he assigned states with n photons the energy $n\hbar\omega$, but not the value

$$(n+\frac{1}{2})\hbar\omega \quad ,$$

which is not compatible with the relativistic co-variant description of photons.

The ground state energy is not measurable. Its chosen value is therefore arbitrarily, triggered only by the fact, to keep calculations as easily as possible, and, mainly, to ensure convergent integrals/series. Energies of freely composed systems should be additive. For photons in a box section (cavity) there are infinite numbers of frequencies ω_i. If one assigns any frequency a ground state energy value $\hbar\omega_i / 2$, then the ground state energy without photons has the infinite energy

$$\frac{1}{2}\sum_i \hbar\omega_i = \infty \cdot$$

The **miss understanding**, that the **ground state energy is fixed** and uniquely defined, starts already in the classical physics: The definition of the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}\omega^2 x^2 =: T + V$$

defines the non-measurable ground state energy in that way, that the state of lowest energy, the point $(x = 0, p = 0)$ in the phase space, that the energy is zero:

the kinetic energy of strings with mass ρ are given by

$$T = \rho\int_0^l \frac{1}{2} u_x^2(x,t)dx \cdot$$

The internal forces of strings (being looked at as mechanical systems) are built on strains, depending proportionally from its lengths:

$$L = \int_0^l \sqrt{1 + u_x^2(x,t)}dx \cdot$$

For small displacements this is replaced by

$$L = l + \Delta l = \int_0^l \left[1 + \frac{1}{2}u_x^2(x,t) + ...\right]dx \quad \text{with} \quad \Delta l \approx \int_0^l \frac{1}{2}u_x^2(x,t)dx \cdot$$

Correspondingly the potential energy $V(x)$ is approximately defined by

$$V(L) = V(l + \Delta l) \approx V(l) + \Delta l \frac{dV}{dL}\Big|_{L=l} \cdot$$

Putting

$$\sigma_s := \frac{dV}{dL}\Big|_{L=l}$$

as “tension” or “strain constant”, the choice

$$V(l) := 0$$

simplifies the algebraic term for the potential energy V in the form:

$$V \approx \sigma_s \int_0^l \frac{1}{2} u_x^2(x,t)dx \cdot$$

Putting the “string velocity”

$$c_s := \sqrt{\frac{\sigma_s}{\rho}}$$

then, for example, the wave equation of strings is given by

$$u_{tt} - c_s^2 u_{xx} = 0 \cdot$$

Alternatively to $V(x)$ in case of the harmonic oscillator one could have chosen instead e.g.

$$V(x) = \frac{1}{2}\omega^2 x^2 - \hbar\omega/2$$

or (with reference to the theory of minimal surfaces, using $1 + \sinh^2 x = \cosh^2 x$)

$$1 + V(x) = \kappa \cosh x \cdot$$

For a single particle in a potential energy $V(x,t)$ the Schrödinger equation is ([RFe] 4-1)

$$\psi_t(x,t) = -\frac{i}{\hbar}\overline{H}\psi(x,t)$$

with

$$\overline{H}\psi(x,t) := -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x} + V(x,t) \cdot$$

With respect to our proposal above we note

$$\overline{H}x - x\overline{H} = -\frac{\hbar^2}{m}\frac{\partial}{\partial x} \quad \text{resp.} \quad \left\|(\overline{H}x - x\overline{H})\psi\right\|_{-1} = c\|\psi\|_0 \quad .$$

Remark: The analogue to the physical state of a particle

$$\psi=\sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n+(\psi,\varphi_n)_{-1/2}\varphi_n+\int\varphi_\lambda[A\psi](\lambda)d\lambda$$

resp. to the a priori representation of a hermitian operator in the form

$$\overline{H}=\overline{H}_{renorm}+\overline{E}_0$$

can be interpreted as "ideal number" or "non-standard number" as analogue to a real number r represented in the form $r+i$, whereby i denotes an infinitely small, finite non-real number, which is not eqaul zero, but smaller than any positive real number $\varepsilon\in R^+$.

The negative-scaled inner product might enable an alternative for current wave package model to "bridge" the gap caused by particle and field dualism to model e.g. scattering phenomena.

In the mathematical world non-zero infinitesimal small numbers exist, as well. Ordered fields (like the real numbers) that have infinitesimal small elements do not fulfill the Archimedean principle. Such fields are called non-Archimedean. The Non-Archimedean extention of real numbers are the Hyperreals (monads, ideal points) and the related analysis is the Nonstandard Analysis [ARo1/2].

The existence of non-standard models of arithmetic was discovered by Th. Skolem in 1938/1938, one year after Heidegger´s publication of „The Age of the World", where we took the following quotation from:

"… *modern physics is called mathematical because, in a remarkable way, it makes use of a quite specific mathematics. But it can proceed mathematically in this way only because, in a deeper sense, it is already itself mathematical.* ..."

Some other related quotations are:

". . . *we observe that the non standard analysis is presented naturally, within the framework of contemporary mathematics, and thus appears to affirm the existence of all sorts of infinitely entities. . . . it appears to us today that the infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, the standard irrational numbers..."* (A. Robinson, 1966)

"..."*We may say a thing is at rest when it has not changed its position between now and then, but there is no 'then' in 'now', so there is no being at rest. Both motion and rest, then, must necessarily occupy time...."* Aristotle, 350 BC

"...*It is probably the last remaining task of the theoretical physics to show us how the term "force" is completely absorbed in the term "number"..."* ([RTa]).

The function

$$L(x) := -\log 2\sinh(x)$$

plays a key role in the context of free energy, vacuum energy of electromagnetic fields, the density matrix for a one-dimensional harmonic oscillator and the Planck black body radiation law (concerning the notations we refer to [RFe]):

the exact value of the free energy F of a linear system of harmonic oscillators is given by

$$\beta F := \sum_{k=1}^{\infty} L(\beta\lambda_k) \qquad \text{with } \frac{1}{\beta} := k_B T \text{ and } \lambda_k := \frac{\hbar\omega_k}{2}$$

with the related probability values in the form

$$a_k = e^{-\beta(\lambda_k - F)} .$$

Due to convergence issues in order to calculate a normalization factor Z the ground state zero term $\beta\lambda_0$ is omitted and F is replaced by

$$\beta F^* = \sum_{k=1}^{\infty} \log(1 - e^{-2\beta\lambda_k}) = -\sum_{k=1}^{\infty} K(2\beta\lambda_k)$$

leading to

$$a_k^* = e^{-\beta(\lambda_k - F^*)} = \frac{1}{Z^*} e^{-\beta\lambda_k} \quad \text{and} \quad \varphi^* := \sum_{k=1}^{\infty} a_k^* \varphi_k \in H_0 .$$

We propose the shift from the underlying Hilbert space H_0 into H_{-1} while keeping the information about the ground state term as part of the physical models, but applying the analysis of this paper to e.g.

$$a_k = e^{-\beta(\lambda_k - F)} = \frac{1}{Z} e^{-\beta\lambda_k} \quad \text{and} \quad \varphi := \sum_{k=1}^{\infty} a_k \varphi_k \in H_{-1} \text{ and } Z := \|\varphi\| .$$

Remark: We recall from [KBr2]: In [CBe] 8, Entry17 (iv) it's mentioned: "*Ramanujan informs us to note that*

$$\sum_{1}^{\infty} \sin(2\pi\nu x) = \frac{1}{2}\cot(\pi x) ,$$

which also is devoid of meaning .... may be formally established by differentiating the well known equality"

$$\sum_{1}^{\infty} \frac{\cos 2\pi\nu x}{\nu} = -\log 2\sin(\pi x) .$$

In this context we refer to §2 and the concept of “logarithmic capacity” of sets and convergence of Fourier series ([AZy], 13.11) with the example ([SGr] 4.384)

$$\lambda(x) \approx \sum_{1}^{\infty} \frac{\cos 2\pi\nu x}{\nu} = -\log 2\sin(\pi x)$$

with

$$\int_{0}^{2\pi} \log 2\sin\frac{y}{2}\,dy = 0 \ .$$

Remark: The polynomial system to build the H_0 are the Hermite polynomials. We sketch the building of a polynomial system for H_{-1}:

Let $\{\xi_k\}$ denote the zeros of $p_{n+1}(x)$. Then the theory of distributions of Stieltjes type with its relation to orthogonal polynomial systems and distribution $d\alpha(x)$ provide a decomposition into partial fractions ([GSz] theorem 3.3.5) in the form

$$\frac{p_n(x)}{p_{n+1}(x)} = \sum_{k=0}^{\infty} \frac{l_k}{x-\xi_k} \quad \text{with} \quad l_k := \frac{p_n(\xi_k)}{p'_{n+1}(\xi_k)} = \frac{p'_{n+1}p_n - p'_n p_{n+1}}{\{p'_{n+1}\}^2}(\xi_k) > 0 \ .$$

We propose to apply the Lommel polynomials $g_{n.\nu+1}(x)$ as corresponding polynomial orthogonal system framework to build the (negatively scaled) Hilbert space. D. Dickinson’s proof ([DDi]) of the orthogonality of the modified Lommel polynomials is built on a properly defined Riemann-Stieltjes integral, enabled by the density function

$$d\psi_\nu = \frac{J_{\nu+1}(2\sqrt{x})}{\sqrt{x}J_\nu(2\sqrt{x})}dx \quad \text{with} \quad \frac{J_{\nu+1}(2\sqrt{x})}{\sqrt{x}J_\nu(2\sqrt{x})} = \lim_{n\to\infty}\frac{g_{n.\nu+1}(x)}{g_{n+1.\nu}(x)} \ ,$$

which is analytic outside any circle that contains the finite zeros of $J_\nu(1/x)$. The prize to be paid to build the orthogonality relation is an only stepwise density (bounded variation) function $d\psi_\nu$.

In [JGr] a (Mittag-Leffler) decomposition into fractions is given in the form

$$2\nu\frac{J_\nu(z)}{J_{\nu-1}(z)} = \sum_{n=1}^{\infty}\frac{d\phi(\frac{1}{z_n})}{\frac{1}{z}-\frac{1}{z_n}} \ .$$

The zeros of continued fraction functions of Stieltjes type are analyzed in [OPe] §69, thm 5.

The stepwise density (bounded variation/fluctuation) function $d\psi := d\psi_0$, whereby

$$\int_0^\infty d\psi(x) < \infty$$

and all "moments"

$$c_k := \int_0^\infty (-x)^{k-1} d\psi(x) < \infty \quad \text{for} \quad k = 1,2,3,....$$

do exist. This leads to the following orthogonal relations (whereby the α_k denote the zeros of $J_0(2\sqrt{x})$)

$$(*) \qquad \sum_{k=1}^{\infty} \frac{g_n(\alpha_k)}{2\alpha_k^{(n+1)/2}} \frac{g_m(\alpha_k)}{2\alpha_k^{(m+1)/2}} = \frac{\delta_{n,m}}{2(n+1)} .$$

The polynomial system defines a (Distribution) Hilbert space, which is less regular than $H_0 = L_2$.

Remark: Favards's theorem (([TCh] 7, II, theorem 6.4) implies that the Lommel polynomials are orthogonal polynomials with respect to a positive weighted, bounded variation measure function. We recall from [DDi]

$$(*) \qquad \sum_{k=1}^{\infty} \frac{1}{j_k{}^2} h_m(\frac{\pm 1}{j_k}) h_n(\frac{\pm 1}{j_k}) = \frac{\delta_{n,m}}{2(n+1)} .$$

With the relations above it follows

Proposition: For the Lommel polynomials the following orthogonality relation holds true

$$(**) \qquad \sum_{k=1}^{\infty} \frac{g_n(\alpha_k)}{2\alpha_k^{(n+1)/2}} \frac{g_m(\alpha_k)}{2\alpha_k^{(m+1)/2}} = \frac{\delta_{n,m}}{2(n+1)} .$$

The proof of the orthogonality of the modified Lommel polynomials is built on a properly defined Riemann-Stieltjes integral [DDi], enabled by the term

$$\frac{d\rho}{dx} := \left[J_1(\frac{1}{x}) \right] / \left[J_0(\frac{1}{x}) \right] ,$$

which is analytic outside any circle that contains the finite zeros of $J_0(1/x)$. Hence it possesses a Laurent expansion about the origin that converges uniformly on and in any annulus, whose inside boundary has the finite zeros of $J_0(1/x)$ in its interior: Let C be the contour that encircles the origin in a positive direction and that lies within the annulus.

Then it holds [DDi]

$$\frac{1}{2\pi i}\int_C x^k h_n(x)d\rho = \begin{cases} 0 & k<n \\ \frac{1}{2^{n+1}(n+1)} & k=n \end{cases}$$

Let $\alpha(x)$ the non-decreasing step function having increase of

$$\frac{1}{j_k^2}=\frac{1}{4\alpha_k} \quad \text{at the point} \quad x=\frac{\pm 1}{j_k}=\frac{1}{2\sqrt{\alpha_k}} \quad \text{for } k=1,2,3,...$$

then it holds [DDi]

$$\int h_n(x)h_m(x)d\tilde{\alpha}(x)=\frac{\delta_{n,m}}{2^{n+1}(n+1)} \quad .$$

Remark: We sketch the link between Riemann-Stieltjes integral densities and hyper functions and distributions (see [KBr2]):

Let $\sigma(\lambda):=\|E_\lambda x\|^2$ in $\lambda\in(-\infty,\infty)$ be a bounded variation spectral function, which builds according to the Green function

$$G(z)=\int\frac{d\sigma(\lambda)}{\lambda-z}$$

the two holomorph Cauchy-Riemann representation in $\mathrm{Re}(s)>0$, $\mathrm{Re}(s)<0$ by

$$G(x+iy)-G(x-iy)=\int\left[\frac{1}{\lambda-(x+iy)}-\frac{1}{\lambda-(x-iy)}\right]d\sigma(\lambda)$$

Then the Stieltjes inverse formula is valid for continuous points a and b, i.e.

$$\sigma(b)-\sigma(a)=\lim_{y\to 0}\frac{1}{2\pi i}\int_a^b G(x+iy)-G(x-iy)dx \quad .$$

If there exists a spectral density functions $\sigma'(\lambda)$, it holds

$$\sigma'(\lambda)=\lim_{\mu\to 0^1}\frac{1}{2\pi i}\left[G(\lambda+i\mu)-G(\lambda-i\mu)\right] \quad .$$

In the one-dimensional case any complex-analytical function, as any distribution f on R, can be realized as the "jump" across the real axis of the corresponding in $C-R$ holomorphic Cauchy integral function

$$F(x):=\frac{1}{2\pi i}\oint\frac{f(t)dt}{t-x},$$

given by

$$(f,\varphi)=\lim_{y\to 0^1}\int_{-\infty}^{\infty}F(x+iy)-F(x-iy))\varphi(x)dx \quad .$$

Remark: The Hilbert space $L^2[0,1]$ is built on the Friedrichs extension of the regular operator

$$L : D(L) \to L_2(R^+, dx)$$

$$L[g(x)] = -g''(x) \quad , \quad D(L) := \{ g | g \in C^2, g(0) = g(1) = 0 \} .$$

For

$$L\psi = \lambda\psi \ , \lambda \in C ,$$

all eigenvalues are real and the eigenfunctions to different eigenvalues are orthogonal. In the case above those eigenvalues and eigenfunctions are

$$\lambda_k = k^2 \pi^2 \qquad \psi_k(x) = \sin(k\pi x) \ .$$

The domain of the Friedrichs self adjoint extension of the singular Bessel operator

$$L : D(L) \to L_2(R^+, dx)$$

$$L[g(x)] = -g''(x) - \frac{1}{4x^2} g(x)$$

requires an additional condition ([WBu]) in the form

$$D(L) := \left\{ g | g, g' \in AC_{loc}, \lim_{x \to 0^+} \frac{g(x)}{\sqrt{x} \log x} = \lim_{x \to 0^+} g(x) \frac{d\sigma(\log x)}{dx} = 0, g, g', \tilde{L}g \in L_2 \right\} \ .$$

Remark: For some further analysis of the Hamiltonian

$$H_{amiltonian}=\frac{\omega}{2}(a^{*}a+aa^{*})$$

with its relation to the operators

$$P_x:=-i\hbar\frac{d}{dx}:=-i\frac{1}{\sqrt{2}}\sqrt{m\omega}(a-a^{*})\ ,\qquad Q_x:=x:=\frac{1}{\sqrt{2}}\sqrt{\frac{1}{m\omega}}(a+a^{*})$$

we note that for any real functions $\hat{\varphi},\varphi\in L^2(R)$ it holds $(\hat{\varphi},\varphi)=0$. We further mention that any real function $\varphi(t)$ and its Hilbert transform $\hat{\varphi}(t)$ are orthogonal, if $\varphi(t),H(\varphi(t))\in L^1$ and the Fourier transform $F(\varphi(t))\in L^1$. This is due to the relation

$$\int_{-\infty}^{+\infty}\varphi(t)\hat{\varphi}(t)dt=\frac{i}{2\pi}\int_{-\infty}^{+\infty}sign(\omega)\left|F(\varphi(\omega))\right|^2d\omega\ ,$$

whereby $|F(\varphi(\omega))|^2$ is an even function. We further note that with $\alpha:=1/(\omega m)$ it holds

$$\alpha P^2+\frac{1}{\alpha}Q^2=\frac{1}{2}(a+a^{*})(a-a^{*})\ .$$

Putting

$$\tilde{A}:=A-(\psi,A\psi)\ ,\ \ \psi:=B-(\psi,B\psi)$$

the variances of the Hermitian operators A,B are defined by

$$(\Delta A)^2:=(\psi,\tilde{A}\psi)\,,\ \ (\Delta B)^2:=(\psi,\tilde{B}\psi)\ .$$

Let $\{A,B\}$ be the anti-commutator and let $[A,B]$ denote the commutator. Putting

$$c:=\frac{1}{2}(\psi,\{\tilde{A}\tilde{B}\}\psi)\in R\ \ ,\ d:=-\frac{1}{2i}(\psi,[\tilde{A}\tilde{B}]\psi)\in R$$

then $$(\Delta A)(\Delta B)\geq(\psi,\tilde{A}\tilde{B}\psi)=c+id\ .$$

This leads to the Heissenberg uncertainty inequality in the form

$$(\Delta A)^2(\Delta B)^2\geq\left|(\psi,\tilde{A}\tilde{B}\psi)\right|^2=\left|c\right|^2+\left|d\right|^2\geq\left|c\right|^2\ .$$

Corollary: Let the harmonic (quantum) oscillator model be defined (only!) as (Hilbert transformed) distributions equation in the form

$$(\tilde{H}_{amiltonian}\psi,\psi)_{-1/2}=(A[\tilde{H}_{amiltonian}\psi],\psi)_0$$

Then (note that $FT[x^2u(x)]=-FT''[u(\xi)]$) the corresponding weak commutator fulfills

$$([P_x,Q_x]\psi,\psi)=0$$

and the anti-communicator is self-adjoint.

§ Appendix

It holds $f_H(x) \notin L_1(o,\infty)$ but $f_H(x) \in H_{-\beta}$ ([BPe]). The dualizing technique of [RDu], building on the Poisson summation formula in combination with the Müntz formula structure, can only be applied, when an appropriate Hilbert space environment guarantees convergent integrals (in a weak sense). The corresponding framework are a Hilbert space with negative scale factor $H_{-\beta}$ where the scale factor is chosen in relation to the kernel of the underlying appropriate singular integral operator, enabling a weak (variational) formulation of the Müntz formula ([AZe]).

The property of vanishing constant Fourier term of a Hilbert-transformed function is used when building appropriate density function $d\sigma(x)$, which enable a self-adjoint-property of the underlying singular integral operator. As our proposed kernels result into singular integral operators a proper framework is required to ensure convergent integrals. In our cases the underlying Hilbert (variational) energy space is $H_{-1/2}$ with the inner product $(u,v)_{-1/2}$ ([KBr2].

The class of distributions, which is defined by divergent integrals, is the class of *oscillatory integrals* ([BPe] 1.19) leading to the concept of pseudo-differential operator. They are in the form

$$A(x) = \int e^{i\phi(x,\theta)} a(x,\theta) d\theta \ ,$$

where the phase function $\phi(x,\theta)$ is a suitable real valued function such that the integrand oscillates rapidly for large $|\theta|$ and the amplitude function $a(x,\theta)$ being allowed to have polynomial growth in θ. It would be too restrictive to require the integral to define a function. Therefore it's interpreted in the distribution sense. Thus one is actually be concerned with integrals of the type

$$\langle A,v \rangle = \int\int e^{i\phi(x,\theta)} a(x,\theta) v(x) dx d\theta \ .$$

The study of the Hilbert transform and the study of operational calculus for non-commuting operators in quantum mechanics contain some basic ingredients of the theory of pseudo-differential operators [BPe] 3.1).The Hilbert transform is a classical pseudo-differential operator with symbol $-isign(s)$. Its salient features enabled the introduction of the algebra of singular integral operators.

Singular distributions can be generated by Hadamard's "finite part" of a divergent integral ([AZe2] 1.4, 2.5); a technique for extracting a finite part from a divergent part, building pseudo functions appying Cauch's principle value concept), where it turns out that this finite part defines a singular distribution. We note (if $\phi(0) \neq 0$) the "finite part" representation

$$Fp\int_{-\infty}^{\infty} \frac{\phi(t)}{|t|} dt = \lim_{\varepsilon \to 0} \left[\left(\int_{-\infty}^{\infty} + \int_{\varepsilon}^{\infty} \right) \frac{\phi(t)}{|t|} dt + 2\phi(0) \log \varepsilon \right] \ .$$

The phase function $\phi(x,\theta)$ of oscillatory integrals is a suitable real valued function such that the integrand oscillates rapidly for large $|\theta|$ and the amplitude function $a(x,\theta)$ being allowed to have polynomial growth in θ. It would be too restrictive to require the integral to define a function. This is also the "dilemma" when trying to build the Riemann duality equation as transform of an integral operator based on the Müntz formula:

Lemma (Müntz formula): For $\omega(x),\omega'(x)$ continuous and bounded in any finite interval with $\omega(x)=o(x^{-\alpha})$ and $\omega(x)=o(x^{-\beta})$ for $x\to\infty$ and $\alpha,\beta>1$ it holds

(M) $$\zeta(s)\int_0^\infty x^s\frac{\omega(x)dx}{x}=\int_0^\infty x^s\left[\sum_1^\infty\omega(nx)-\frac{1}{x}\int_0^\infty\omega(t)dt\right]\frac{dx}{x}\qquad \text{for}\quad 0<\mathrm{Re}(s)<1.$$

Proof: we recall the proof from [ETi] 2.11:

i) because $\omega(x)$ is continuous and bounded in any finite interval with $\omega(x)=o(x^{-\alpha})$ it holds

$$\sum_1^\infty\frac{1}{n^s}\left|\int_0^\infty x^{s-1}\omega(x)\right|dx\qquad \text{exists for}\quad 1<\sigma<\alpha\ .$$

ii) $$\sum_1^\infty\omega(nx)-\int_0^\infty\omega(xt)dt=x\int_0^\infty\omega'(t)(t-[t])dt=x\int_0^{1/x}O(1)dt+x\int_{1/x}^\infty O((xt)^{-\beta})dt=O(1)$$

The first summand is justified, because $\omega(x)$ is continuous and bounded in any finite interval the second summand is justified, because $\omega(x)=o(x^{-\alpha})$, i.e. it holds

$$\sum_1^\infty\omega(nx)=O(1)+\frac{c}{x}\qquad \text{with}\qquad c=\int_0^\infty\omega(t)dt=:\omega_0\ .$$

Hence $$\int_0^\infty x^s\sum_1^\infty\omega(nx)+\frac{dx}{x}=\int_0^1 x^s\left[\sum_1^\infty\omega(nx)-\frac{c}{x}\right]\frac{dx}{x}+\int_1^\infty x^s\sum_1^\infty\omega(nx)\frac{dx}{x}+\frac{c}{s-1}$$

for $\sigma>0$ except $s=1$. It also holds for $\sigma<1$

$$-c\int_1^\infty x^{s-2}dx=\frac{c}{s-1}\ ,$$

and therefore the result for $0<\sigma=\mathrm{Re}(s)<1$.

Lemma: For the Fourier transform of $f_H(x)$ it holds

$$\hat{f}_H(x) = -\hat{f}(x) \quad \text{weak in the } L_2 - \text{ sense, i.e.}$$

$$(\hat{f}_H, v) = -(\hat{f}, v) \quad \text{for all} \quad v \in H := \left\{ v \in W_2^1 \middle| v(0) = v'(0) = 0 \right\}.$$

Proof: We recall the definition $\omega_1(x) := f(x) := e^{-\pi x^2}$. It holds in a weak $L_2 -$ sense

$$f_H(x) = 4\pi \int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi = -4\pi \left[\frac{f(\xi)}{2\pi\xi}\cos(2\pi\xi x)\Big|_{\xi=0}^{\xi=\infty} - \int_0^\infty f'(\xi)\frac{\cos(2\pi\xi x)}{2\pi\xi}d\xi \right]$$

$$f_H(x) = \lim_{\xi\to 0} e^{-\pi\xi^2}\frac{\cos(2\pi\xi x)}{\xi} + 2\int_0^\infty f'(\xi)\frac{\cos(2\pi\xi x)}{\xi}d\xi$$

$$\frac{1}{2} f_H(x) = \lim_{\xi\to 0} e^{-\pi\xi^2}\frac{\cos(2\pi\xi x)}{\xi} - 2\pi\int_0^\infty f(\xi)\cos(2\pi\xi x)d\xi \; .$$

With

$$2\int_0^\infty e^{-\pi\xi^2}\cos(b\xi)d\xi = e^{-\pi x^2}$$

it follows

$$\frac{1}{2\pi} f_H(x) = -f(x) + \lim_{\xi\to 0}\frac{e^{-\pi\xi^2}}{\xi}\cos(2\pi\xi x) \; .$$

We note the Fourier transform of $g_x^1(\xi) := \sin(\xi x)$ and $g_x^2(\xi) := \cos(\xi x)$:

$$\hat{g}_x^1(\omega) = \frac{1}{2i}\left[\delta(\omega - \frac{x}{2\pi}) - \delta(\omega + \frac{x}{2\pi})\right], \quad \hat{g}_x^2(\omega) = \frac{1}{2}\left[\delta(\omega - \frac{x}{2\pi}) - \delta(\omega + \frac{x}{2\pi})\right].$$

Hence

$$\frac{1}{2\pi}\hat{f}_H(x) = -\hat{f}(x) + \lim_{\xi\to 0}\frac{e^{-\pi\xi^2}}{2\xi}\left[\delta(x - 2\pi\xi) + \delta(x + 2\pi\xi)\right]$$

and therefore

$$(\hat{f}_H, v) = -(\hat{f}, v) + \lim_{\xi\to 0}\frac{v(0)}{\xi} = -(\hat{f}, v) \quad \text{for all} \quad v \in H := \left\{ v \in W_2^1 \middle| v(0) = v'(0) = 0 \right\}$$ **q.e.d.**

Remark: From [SGr] 3.896 we recall

$$\int_0^\infty e^{-ax^2}\sin(bx)dx = \frac{b}{2a}e^{-\frac{b^2}{4a}}{}_1F_1(\frac{1}{2};\frac{3}{2};\frac{b^2}{4a}) = \frac{b}{2a}{}_1F_1(1;\frac{3}{2};-\frac{b^2}{4a}) = \frac{b}{2a}\sum_{k=1}^\infty \frac{1}{(2k-1)!!}(-\frac{b^2}{4a})^{k-1}$$

$$\int_0^\infty e^{-ax^2}\cos(bx)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{b^2}{4a}} \; , \quad \mathrm{Re}(a) > 0 \; .$$

Remark: We give some further background and analysis of the even function

$$k(x) := -\ln\left|2\sin\frac{x}{2}\right| =: -\log\left|2\sin\frac{x}{2}\right| .$$

Consider the model problem

$$-\Delta U = 0 \qquad \text{in } \Omega$$

$$U = f \qquad \text{on } \Gamma := \partial\Omega \ ,$$

whereby the area Ω is simply connected with sufficiently smooth boundary. Let $y = y(s) - s \in (0,1]$ be a parametrization of the boundary $\partial\Omega$. Then for fixed $\bar{z}$ the functions

$$U(\bar{x}) = -\log|\bar{x} - \bar{z}|$$

Are solutions of the Lapace equation and for any $L_1(\partial\Omega)-$ integrable function $u = u(t)$ the function

$$(Au)(\bar{x}) := \oint_{\partial\Omega} \log|\bar{x} - u(t)| dt$$

is a solution of the model problem. In an appropriate Hilbert space H this defines an integral operator ,which is coercive for certain areas Ω and which fulfills the Garding inequality for general areas Ω . We give the Fourier coefficient analysis in case of $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $x(s) := (\cos(s), \sin(s))$ be a parametrization of $\Gamma := S^1(R^2)$ then it holds

$$|x(s) - x(t)|^2 = \left|\begin{pmatrix}\cos(s) - \cos(t)\\ \sin(s) - \sin(t)\end{pmatrix}\right|^2 = 2 - 2\cos(s-t) = 2(1 - \cos(2\frac{s-t}{2})) = 2\left[2\sin^2\frac{s-t}{2}\right] = 4\sin^2\frac{s-t}{2}$$

and therefore

$$-\log|x(s) - x(t)| = -\log 2\left|\sin\frac{s-t}{2}\right| = k(s-t) \ .$$

The Fourier coefficients k_ν of the kernel $k(x)$ are calculated as follows

$$k_\nu := \frac{1}{2\pi}\oint k(x)e^{-i\nu x}dx = \frac{1}{2\pi}\int_0^{2\pi}\log\left|2\sin\frac{t}{2}\right|e^{-i\nu t}dt = \frac{2}{2\pi}\int_0^{\pi}\log\left|2\sin\frac{t}{2}\right|\cos(\nu t)dt = k_{-\nu}$$

As $\varepsilon\log 2\sin\frac{\varepsilon}{2} \underset{\varepsilon\to 0}{\to} 0$ partial integration leads to

$$k_\nu = \frac{1}{\nu\pi}\sin(\nu t)\Big|_0^{\pi} - \frac{1}{\nu\pi}\int_0^{\pi}\frac{2\sin(\nu t)\cos\frac{t}{2}}{2\sin\frac{t}{2}}dt = -\frac{1}{\nu\pi}\int_0^{\pi}\frac{\sin(\frac{2\nu+1}{2}t) - \sin(\frac{2\nu-1}{2}t)}{2\sin\frac{t}{2}}dt$$

$$k_\nu = -\frac{1}{\nu\pi}\int_0^{\pi}(\left[\frac{1}{2} + \cos(t) .. + \cos(\nu t)\right] - \left[\frac{1}{2} + \cos(t) .. + \cos((\nu-1)t)\right])dt = -\frac{1}{\nu} \ .$$

Remark: From [GWa] 17-22, we note

$$\frac{x}{1-x} = 2\sum_{k=1}^{\infty} J_k(kx) \quad \text{and} \quad \frac{x}{1+x} = 2\sum_{k=1}^{\infty} (-1)^{k-1} J_k(kx) \cdot$$

Corollary: It holds the representation as a Kapteyn series ([GWa] 17) in the form

$$g(x) := \frac{1}{1-x} = -\frac{1}{x} g(\frac{1}{x}) = \frac{1}{x}\frac{x}{1-x} = \frac{2}{x}\sum_{k=1}^{\infty} J_k(kx)$$

resp.

$$\log(g(x)) - \log(g(1-x)) = \log\frac{1}{1-x} - \log\frac{1}{x} = \log\frac{1}{\pi}\int_0^{\pi}\sum_{k=1}^{\infty} 2\sin(k(x\sin\varphi - \varphi))d\varphi$$

whereby ([GWa] 8-5, "Descriptive properties of $J_k(kx)$ for $0 < x \leq 1$)

$$J_k(kx) = \frac{1}{\pi}\int_0^{\pi} e^{k(x\sinh\varphi - \varphi)} d\varphi = \frac{1}{\pi}\int_0^{\pi} \sin(k(x\sin\varphi - \varphi))d\varphi \cdot$$

Remark: The relation of $\log[g(x)]$ to the Gamma function in the form

$$\frac{d}{ds}[\log\Gamma(s)] + \gamma = \sum_0^{\infty} \frac{(-1)^k}{k+1}\binom{s-1}{k+1} = \sum_0^{\infty}\left[\frac{1}{k+1} - \frac{1}{k+s}\right]$$

and its relation to the 1st and 2nd Mellin inverse problem is given in [NNi], §92.

Let

$$\Pi(s) = \int_0^\infty x^s d\lambda(x)$$

with

$$d\lambda(x) := -l(x)\frac{d\log(l(x))}{x} \quad \text{and} \quad l(x) := \frac{1}{2\sinh(x)} .$$

Lemma: For the transform $\Pi(s)$ of $d\lambda(x)$ it holds $\Pi(0)$ is divergent and

$$\Pi(2m+1) = (2^{21}-1)\pi^{2m}|B_{2m}| \quad , \quad \Pi(2m+2) = \frac{2^{2m+1}-1}{2^{2m}}(2m+1)!\zeta(2m+1) .$$

Proof: It holds

$$l(x) = \frac{1}{2\sinh(x)} \quad \text{and} \quad \frac{d}{dx}\log(l(x)) = -L'(x) = -\log' 2\sinh(x) = \frac{2\cosh(x)}{2\sinh(x)} = \coth(x)$$

and therefore

$$\Pi(s) = \int_0^\infty x^s d\lambda(x) = \int_0^\infty x^s l(x)\frac{-dL(x)}{x} = \frac{1}{2}\int_0^\infty x^s \coth(x)\frac{dx}{x\sinh(x)} = \frac{1}{2}\int_0^\infty x^s \frac{\cosh(x)}{\sinh^2(x)}\frac{dx}{x} .$$

From [SGr] 3.527 we recall

$$\Pi(2m+1) = \int_0^\infty x^{2m+1} d\lambda(x) = \frac{1}{2}\int_0^\infty x^{2m} \frac{\cosh(x)}{\sinh^2(x)} dx = (2^{21}-1)\pi^{2m}|B_{2m}|$$

$$\Pi(2m+2) = \int_0^\infty x^{2m+2} d\lambda(x) = \frac{1}{2}\int_0^\infty x^{2m+1} \frac{\cosh(x)}{\sinh^2(x)} dx = \frac{2^{2m+1}-1}{2^{2m}}(2m+1)!\zeta(2m+1) ,$$

which proves the lemma. **q.e.d.**

From [LGa] we recall the two versions of Ikehara theorem:

Lemma (Ikehara 1): Let μ be a monotone nondecreasing function on $(0,\infty)$ and let

$$F(s)=\int_1^\infty x^{1-s}\frac{d\mu(x)}{x} .$$

If the integral converges absolutely for $\operatorname{Re}(s)>1$ and there is a constant A such that

$$F(s)-\frac{A}{s-1}$$

extents to a continuous function in $\operatorname{Re}(s)\geq 1$, then $\mu(x)\approx Ax$.

Lemma (Ikehara 2): Let the Dirichlets series

$$F(s)=\sum_1^\infty \frac{c_n}{n^s}$$

be convergent for $\operatorname{Re}(s)>1$. If there exists a constant A such that

$$F(s)-\frac{A}{s-1}$$

admits a continuous extension to the line $\operatorname{Re}(s)\geq 1$, then

$$\sum_1^N c_n \approx A*N \qquad \text{as} \quad N\to\infty .$$

Remark: The Montgomery Zeta function density theorem ([HMo]) states

$$N(\sigma,T):=\sum_{\substack{\rho=\alpha+i\beta \\ \alpha\geq\sigma \\ 0<\beta\leq T}} 1 = O(T^{\frac{4(1-\sigma)}{3-2\sigma}}\log^9 T) .$$

for $1/2\leq\sigma:=\operatorname{Re}(s)\leq 4/5$ and $T\geq 2$.

In the following we give some further formulas and relationships to $L(x)$:

Let

$$K(2x):=\sum_{k=1}^{\infty}\frac{1}{k}e^{-2kx}\ ,\ \ x>0$$

with

$$\frac{1}{2}\operatorname{cos}ech(x):=\frac{1}{2\sinh(x)}=\frac{1}{2x}-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x}{\pi^2k^2+x^2}=\frac{e^{-x}}{1-e^{-2x}}=\sum_{1}^{\infty}e^{-(2k+1)x}\quad .$$

It holds with

$$L(x)=-\log\frac{e^{-x}}{1-e^{-2x}}\qquad\text{and}\qquad\log\frac{1}{1-y}=\sum_{k=1}^{\infty}\frac{y^k}{k}$$

the relation

$$L(x)=x+K(2x)\ .$$

Remark: The functions $L(x)$ and $l(x)$ are related to the Exponential- and the Bessel-function by the formulas ([SGr] 6.622)

$$L(x)=\log(2\sinh(x))=\int_0^{\infty}(J_0(t)-e^{-t\sinh x})\frac{dt}{t}$$

and

$$2L'(x)=2\coth(x)=\int_0^{\infty}(J_0(t)+t\cosh xe^{-t\sinh x})\frac{dt}{t}\cdot$$

Remark: An equivalent formula to the Riemann duality equation is given by ([HHa], see also [SGr] 1.421)

$$\coth(\pi x)=1+2\sum_{1}^{\infty}e^{-2\pi nx}=\frac{1}{\pi x}+\frac{2x}{\pi}\sum_{1}^{\infty}\frac{1}{x^2+n^2}$$

resp.

$$\frac{x\coth(x)-1}{x}=\sum_{1}^{\infty}\frac{2x}{\pi^2k^2+x^2}=\sum_{1}^{\infty}\log'(\pi^2k^2+x^2)\qquad .$$

We note (e.g. [CBe], [SGr] 4.384)

$$x\coth(x)-1=\sum_{1}^{\infty}B_{2n}\frac{(2x)^n}{(2n)!}$$

and

$$\sin(ix)=i\sinh x\quad\text{and}\quad\cos(ix)=\cosh x\,.$$

For

$$d\rho(x) := (xl(x))\frac{d\log(xl(x))}{x} =: \omega'(x)\frac{dx}{x}$$

the corresponding Mellin transform is given by

$$\Pi^*(s) := \int_0^\infty x^s \frac{d\rho(x)}{x} = \frac{1}{2}\int_0^\infty x^s\left[\frac{1}{\sinh x} - x\coth x\right]\frac{dx}{x} \ .$$

Let

$$\Pi(s) = \int_0^\infty x^s d\lambda(x) \quad \text{with} \quad d\lambda(x) := -xl(x)\frac{d\log(xl(x))}{x}$$

then it holds:

Lemma: For the transform $\Pi(s)$ of $d\lambda(x)$ it holds $\Pi(0)$ is divergent and

$$\Pi(2m+1) = (2^{21}-1)\pi^{2m}\left|B_{2m}\right| \quad , \quad \Pi(2m+2) = \frac{2^{2m+1}-1}{2^{2m}}(2m+1)!\zeta(2m+1) \ .$$

Proof: is given in the appendix.

Lemma: The Fourier transform of $K(x)$ is given by

$$\hat{K}(2x) = \frac{1-\pi x\coth(\pi x)}{2x^2} = \sum_{k=1}^\infty \frac{1}{k^2+|x|^2} \ .$$

Proof: The Fourier transform of $F_\varepsilon(x) = e^{-\varepsilon|x|}$ is given by ([BPe] 2.3)

$$\hat{F}_\varepsilon(\xi) = \frac{2\varepsilon}{\varepsilon^2+|\xi|^2} \ .$$

Therefore the Fourier transform of $K(x)$ is calculated by

$$\hat{K}(\xi) = 2\sum_{k=1}^\infty \frac{1}{k}\frac{2k}{4k^2+|\xi|^2} = 4\sum_{k=1}^\infty \frac{1}{4k^2+|\xi|^2} \ .$$

q.e.d.

Lemma: The function $K(x)$ is linked to the Gamma function by the relation

$$2\zeta(1-s)\frac{2\Gamma(1-s)}{(-s)}=\int_0^\infty x^{1-s}K(x)\frac{dx}{x}\approx\frac{2\zeta(1-s)}{s(s-1)} \quad \text{for} \quad \mathrm{Re}(1-s)>1$$

Proof: It holds

$$\Gamma(-s)=\int_0^\infty x^{1-s}\frac{1}{x}e^{-x}\frac{dx}{x}$$

and therefore by substitution $y=2nx$ with $\frac{dy}{y}=\frac{2ndx}{2nx}=\frac{dx}{x}$

$$\Gamma(-s)=(2n)^{1-s}\int_0^\infty y^{1-s}\frac{1}{2ny}e^{-2ny}\frac{dy}{y} \quad .$$

Following the same arguments as in [HEd] 1.2, this enables the definition of the Zeta function in the form

$$2\Gamma(-s)\zeta(1-s)=2^{1-s}\int_0^\infty y^{1-s}K(2x)\frac{dy}{y}=\int_0^\infty y^{1-s}K(x)\frac{dy}{y} \quad .$$

From [ETi] 2.4, we recall

$$\frac{2}{s(s-1)}=\frac{2}{(-s)(1-s)}\approx\frac{2\Gamma(1-s)}{(-s)}=2\Gamma(-s) \quad ,$$

which proves the lemma. **q.e.d.**

Remark: For $x=0$ the function $K(x)$ is a divergent series, i.e. in case the Zeta function is defined by the approach above, where the Zeta function at the critical point $s=1$ is $K(0)=\zeta(1)$; the corresponding value of its Fourier transform is given by

$$\hat{L}(0)=\hat{K}(0)=\sum_{k=1}^\infty\frac{1}{k^2}=\frac{\pi^2}{6} \quad .$$

Remark: From [GWa], 3-4, 13-24 and[SGr] 2.632, 3.761 we recall the relations

i) $$\int_0^\infty x^s J_0(x)\frac{dx}{x}=\frac{2^{\frac{s}{2}}}{2^{\frac{2-s}{2}}}\frac{\Gamma(\frac{s}{2})}{\Gamma(1-\frac{s}{2})} \quad \text{for} \quad 0<\mathrm{Re}(s)<\frac{3}{2}$$

ii) $$\int_0^\infty x^{s+1/2}J_{-1/2}(x)\frac{dx}{x}=\int_0^\infty x^{s+1/2}\left[\sqrt{\frac{2}{\pi}}\frac{\cos x}{\sqrt{x}}\right]\frac{dx}{x}=\frac{2^{\frac{s}{2}}}{2^{\frac{2-s}{2}}}\frac{\Gamma(\frac{s}{2})}{\Gamma(1\frac{1-s}{2})} \quad \text{for} \quad 0<\mathrm{Re}(s)<1$$

iii) $$\int x^\mu\cos(ax)\frac{dx}{x}=-\frac{1}{2a^\mu}\left[e^{i\frac{\pi}{2}\mu}\Gamma(\mu,-iax)+e^{-i\frac{\pi}{2}\mu}\Gamma(\mu,iax)\right] \quad \text{with} \quad \Gamma(a,x):=\int_x^\infty t^a e^{-t}\frac{dt}{t}$$

iv) $$\int x^s\cos(ax)\frac{dx}{x}=\frac{\Gamma(s)}{a^s}\cos(\frac{\pi}{2}s) \quad \text{for} \quad 0<\mathrm{Re}(s)<1 \quad \text{and} \quad a>0.$$

References

[TAm] T. Amdeberhan, M. Coffey, O. Espinosa, C. Koutschan, D. Manna, V. Moll, integrals of powers of log-gamma, to appear in Proc. Amer. Math. Soc.

[MBe] M. V. Berry, J. P. Keating, $H = xp$ and the Riemann zeros, in Supersymmetry and Trace Formulae: Chaos and Disorder (Ed. I.V. Lerner, J.P. Keating, D.E.Khmelnitski), Kluver, New York (1999) 355-367

[CBe] B. C. Berndt, Ramanujan´s Notebooks, Part I, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985

[CBe2] B. C. Berndt, Ramanujan´s Lost Notebooks, Part II, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 2000

[IBo] I. Bogdanova, P. Vandergheynst, J.-P. Gazeau, Wavelets on the 2-Hyperboloid, Nov. 24, 2004

[KBr1] K. Braun, A spectral analysis argument to prove the Riemann Hypothesis, www.riemann-hypothesis.de

[KBr2] K. Braun, A Note to the Bagchi Formulation of the Nyman-RH-Criterion, www.riemann-hypothesis.de

[KBr3] K. Braun, Interior Error Estimates of the Ritz Method for Pseudo-Differential Euquations, Jap. Journal of Applied Mathematics, 3, 1 (1986) 59-72

[WBu] W. Bulla, F. Gesztesy, Deficiency indices and singular boundary conditions in quantum mechanics, J.Math. Phys. 26 (1985) 2520-2528

[DCa] D. A. Cardon, Convolution Operators and zeros of entire functions, Proc. Amer. Math. Soc. 130, 6 (2002) 1725-1734

[TCh] T.S. Chihara, An introduction to Orthogonal Polynomials, Mathematics and its Applications 13, Gordon and Breach, New York, 1978

[RCo] R. Courant, Dirichlet's principle, conformal mapping, and minimal surfaces, New York: Interscience 1950

[DDi] D. Dickinson, On Lommel and Bessel polynomials, Proc. Amer. Math. Soc. 5 (1954) 946-956

[GDo] G. Doetsch, Summatorische Eigenschaften der Besselschen Funktionen und andere Funktionalrelationen, die mit der linearen Transformationsformel der Thetafunktion äquivalent sind, Comp. Math., Tome 1 (1935) 85-97

[RDu] R.J. Duffin, H.F. Weinberger, Dualizing the Poisson summation formula, Proc. Natl. Acad. Sci. 88 (1991) 7348-7350

[HEd] H.M. Edwards, Riemann`s Zeta Function, Dover Publications, Inc., Mineola, New York, 1974

[GEs] G. Eskin, Boundary value problems for elliptic pseudodifferential operators, Amer. Math. Soc., Providence, Rhode Island, translation of mathematical monographs, vol 52, 1973

[RFe] R.P. Feyman, A.R. Hibbs, Quantum Mechanics and Path Integrals, Dover Publications, Inc., Mineola, New York, 1965

[LGa] L. Garding, Some points of analysis and their history, University Lecture Series, no. 11, American Mathematical society, Providence, R.I., 1991

[SGr] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals Series and Products, Fourth Edition, Academic Press, New York, San Francisco, London, 1965

[HHa] H. Hamburger, Über einige Beziehungen, die mit der Funktionalgleichung der Riemannschen ζ -Funktion äquivalent sind, Math. Ann. 85 (1922) 129-140

[JHa] J. Havil, Gamma: Exploring Euler's Constant, Princeton University Press, 2003

[MHe] M. Heidegger, (1950), „Holzwege", (Forest Paths), Fankfurt: Vittorio Klostermann:

[MHe1] (1935) "the Origin of the Work of Art", transl. A. Hofstadter, in D.F. Krell (ed.) Basic Wirtings, revised edn, San Franscisco: Harper, 1993

[MHe2] (1938) "The Age of the World Picture", in W. Lovitt (trans. and ed.) The Question Concerning Technology and Other Essays, New York: Harper & Row, 1977

[THe] T. Heikonen, Notes on Differential forms and Spacetime, Lecture Notes, (2006)

[EHe] E. Hellinger, Zur Stieltjesschen Kettenbruchtheorie, Math. Ann. 86 (1922) 18-37

[PHu] P. Humbert, Bessel-integral functions, Proceedings of the Edinburgh Mathematical Society (series 2), 3 (1933) 276-285

[ELa] E. de Laguerre, Sur l'intégrale $\int_x^\infty x^{-1}e^{-x}dx$, Bull. Soc.Math. France, 7 (1879) 72-81

[XLi] X.-J. Li, The positivity of a sequence of numbers and the Riemann Hypothesis, J. Number Theory 65 (1997) 325-333

[HMo] H.L. Montgomery, Topics in multiplicative Number Theory, Springer Lecture Notes, 227, 1971

[NMu] N.I. Muskhelishvili, Singular Integral Equations, P. Noordhoff N. V., Groningen, 1969

[JNe] J. von Neumann, Mathematische Grundlagen der Quantenmechanik, Berlin, Springer, 1932

[NNi] N. Nielsen, Handbuch der Theorie der Gammafunktion, Leipzig, B.G. Teubner Verlag, 1906

[ROs] R. Ossermann, A proof of the regularity everywhere of the classical solution to Plateau's problem, Ann. Mat. 2, 91 (1970) 550-560

[OPe] O. Peron, Die Lehre von den Kettenbrüchen, Anonymous, copy rights by B.G. Teubner Verlag, Leipzig 1912

[BPe] B. E. Petersen, Introduction to the Fourier Transform & Pseudo-Differential Operators, Pitman Publishing Limited, Boston, London, Melbourne

[JPl] J. Plemelj, Potentialtheoretische Untersuchungen, B.G. Teubner, Leipzig, 1911

[GPo] G. Polya, Bemerkungen über die Integraldarstellung der Riemannschen Zeta-Funktion, Acta Mathematica 48 (1926) 305-317

[GPo2] G. Polya, Ueber die Nullstellen gewisser ganzer Funktionen, Math. Zeit., 2 (1918) 352-383

[GPo3] G. Polya, Ueber diue algebraisch-funktionentheoretischen Untersuchungen von J.L.W.V. Jensen, reprinted in Collected Papers, Vol. II, 278-308

[ARo1] A. Robinson: "Non-standard Analysis", Amsterdam, 1966

[ARo2] A. Robinson: "The metaphysics of the calculus", University of California, LA

[ESh] E. Shamir, Reduced Hilbert transforms and singular integral equations, J. Analyse Math., 12, 277-305, 1964

[GSz] G. Szegö, Orthogonal Polynomials, Amer. Math. Soc., Colloquium Publications, vol. 23, 1938

[RTa] R. Taschner, "Der Zahlen gigantische Schatten", dtv, München, 2009

[ETi] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Oxford University Press Inc., New York, first published 1951, Second Edition 1986

[GWa] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 2nd edition first published 1944, reprinted 1996, 2003, 2004, 2006

[AZe] A. Zemanian, Generalized Integral Transformations, John Wiley and sons, New York, 1968

[AZe2] A. Zemanian, Distribution Theory and Transform Analysis, Dover Publications, Inc., New York, 1987

[AZy] A. Zygmund, Trigonometric series, Vol. I, Cambridge University Press, 1968