REDUCED HILBERT TRANSFORMS AND SINGULAR INTEGRAL EQUATIONS

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Introduction

The properties of Hilbert transform on the whole real line R

$$(1) \qquad (\mathscr{H}u)(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy$$

are quite well known. We mention in particular that $\mathscr{H}$ can be extended to a bounded operator in $L^p(R)$ and that, being a convolution transform, its Fourier transform is a multiplication operator. We have in fact

$$(2) \qquad \mathscr{H}^{\wedge}(\xi) = -\frac{1}{2} \operatorname{sign} \xi \, .$$

Reduced Hilbert transforms are obtained by restricting the variable x and the domain of integration in (1) to a subset R' (or two different subsets) of R. The case $R' = R_+$, the positive half-line, is studied here in detail. In this case the theory becomes quite simple after we prove in Section 1 suitable diagonalization formulas. The case of a finite interval is treated briefly in the last section.

We shall be interested in solving the following system of singular integral equation

$$(3) \qquad A\phi + B \int_0^{\infty} \frac{\phi(y)}{x-y} dy = \psi(x), \qquad x > 0$$

where ϕ and ψ are m dimensional vector functions, A and B are constant $m \times m$ matrices.

The system (3) will be studied in the framework of two scales of spaces, $W^{s,p}$ and $H^{s,p}$, $s \geqq 0$. For integral s, both spaces coincide with the space of functions with derivatives up to order s in L^p. For fractional s, they are obtained by (different) interpolation methods.

Our main result is that, except for at most m values of s (mod 1), the system (3) admits a solution for every ψ in a closed, finite codimensional subspace. The codimension increases with s, more precisely, it jumps upward at the exceptional values of s, and the total jump is m for a unit increase in s. The results are first obtained for $s = \sigma$, $0 \leqq \sigma < 1$, and are then easily extended to $s \geqq 1$. Applications to the study of mixed boundary value problems for elliptic partial differential equations were given in [14], [15].

Reduced Hilbert transformations in L^2 were studied by Koppelman-Pincus [5] and J. Schwartz [13]. It is Schwartz's diagonalization formula for $p = 2$ which we extend to $1 < p < \infty$. An L^p approach was given by Widom [17], who obtained some of our results (the L^p case for a scalar equation, $m = 1$). Widom's methods are different and considerably more complicated, and their extension to the vectorial case, $m > 1$, seems difficult.

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Notations.

R is the real line. R_+ (resp. R_-) is the ray $x > 0$ (resp. $x < 0$), and Y_+ (resp. Y_-) its characteristic function.

$C_0^\infty(\Omega)$ – the class of infinitely differentiable functions with compact support in the domain Ω.

We always assume that $1 < p < \infty$ and $q = p/(p-1)$, $(1/p + 1/q = 1)$. Unless otherwise stated, σ will satisfy $0 \leqq \sigma < 1$.

Fourier transformation are denoted by $\mathscr{F}$, $\mathscr{F}^{-1}$ and also

$$(u^\wedge)(\xi) = (\mathscr{F}u)(\xi), \qquad (v^\vee)(x) = (\mathscr{F}^{-1}v)(x).$$

Norms of functions in a domain Ω will be denoted by $\| u, \Omega \|$ with suitable indices. Semi-norms will be denoted by $[u, \Omega]$. The notation $\| u \|_1 \sim \| u \|_2$ is used for equivalence between the norms, i.e.

$$\| u \|_1 \leqq K \| u \|_2, \quad \| u \|_2 \leqq K \| u \|_1,$$

where K is independent of u. The phrase "K independent of u" will usually be omitted in the statement of such estimates.

§1. Diagonalization formulas.

Hilbert transforms on R_+ are defined by

$$(1.1) \qquad (H^{\pm}\phi)(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_0^\infty \frac{\phi(y)}{x \pm i\varepsilon - y} dy, \qquad x > 0,$$

$$(H\phi)(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\substack{|x-y|>\varepsilon \\ y>0}} \frac{\phi(y)}{x - y} dy, \qquad x > 0.$$

By Plemelj's formulas we have the relations

$$(1.2) \qquad H^{\pm}\phi = \frac{1}{2}\phi \mp H\phi$$

and by M. Riesz's result, $H^{\pm}$ and H can be extended to bounded operators in $L^p(R_+)$, $1 < p < \infty$.

For a fixed p and $\phi \in C_0^\infty(R_+)$, let

$$(1.3) \qquad (U\phi)(t) = e^{t/p}\phi(e^t), \qquad -\infty < t < \infty$$

$$(1.4) \qquad M\phi = \mathscr{F}U\phi = (e^{t/p}\phi(e^t))^{\wedge}.$$

Note that U is an isometry between $L^p(R_+)$ and $L^p(R)$. The inverse of M is given by

$$(1.5) \qquad (M^{-1}f)(x) = (U^{-1}\mathscr{F}^{-1}f)(x) = x^{-1/p}(\mathscr{F}^{-1}f)(\log x), \qquad x > 0.$$

By writing explicitely by Fourier transform and using the substitutions $x = e^t$ $t = \log x$ we obtain

$$(1.6)\qquad (M\phi)(\tau) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \phi(x) x^{-1/q - i\tau} dx, \qquad -\infty < \tau < \infty.$$

$$(1.7)\qquad (M^{-1}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x^{i\tau - 1/p} f(\tau) d\tau, \qquad x > 0.$$

If the Fourier inversion formula holds for $U\alpha$, then $M^{-1}M\alpha = \alpha$. This is the case if $\alpha(x)x^\sigma \in C^\infty(R_+)$, $0 \leqq \sigma < 1/p$, and α satisfies a very mild growth condition at $+\infty$ (for instance $\alpha(x) \sim x^\gamma$, $\gamma < 0$); for then $U\alpha$ is exponentially decreasing at $\pm\infty$ and $M\alpha = \mathscr{F}U\alpha$ is also fastly decreasing. In any case, if $\alpha = M^{-1}M\alpha$ then $\alpha(x)$ has the following representation

$$(1.8)\qquad \alpha(x) = M^{-1}M\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x^{i\tau - 1/p} a(\tau) d\tau, \qquad a = M\alpha.$$

Theorem 1.1. *Let* $-1 + 1/p < \sigma < 1/p$ *and let* $a = M\alpha$ *be fastly decreasing* ($a \in \mathscr{S}$). *Then the transformation* M *diagonalizes the operators*

$$(1.9)\qquad (H_\sigma^\pm \alpha)(x) = x^{-\sigma} H^\pm x^\sigma \alpha(x)$$

and we have

$$(1.10)\qquad \begin{aligned} MH_\sigma^+\alpha &= \rho_\sigma M\alpha \\ MH_\sigma^-\alpha &= (1 + \rho_\sigma) M\alpha \end{aligned}$$

where

$$(1.11)\qquad \rho_\sigma(\tau) = [\exp(2\pi i(\sigma - 1/p + i\tau)) - 1]^{-1}.$$

Proof. Using (1.8) we obtain

$$(H_\sigma^+\alpha)(x) = x^{-\sigma}H^+x^\sigma = \lim_{\varepsilon \downarrow 0} \frac{x^{-\sigma}}{2\pi i} \int_0^\infty \frac{\eta^\sigma}{x + i\varepsilon - \eta} d\eta \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \alpha(\tau) \eta^{i\tau - 1/p} d\tau.$$

Since $a(\tau) \in L^1(R)$, the inner integral is estimated by $K\eta^{-1/p}$, and the integrand with respect to η is estimated by $K\eta^{\sigma-1/p}/|x+i\varepsilon-\eta|$. This expression belongs to $L^1(R_+)$, since $-1<\sigma-1/p<0$. We can therefore change the order of integration

$$(H_\sigma^+\alpha)(x) = \lim_{\varepsilon\downarrow 0} \frac{x^{-\sigma}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(\tau)\, I_\varepsilon d\tau \tag{1.12}$$

where

$$I_\varepsilon = \frac{1}{2\pi i} \int_0^\infty \frac{\eta^{\sigma-1/p+i\tau}}{x+i\varepsilon-\eta} d\eta.$$

To compute I_ε, we integrate $-z^{\sigma-1/p+i\tau}\cdot(z-i\varepsilon-x)^{-1}$ over a path consisting of two circles of radii r and R around the origin and connected above and below the positive semi axis. (We choose the branch of $\log z$ which approaches $\log|x|$ on the upper positive axis). By Cauchy's formula, this complex integral is $-(x+i\varepsilon)^{\sigma-1/p+i\tau}$. Letting $r\to 0$ and $R\to\infty$, the integrals over $|z|=r$ and $|z|=R$ tend to zero, due to the assumption on σ. The integral on the upper positive axis is I_ε, and the lower one is $I_\varepsilon \exp 2\pi i(\sigma - 1/P + i\tau)$. Hence

$$I_\varepsilon = (x+i\varepsilon)^{\sigma-1/p+i\tau}[\exp(2\pi i(\sigma-1/p+i\tau))-1]^{-1}.$$

For each $x>0$, $|I_\varepsilon|$ is easily estimated by a constant (independent of τ and ε) times $x^{\sigma-1/p}$, and since $-1<\sigma-1/p<0$, we can shift the limit in (1.12) inside the integral and obtain, using (1.11)

$$H_\sigma^+(x) = \frac{x^{-\sigma}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{\sigma-1/p+i\tau} \rho_\sigma(\tau)\, a(\tau)\, d\tau = M^{-1}(\rho_\sigma\cdot a) = M^{-1}(\rho_\sigma M\alpha).$$

Since $-1<\sigma-1/p<0$, $\rho_\sigma(\tau)$ is a well behaved function and $\rho_\sigma M\alpha \in \mathscr{S}$, like $M\alpha$. We can thus use the inversion formula and obtain $MH_\sigma^+ = \rho_\sigma M\alpha$, which is the desired result for H_σ^+. From (1.2) we obtain the result for H_σ^-.

Remark 1.1. Similar formulas can be obtained for the transforms- $|x|^{-\sigma}J_1x^{\sigma}$ and $x^{-\sigma}J_2|x|^{\sigma}$ where J_1 (resp. J_2) is the Hilbert transform from R_+ to R_- (resp. from R_- to R_+). The integrals are non-singular now, hence the three variants $J, J^{\pm}$ coincide. In the definition of M one has only to change U (or U^{-1}) so that it will be an isometry between $L^p(R_-)$ and $L^p(R)$. For $|x|^{-\sigma}J_1x^{\sigma}$ one obtains the diagonalization factor

$$\rho_\sigma^*(\tau) = [\exp(2\pi i(\sigma - 1/p + i\tau))]^{-1}.\exp(\pi i(\sigma - 1/p + i\tau)).$$

Remark 1.2. For $p=2$, $\mathscr{F}$, hence also M, is an isometry and the formulas above yield in fact the spectral decompositions of these bounded symmetric operators (cf. [5], [13]).

Remark 1.3. By using the fact that

$$\int_0^\infty x^{-1}\phi(x)dx = 0 \;\Rightarrow\; H^{\pm}(x^{-1}\phi(x)) = x^{-1}H^{\pm}\phi(x)$$

we can obtain formulas (1.10) in an extended range of σ. For instance, if $1/p < \sigma < 1 + 1/p$ and $\int_0^\infty x^{\sigma-1}\alpha(x)dx = 0$ then

$$x^{-(\sigma-1)}H^+x^{\sigma-1}\alpha(x) = x^{-\sigma}H^+x^{\sigma}\alpha(x).$$

Since $\sigma - 1$ is in the original range, we obtain formula (1.10) for σ upon noticing that $\rho_{\sigma-1} = \rho_\sigma$.

§2. Estimates in the $E^{\sigma,p}$ norm

Using the identities $2H = H^+ + H^-$ and $I = H^- - H^+$ (I is the identity operator), the system (3) can be written in the form

$$A\phi \equiv (CH^+ + DH^-)\phi = \psi, \tag{2.1}$$

where the given ψ and the unknown ϕ are m-vectors of functions on R_+, C and D are constant $m \times m$ matrices. To avoid the trivial case where A is a constant times the identity, and does not contain H, we assume that $C + D \neq 0$.

We introduce a norm $E^{\sigma,p}$ by

$$(2.2)\qquad \| \phi, R_\pm \|_{E^{\sigma,p}} = \Big(\int_{R_\pm} |x|^{-\sigma p} |\phi(x)|^p dx)^{1/p}.$$

If $\phi = (\phi_1, \dots, \phi_m)$, then $\| \phi \| = \sum \| \phi_j \|$. Moreover $\phi \in L^p$ (or any other space) will be an abbreviation for $\phi_j \in L^p$.

Theorem 2.1. *Let* $-1 + 1/p < \sigma < 1 + 1/p, \sigma \neq 1/p$ *and let* C *and* D *be non-singular matrices. If* $\phi \in C_0^\infty(R_+)$ *and* $\int_0^\infty x^{-1}\phi(x)dx = 0$ *in case* $\sigma > 1/p$, *then*

$$(2.3)\qquad \| H^+ \phi, R_+ \|_{E^{\sigma,p}} \leqq K \| \phi, R_+ \|_{E^{\sigma,p}}$$

$$(2.4)\qquad \| J_1 \phi, R_- \|_{E^{\sigma,p}} \leqq K \| \phi, R_+ \|_{E^{\sigma,p}}$$

(*and similarly for* $J_2\psi, \quad \psi \in C_0^\infty(R_-)$).

Moreover, if the eigenvalues λ_j *of* $-C^{-1}D$ *are outside the ray* $\arg \lambda = 2\pi(1/p - \sigma)$ *then*

$$(2.5)\qquad \| \phi, R_+ \|_{E^{\sigma,p}} \leqq K \| A\phi, R_+ \|_{E^{\sigma,p}} .$$

Proof. We note first that the condition $\int_0^\infty x^{-1}\phi(x)dx = 0$, imposed for $\sigma > 1/p$, implies in particular the existence of the integral, hence $\phi(0) = 0$. This implies in turn that $\| \phi \|_{E^{\sigma,p}} < \infty$.

Denoting $\alpha(x) = x^{-\sigma}\phi(x)$ we have $\| \phi, R_+ \|_{E^{\sigma,p}} = \| \alpha, R_+ \|_{L^p}$. Since $H_\sigma^\pm = x^{-\sigma} H^\pm x^\sigma$, (2.3) is equivalent to

$$\| H_\sigma^\pm \alpha(x), R_+ \|_{L^p} \leqq K \| \alpha(x), R_+ \|_{L^p} .$$

By Theorem 1.1.

$$H_\sigma^+ \alpha = M^{-1} \rho_\sigma M \alpha = U^{-1} \mathscr{F}^{-1} \rho_\sigma \mathscr{F} U \alpha.$$

Since U and U^{-1} are isometries, H_σ^+ is bounded in $L^p(R_+)$ if and only if $\mathscr{F}^{-1}\rho_\sigma \mathscr{F}$ is bounded in $L^p(R)$. Or, using the conventional terminology, if $\rho_\sigma(\tau)$ is an *L^p-multiplier* (Notation: $\rho_\sigma \in \mathscr{M}_p$).

Now by M. Riesz's theorem, $H_0^+ = H^+$ is bounded in $L^p(R_+)$, hence $\rho_0(\tau) \in \mathscr{M}_p$. But

$$\rho_\sigma = [\exp(2\pi i\sigma)\cdot\exp(2\pi i(i\tau - 1/p)) - 1]^{-1}$$

has the same limit values as ρ_0 at $\tau = \pm\infty$ and the difference $\rho_\sigma - \rho_0$ dies down exponentially. It follows that ρ_σ will be in $\mathcal{M}_p$ if its denominator does not vanish for $-\infty \leqq \tau \leqq \infty$, that is, if $\sigma \neq 1/p$. (One could also use here Michlin's result for multipliers [8], [4].)

Clearly also $1 + \rho_\sigma \in \mathcal{M}_p$ if $\sigma \neq 1/p$, so that H_σ^- is bounded in $L^p(R_+)$. Thus (2.3) is proved. For (2.4) we have to show that $\rho_\sigma^* \in \mathcal{M}_p$, $\sigma \neq 1/p$. This is obvious since ρ_σ^* dies down exponentially at $\tau = \pm\infty$. (Since $J_1\phi$ is no more singular, no wonder that we need not use the fact that $\rho_0 \in \mathcal{M}_p$, which is equivalent to Riesz's theorem.)

By (2.3), $A = CH^+ + DH^-$ is also a bounded operator in the $E^{\sigma,p}$ norm, $\sigma \neq 1/p$. We turn now to prove (2.5), which means that A is $1-1$ and with closed range in the appropriate $E^{\sigma,p}$space. Consider the system

$$A\phi \equiv CH^+\phi + DH^-\phi = \psi\ .$$

Multiplying by x^σ and setting $\alpha = x^{-\sigma}\phi,\ \beta = x^{-\sigma}\psi$, we obtain $A_\sigma\alpha \equiv CH_\sigma^+\alpha + DH_\sigma^-\alpha = \beta$.

By Theorem 1.1 and Remark 1.3,

$$\alpha = M^{-1}[C\rho_\sigma + D(1+\rho_\sigma)]^{-1}M\beta,$$

provided that $\int_0^\infty x^{\sigma-1}\alpha(x)dx = 0$ if $\sigma > 1/p$. Now (2.5) is equivalent to $\|\alpha, R_+\|_{L^p} \leqq K\|\beta, R_+\|_{L^p}$ and this is true if and only if (each element of) the matrice

$$G_\sigma^{-1}(\tau) = [C\rho_\sigma(\tau) + D(1+\rho_\sigma(\tau))]^{-1} \tag{2.6}$$

belongs to $\mathcal{M}_p$. Thus our theorem will be proved after we establish the following

Lemma 2.1. *For $\sigma \neq 1/p$, $G_\sigma^{-1} \in \mathcal{M}_p$ if and only if $G_\sigma(\tau)$ is non singular for $-\infty \leqq \tau \leqq \infty$. This amounts to the non-singularity of C and D and to the eigenvalues condition in Theorem 2.1.*

Proof. If $G_\sigma(\tau)$ is singular for some real τ_0 or $\tau = \pm\infty$, then some elements of $G_\sigma^{-1}(\tau)$ are not bounded on the real line. Hence they are not in $\mathcal{M}_2$

and a fortiori not in $\mathcal{M}_p$. Since $\rho_\sigma(\infty) = -1$ and $\rho_\sigma(-\infty) = 0$, $G_\sigma(\tau)$ is non-singular at $\pm\infty$ if and only if C and D are non-singular. If we denote now

$$E = -C^{-1}D, \quad \mu(\tau) = 2\pi i(\sigma - 1/p + i\tau) \tag{2.7}$$

then

$$G_\sigma^{-1} = (e^\mu - 1)(C + De^\mu)^{-1} = (e^\mu - 1)(I - Ee^\mu)^{-1}C^{-1}. \tag{2.8}$$

It is sufficient to examine whether $-(I - Ee^\mu)^{-1} \in \mathcal{M}_p$, for then

$$e^\mu(I - Ee^\mu)^{-1} = E^{-1}[(I - Ee^\mu)^{-1} - I] \in \mathcal{M}_p \quad ,$$

and also $G_\sigma^{-1} \in \mathcal{M}_p$, since $G_\sigma^{-1}C$ is the sum of the last two matrices. We shall treat the case $m = 1$ first. In this case C, D and E are scalars. The function $(Ee^\mu - 1)^{-1}$ is equal to 1 at $\tau = \infty$ and vanishes at $\tau = -\infty$. These respective values are also assumed by $\rho_\sigma = (e^\mu - 1)^{-1}$, which belongs to $\mathcal{M}_p$ if $\sigma \neq 1/p$. The difference between the two expressions dies down exponentially, hence the first one belongs to $\mathcal{M}_p$ provided that its denominator $Ee^\mu - 1$ does not vanish for real τ, and in view of (2.7), this means that

$$\arg(-C^{-1}D) \neq 2\pi(1/p - \sigma) \ .$$

This proves the lemma for $m = 1$. In case $m > 1$, we utilize a similarity transformation which carries E to its Jordan's canonical form E_1. Then $I - Ee^\mu$ is carried to $B = I - E_1 e^\mu$ which is constructed of diagonal blocks

$$\begin{bmatrix} 1 - \lambda e^\mu, & 0, & \dots & & 0 \\ 1 & & & & \cdot \\ 0 & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 0 \\ 0, \dots, 0, & & 1, & & 1 - \lambda e^\mu \end{bmatrix}$$

(λ is the generic eigenvalue of E) The elements of B^{-1} will be sums of products of the form

$$\prod_{(k)}(1-\lambda e^{\mu})\Big/\prod_{(m)}(1-\lambda e^{\mu}), \qquad k<m. \tag{2.9}$$

The denominator here, being the product of all the diagonal elements, is equal to $\det(B) = \det(I - Ee^{\mu})$ The numerator contains $k < m$ such elements. A necessary condition that (the elements of) $(I - Ee^{\mu})^{-1} \in \mathcal{M}_p$ is therefore that $1 - \lambda e^{\mu}$ does not vanish for real τ (hence also $\det(I - Ee^{\mu}) \neq 0$). This means, (as in the case $m = 1$), that

$$\arg\lambda \neq 2\pi(1/p - \sigma), \qquad \lambda \text{ is any eigenvalue of } E.$$

But this condition is also sufficient, because, after cancelling k factors, (2.9) is $\prod_{(m-k)}(1 - \lambda e^{\mu})^{-1}$. No λ is zero, since E is non singular, and this expression belongs to $\mathcal{M}_p$ since it assumes at $\tau = \pm\infty$ the same values as $-\rho_\sigma = (1-e^{\mu})^{-1}$.

§3. Estimates in $W^{s,p}$.

In this section we shall study the operators $H^{\pm}$ and A in the Sobolev spaces $W^{s,p}$, $s \geqq 0$, and related spaces of functions defined in R or $R_{\pm}$. The definitions and some basic properties of the spaces will be sketched first (Cf. also [6], [10], [15]).

For integral $r \geqq 0$ $W^{r,p}(R)$ is the completion of $C_0^\infty(R)$ with respect to the norm

$$\| u, R \|_{r,p} = \sum_{|\alpha| \leqq r} \| D^\alpha u, R \|_{L^p}.$$

For real $s \geqq 0$ we define first a semi-norm of order s. For $s = 0$, the semi-norm is equal to the norm. For $s > 0$

$$[u, R]_{s,p} = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|u(x) - u(y)|^p}{|x - y|^{1+sp}}\,dxdy\right]^{1/p}, \qquad 0 < s < 1 \tag{3.1}$$

$$[u, R]_{s,p} = \sum_{|\alpha| = [s]} [D^\alpha u, R]_{s-[s],p}\,, \qquad 1 \leqq s, \tag{3.2}$$

where $[s]$ is the integral part of s.

The space $W^{s,p}(R)$ is defined now as the completion of $C_0^\infty(R)$ with respect to the norm

$$\| u, R \|_{s,p} = \| u, R \|_{[s],p} + [u, R]_{s,p} \ .$$

The space $W^{s,p}(R_\pm)$ is defined as the class of all the restrictions to $R_\pm$ of functions in $W^{s,p}(R)$. The norms and the semi-norms in $W^{s,p}(R_\pm)$ are obtained either by restricting the integrals in (3 1) to $R_\pm$, or by taking the "quotient norm". E.g. for R_+

$$(3.3) \quad \| u, R_+ \|_{s,p} = \operatorname*{Inf}_{u^*} \| u^*, R \|_{s,p}, \quad u^* \in W^{s,p}(R), \quad u^* = u \quad \text{in } R_+.$$

The two definitions yield equivalent norms. We also note that $C_0^\infty(R_\pm)$ is dense in $W^{s,p}(R_\pm)$.

The definitions above make it clear that in general, assertions about $W^{s,p}$, $s \geqq 1$, are reduced to the case $0 \leqq s < 1$. Accordingly, we shall study the operators $H^\pm$ and A in $W^{\sigma,p}$, $0 \leqq \sigma < 1$, and in each case indicate briefly how the results extend to $W^{s,p}$, $s \geqq 1$.

For $\Omega = R$, $R_\pm$ and $\phi \in C_0^\infty(\bar{\Omega})$, the semi-norm $[\phi, \Omega]_{\sigma,p}$ vanishes only if $\phi = 0$. Hence the completion of $C_0^\infty(\bar{\Omega})$ with respect to this semi-norm is also a Banach space, which we denote by $\mathbf{W}^{\sigma,p}(\Omega)$. It can be shown that $\| u \|_{\sigma,p}$ and $[u]_{\sigma,p}$ are equivalent for all the functions u supported in a fixed compact (*locally equivalent*), hence a function in $\mathbf{W}^{\sigma,p}$ is locally in $W^{\sigma,p}$. Applying the same procedure for $s = r + \sigma$, we obtain the space $\mathbf{W}^{s,p}$ which contains all the functions u such that $D^r u \in \mathbf{W}^{\sigma,p}$; but functions differing in polynomial of degree $\leqq r$ ($< r$ if $\sigma = 0$) are to be identified.

The closure of $C_0^\infty(R_\pm)$ in $W^{\sigma,p}(R_\pm)$ [resp. in $\mathbf{W}^{\sigma,p}(R_\pm)$] is denoted by $W_0^{\sigma,p}(R_\pm)$ [resp. $\mathbf{W}_0^{\sigma,p}(R_\pm)$].

The results of the next theorem are contained or easily derived from [7].

Theorem 3.1. *a)* $\mathbf{W}_0^{\sigma,p}(R_+) = \mathbf{W}^{\sigma,p}(R_+)$, $\sigma \leqq 1/p$. *For* $\sigma > 1/p$ *there is a proper inclusion.*

b) The functional $\gamma: u \to u(0)$, *defined for continuous functions on* R_+, *is bounded in the* $\| \ \|_{\sigma,p}$ *norm*, $\sigma > 1/p$. *By extension*, $\gamma u = u(0)$ *is defined and continuous for all* $u \in W^{\sigma,p}$, $\sigma > 1/p$. *We note that* $u(0) = 0$ *is equivalent to* $u \in W_0^{\sigma,p}(R_+)$.

c) For $\sigma \neq 1/p$, $\mathbf{W}_0^{\sigma,p}(R_+) \subset E^{\sigma,p}(R_+)$ (*cf.* (2.2)), *and the imbedding is continuous.*

d) For $u \in \mathbf{W}^{\sigma,p}(R_+)$, *let* $\tilde{u}(x) = u(x)$, $x > 0$ *and* $\tilde{u}(x) = 0$, $x < 0$. *Then*

$$\tilde{u}(x) \in \mathbf{W}^{\sigma,p}(R) \Leftrightarrow u \in \mathbf{W}_0^{\sigma,p}(R_+), \qquad \sigma \neq 1/p.$$

e) If $u \in \mathbf{W}^{\sigma,p}(R)$ *and* $\sigma \neq 1/p$ *then*

$$[u, R]_{\sigma,p} \leqq K([u, R_-]_{\sigma,p} + [u, R_+]_{\sigma,p}). \tag{3.4}$$

Remark 3.1. All the results of this theorem are true for $W^{\sigma,p}$ instead of $\mathbf{W}^{\sigma,p}$, and in fact in this form were (*a*)–(*d*) proved in [7]. The (stronger) results for $\mathbf{W}^{\sigma,p}$ are obtained by a standard homogenity argument: We use the results for $u(\lambda x)$ and let $\lambda \to \infty$. The proof of (*e*) is obtained from (*d*) by considering the map $u \to (Y_- u, Y_+ u)$ and using the closed graph theorem.

The connection between $\mathbf{W}^{\sigma,p}$ and $E^{\sigma,p}$ reappears in the following easily verified formula

$$[u, R_\pm]^p_{\sigma,p} = \int_0^\infty |1 - y|^{\sigma p - 1} \| u(x) - u(xy), R_\pm \|^p_{E_x^{\sigma,p}} \, dy, \quad 0 < \sigma < 1. \tag{3.5}$$

We also note that for a fixed $y > 0$

$$H^\pm(\phi(xy)) = (H^\pm \phi)(xy), \tag{3.6}$$

and similarly for J_1, J_2 and A. From the last two formulas, we see that every estimate for these operators in the $E^{\sigma,p}$ norm carries over to the $[\,,\,]_{\sigma,p}$ norm. In particular we have:

Theorem 3.2. *Let* $0 \leqq \sigma < 1$ *and* $\sigma \neq 1/p$. *The estimates* (2.3) *and* (2.4) *are true for the* $\mathbf{W}^{\sigma,p}$ *norm. Moreover if the eigenvalues condition of Theorem* 2.1 *is satisfied, then* (2.5) *is true in* $\mathbf{W}^{\sigma,p}(R_+)$.

We note that the condition $\int_0^\infty x^{-1} \phi(x) dx = 0$ if $\sigma > 1/p$, which was necessary in Theorem 2.1, becomes superfluous, since we use here the $E^{\sigma,p}$ estimates only for functions $\psi(x) = \phi(x) - \phi(xy)$, which satisfy $\int_0^\infty x^{-1} \psi(x) dx = 0$.

Theorem 3.3. *Let* $0 \leqq \sigma < 1$, $\sigma \neq 1/p$. *Then*

$$\| H^\pm \phi, R_\pm \|_{\sigma,p} \leqq K \| \phi, R_\pm \| \tag{3.7}$$

and similarly for J_1, J_2 *and* $A = CH^+ + DH^-$. *Moreover if* C *and* D *are non-singular and the eigenvalues of* $-C^{-1}D$ *are outside the rays* $\arg\lambda = 2\pi(1/p - \sigma)$ *and* $\arg\lambda = 2\pi/p$, *then*

$$\| \phi, R_+ \|_{\sigma,p} \leqq K \| A\phi, R_+ \|_{\sigma,p} \ . \tag{3.8}$$

Proof. By homogeneity arguments we obtain that the $W^{\sigma,p}$ estimates are true if and only if they are true in $L^p(\sigma = 0)$ and $\mathbf{W}^{\sigma,p}$. For (3.7), the case $\sigma = 0$ is clearly true and the $\mathbf{W}^{\sigma,p}$ estimate is given in Theorem 3.2. For (3.8) we have to require the eigenvalues condition for both σ and 0.

Remark 3.2. The estimates of the last two theorems are easily extended to $W^{r+\sigma,p}$ (resp. $\mathbf{W}^{r+\sigma,p}$), $r > 0$. Indeed, one has $D^r H^+ \phi = H^+ D^r \phi$ (and similarly for the other operators) provided that

$$\phi(0) = \phi'(0) = \dots = \phi^{(r-1)}(0) = 0.$$

These functions ϕ constitute a closed subspace of a finite codimension in $W^{r+\sigma,p}(R_+)$. The estimates are true for this subspace. Thus the operator H^+ (and the other ones) which is bounded in this subspace, must be bounded in the whole of $W^{r+\sigma,p}(R_+)$. On the other hand, A is a $1-1$ operator in $W^{\sigma,p}(R_+)$, hence also in $W^{r+\sigma,p}(R_+)$. The converse estimate (3.8) for the whole space follows now immediately from its validity in the subspace.

We consider now in $W^{\sigma,p}(R)$ the operator

$$\mathscr{A}u = \lim_{\varepsilon \downarrow 0} \left(C \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(t)}{x + i\varepsilon - t} dt + D \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(t)}{x - i\varepsilon - t} dt \right). \tag{3.9}$$

If $u = 0$ in R_-, then $\mathscr{A}u = Au$ in R_+.

Theorem 3.4. *Let* $\sigma \neq 1/p$ *and let* C, D *be non singular. If the eigenvalues of* $-C^{-1}D$ *are outside the ray* $\arg\lambda = 2\pi(1/p - \sigma)$, *then*

$$[u, R]_{\sigma,p} \leqq K([u, R_-]_{\sigma,p} + [\mathscr{A}u, R_+]_{\sigma,p}). \tag{3.10}$$

(We get the similar estimates in $W^{\sigma,p}$ *if we require also the eigenvalues condition for* $\sigma = 0$.*)*

Proof. By (3.4) we have

$$[u, R]_{\sigma,p} \leqq K([u, R_-]_{\sigma,p} + [u, R_+]_{\sigma,p}), \qquad \sigma \neq 1/p,$$

hence it is sufficient to estimate $[u, R_+]_{\sigma,p}$ by the right hand side of (3.10). By Theorem 3.2 we have

$$(3.11) \qquad [u, R_+]_{\sigma,p} = [Y_+ u, R_+]_{\sigma,p} \leqq K[A(Y_+ u), R_+]_{\sigma,p}.$$

But for $x > 0$

$$\mathscr{A}u = Y_+ \mathscr{A}u = Y_+ \mathscr{A}(Y_+ u + Y_- u) = A(Y_+ u) + Y_+ \mathscr{A} Y_- u .$$

Using the triangle inequality

$$(3.12) \quad [A(Y_+ u), R_+]_{\sigma,p} \leqq [\mathscr{A}u, R_+]_{\sigma,p} + [Y_+ \mathscr{A} Y_- u, R_+]_\sigma$$

$$\leqq [\mathscr{A}u, R_+]_{\sigma,p} + K[u, R_-]_{\sigma,p} .$$

The last step is true since $Y_+ \mathscr{A} Y_-$ is a linear combination of $\int_{-\infty}^{0} \frac{u(y)}{x-y}\, dy$, namely of $J_2 u$, which is bounded in the $W^{\sigma,p}$-norm from R_- to R_+. Combining (3.11) and (3.12), wo obtain the desired estimate.

Remark 3.3. For a unit increase in σ, the ray $\arg \lambda = 2\pi(1/p - \sigma)$ makes a complete circuit. The $m \times m$ matrice $E = -C^{-1}D$ has at most m eigenvalues. It follows that the eigenvalues condition (and all the estimates which depend on it) is satisfied for every σ except for at most m values of σ(mod 1).

§4. The range of A.

We shall determine now the range of $A = CH^+ + DH^-$ as an operator in $\{W^{\sigma,p}(R_+)\}^m$. We assume throughout that C and D are non-singular. $C + D \neq 0$ (to avoid the trivial case) and that λ_j are the eigenvalues of $E = -C^{-1}D$. We recall that the diagonalization factor of $x^{-\sigma}Ax^{\sigma}$ is given by

$$G_\sigma(\tau) = C\rho_\sigma(\tau) + D(1 + \rho_\sigma(\tau)),$$

$$\rho_\sigma(\tau) = [\exp(2\pi i(\sigma - 1/p + i\tau)) - 1]^{-1}.$$

$G_\sigma(\tau)$ is a periodic, matrix-valued analytic function of τ. Its poles are the poles of $\rho_\sigma(\tau)$, $\tau = i(\sigma - 1/p + k)$, k integer. We note that

$$G_\sigma(\tau) = G_0(\tau - i\sigma), \qquad \rho_\sigma(\tau) = \rho_0(\tau - i\sigma).$$

Consider now $G_\sigma^{-1}(\tau)$. Its singularities are the points where $G_\sigma(\tau)$ is a singular matrix. Since

$$G_\sigma(\tau) = [\exp(2\pi i(\sigma - 1/p + i\tau)) - 1]^{-1} C[I - E\exp(2\pi i(\sigma - 1/p + i\tau))]$$

and C is non singular, the only singularities are those of the rightmost factor, which is singular if one of its eigenvalues vanishes. Thus τ_0 is a *singularity (of multiplicity r)* if and only if for some λ_j (of multiplicity r):

$$1 - \lambda_j \exp(2\pi i(\sigma - 1/p - i\tau_0)) = 0.$$

For a fixed λ_j, this equation has a single solution at each period strip. Thus, counting multiplicities, each period strip contains m singularities of $G_\sigma^{-1}(\tau)$.

We have already established (in Theorem 3.3 and Lemma 2.1) that the range of A, $R(A)$, is closed in $W^{\sigma,p}$ if and only if $G_\sigma^{-1}(\tau)$ and $G_0^{-1}(\tau)$ have no real singularities, or equivalently, if $G_0^{-1}(\tau)$ has no singularities on $\operatorname{Im}\tau = 0$ and $\operatorname{Im}\tau = -\sigma$. We prove now

Theorem 4.1. *If $G_0^{-1}(\tau)$ has l singularities inside the strip $-\sigma < \operatorname{Im}\tau < 0$ and no singularity on its boundary, then $R(A) \subset \{W^{\sigma,p}(R_+)\}^m$ is closed and of codimension l. In particular, as σ increases and the line $\operatorname{Im}\tau = -\sigma$ passes a k-multiple singularity, the codimension jumps by k.*

Proof. Consider first the scalar case, $m = 1$. Since the range is closed, we have to examine the possibility of solving $A\phi = \psi$ for $\psi \in C_0^\infty(R_+)$ only. If we could use diagonalization, then, using also formulas (1.3) – (1.7):

$$\phi(x) = (M^{-1} G_0^{-1} M\psi)(x) = x^{-1/p}(\mathscr{F}^{-1} G_0^{-1} M\psi)(\log x).$$

We shall study this expression. The function $(M\psi)(\tau)$ is analytic in $\operatorname{Im}\tau > -1/p$ and dies exponentially on each line $\operatorname{Im}\tau = $ const.. If $\psi(0) = 0$, $M\psi$ is analytic also in $\operatorname{Im}\tau > -1/p - 1$, for then $e^{t/p}\psi(e^t)$ dies down at $t = -\infty$ at least as fast as $e^{t(1+1/p)}$, and $M\psi = \mathscr{F}(e^{t/p}\psi(e^t))$.

We claim that $\psi \in R(A)$ if $G_0^{-1}(\tau)M\psi$ has no singularities (poles) in $-\sigma \leqq \operatorname{Im}\tau \leqq 0$ Indeed, in this case $G_0^{-1}(\tau) \cdot (M\psi)(\tau)$ dies down exponentially on every parallel in this strip (since $M\psi$ has this property and $G_0{}^{-1}(\tau)$ is bounded). Its Fourier transform

$$(\mathscr{F}^{-1}G_0^{-1}M\psi)(t) = O(e^{t(\sigma+\varepsilon)}), \qquad t \to -\infty$$

$$= O(x^{\sigma+\varepsilon}), \qquad x \to 0 \qquad (t = \log x)$$

for $\varepsilon > 0$ sufficiently small. Multiplying by $x^{-1/p}$ wee see that $\phi = O(|x|^{\sigma-1/p+\varepsilon})$, so that $\| \phi, R_+ \|_{E^{\sigma,p}}$, and thus also $[\phi, R_+]_{\sigma,p}$ are finite near $x = 0$ and $\phi(0) = 0$ if $\sigma > 1/p$. Since $G_0^{-1}(\tau) \in \mathscr{M}_p$, we have also $\phi \in L^p(R_+)$. Moreover, ϕ is easily seen to be a C^∞ function of exponential decay at $x = \infty$. (In particular, $\| \phi, R_+ \|_{\sigma,p} < \infty$). Now we can diagonalize $A\phi$ and obtain $A\phi = \psi$, so that $\psi \in R(A)$.

Now $G_0^{-1}(\tau) \cdot (M\psi)(\tau)$ will have no poles in $-\sigma \leqq \operatorname{Im}\tau \leqq 0$ if

(i) $M\psi = 0$ if $\sigma > 1/p$ (i.e. $\psi \in W_0^{\sigma,p}(R_+)$);

(ii) If τ_0 is a pole of $G_0^{-1}(\tau)$ in the strip then

$$(M\psi)(\tau_0) = \int_0^\infty \psi(x)x^{-1/q-i\tau_0}dx = 0.$$

If we examine now what necessary conditions should a function $\psi \in R(A)$ satisfy, we find that $\psi(0) = 0$ is not one of them. For suppose that $\psi = A\phi$ does satisfy $\psi(0) = 0$, and let $\phi^* = \phi + \phi_1$ where ϕ_1 is smooth, $\phi_1(0) = 0$, and $\int_0^\infty t^{-1}\phi_1(t)dt \neq 0$. Since $A = (D - C)I + (C + D)H$ and $C + D \neq 0$,

$$\psi^*(0) = (A\phi^*)(0) = (A\phi_1)(0) = (C + D)\int_0^\infty t^{-1}\phi_1(t)dt \neq 0.$$

Thus if there is no pole τ_0 of $G_0^{-1}(\tau)$ in the strip, every $\psi \in C_0^\infty(R_+)$ is in $R(A)$ and the codimension is zero.

If τ_0 is a pole in the strip and $-\sigma < \operatorname{Im}\tau_0 < 0$, then the condition $(M\psi)(\tau_0) = 0$ is meaningful if $\psi(0) = 0$ and it is clearly satisfied by $\psi = A\phi \in R(A)$. At the end of this section we shall show that $(M\psi)(\tau_0)$ is a continuous functional on $W_0^{\sigma,p}(R_+)$. If $\sigma < 1/p$, then $W_0^{\sigma,p}(R_+) = W^{\sigma,p}(R_+)$ and the condition $(M\psi)(\tau_0) = 0$ characterizes the range, which is therefore of codimension 1. If however $\sigma > 1/p$ then we have:

$$\psi(0) = 0 \Rightarrow (M\psi)(\tau_0) = 0.$$

But $\psi(0) = 0$ is not necessary. Hence we obtain that $R(A)$ is a 1-codimensional subspace containing the 2-codimensional subspace characterized by $\psi(0) = 0$ and $(M\psi)(\tau_0) = 0$.

Having proved the assertion for $m = 1$, we pass to the case $m > 1$. If all the eigenvalues of E are distinct, then a suitable similarity transformation carries E, $I - E\exp(2\pi i(\sigma - 1/p + i\tau))$ and its invesrse to a diagonal form. After this change of base we obtain m disconnected 1-dimensional equations and the desired result follows immediately. If some eigenvalues of E coalesce, we consider

$$A_\lambda = A + \lambda I = (C - \lambda I)H^+ + (D + \lambda I)H^-.$$

For $|\lambda|$ small, all the properties of A are preserved, (in particular the relevant singularities remain in the same strip), except that the eigenvalues of $E_\lambda = -(C - \lambda I)^{-1}(D + \lambda I)$ become distinct. Our theorem is then true for A_λ, and since the index (which here is the codimension of the range) of A is stable under small perturbations [3], $R(A)$ and $R(A_\lambda)$ have the same codimension and our theorem is proved.

Remark 4.1. Consider the range in $\mathbf{W}^{\sigma,p}$ instead of $W^{\sigma,p}$. The range is closed if $G_0(\tau)$ has no singularities on $\operatorname{Im}\tau = -\sigma$ (i.e. $G_\sigma(\tau)$ has no real singularities). The codimension is clearly the number of singularities in $-\sigma < \operatorname{Im}\tau \leqq 0$.

Remark 4.2. Consider now $W^{s,p}(R_+)$, $s = r + \sigma$. The codimension here is the number of singularities of $G_0(\tau)$ in $-s < \operatorname{Im}\tau < 0$. It is sufficient to treat the case $m = 1$, and we suppose first that $r = 1$. Then $\psi \in C_0^\infty(R_+)$ is in the range for $1 + \sigma$ if both ψ and ψ' belong to the range in $W^{\sigma,p}(R_+)$. For then

$$\exists\ \phi_1, \phi_2 \in W^{\sigma,p}(R_+),\ \ A\phi_1 = \psi',\ A\phi = \psi.$$

But

$$A\phi_1 = \psi' = (A\phi)' = A\phi' + \text{const.} \cdot \phi(0).$$

However, $\phi(0) = 0$, since ψ' and $A\phi'$ vanish at $x = \infty$. Hence $A\phi_1 = A\phi'$ and $\phi_1 = \phi'$, so that $\phi \in W^{1+\sigma,p}(R_+)$. It is clear now how to proceed for higher r. (The same remark for $\mathbf{W}^{s,p}$).

Lemma 4.1. *Let $\gamma(x) \in L^q(\varepsilon, \infty)$ for some $\varepsilon > 0$ and $x^\sigma\gamma(x) \in L^q(0, \varepsilon)$. Then the functional*

$$\psi(x) \to \int_0^\infty \psi(x)\gamma(x)dx$$

is bounded in $W_0^{\sigma,p}(R_+)$, $\sigma \neq 1/p$.

Proof. We write $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1(x) = 0$ for $0 < x < \varepsilon$, $\gamma_2(x) = 0$ for $x \geqq 2\varepsilon$, and $\gamma_1, \gamma_2 x^\sigma \in L^q(R_+)$. The functional $\psi \to \int_0^\infty \gamma_1\psi dx$ is bounded in $L^p(R_+)$ and a-fortiori in $W_0^{\sigma,p}(R_+)$ while

$$\int_0^\infty \psi\gamma_2 dx = \int_0^\varepsilon (x^{-\sigma}\psi)(\gamma_2 x^\sigma)dx \leqq \| x^{-\sigma}\psi \|_{L^p} \| \gamma_2 x_\sigma \|_{L^q}$$
$$\leqq K \| \psi, R_+ \|_{E^{\sigma,p}} \leqq K \| \psi, R_+ \|_{\sigma,p}\ ,$$

where the last step follows from Theorem 3.1 (c).

As a result we obtain that if $-\sigma < \operatorname{Im} \tau_0 < 0$, then

$$(M\psi)(\tau_0) = \int_0^\infty \psi(x)x^{-1/q-i\tau_0}dx$$

is continuous on $W_0^{\sigma,p}(R_+)$. (Indeed $\gamma(x) = x^{-1/q-i\tau_0}$ satisfies the conditions of the lemma). This result was used in the proof of Theorem 4.1.

§ 5. Estimates in $H^{s,p}$

In this section we shall study the operators $H^{\pm}$, A in a new scale of spaces, $H^{\sigma,p}(R_+)$. The results are nearly identical though the methods of proof (for $\sigma > 0$) are quite different.

Let $s \geqq 0$ and

$$J^s u = [(1 + |\xi|^2)^{s/2} u^\wedge(\xi)]^\vee \ .$$

The space $H^{s,p}(R^n)$ is the set of all functions for which $J^s u \in L^p(R^n)$. This is a Banach space with respect to the norm

$$\| u, R^n \|_{H^{s,p}} = \| J^s u, R^n \|_{L^p}.$$

By utilizing Michlin's multipliers theorem [8], it is easily shown (e.g. in [6]) that $H^{s,p}(R^n) = W^{s,p}(R^n)$ for an integral r. It is also verified that for $p = 2$ and any s, $H^{s,2}(R^n) = W^{s,2}(R^n)$. However in other cases the spaces $H^{s,p}$ and $W^{s,p}$ are different. (Kahane uses lacunary Fourier series to show this), but they are still contiguous in the sense that for $\varepsilon > 0$

$$H^{s+\varepsilon,p}(R^n) \subset W^{s,p}(R^n) \subset H^{s-\varepsilon,p}(R^n)$$

the imbeddings being continuous. (Cf. [6], for other properties of $H^{s,p}$, cf. [1], [2].)

If $I^s u = (|\xi|^s u^\wedge(\xi))^\vee$, then

$$[u, R^n]_{H^{s,p}} = \| I^s u, R^n \|_{L^p}$$

plays the rôle of a semi-norm in $H^{s,p}$ and we have

$$\| u, R^n \|_{H^{s,p}} \sim \| u, R^n \|_{L^p} + [u, R^n]_{H^{s,p}} \ .$$

For $\Omega \subset R^n$ with smooth boundary we define

$$\| u, \Omega \|_{H^{s,p}} = \operatorname*{Inf}_{u_1} \| u_1, R^n \|_{H^{s,p}} \ , \qquad u_1 \in H^{s,p}(R^n), \quad u_1 = u \text{ in } \Omega;$$

and similarly for $[u, \Omega]_{H^{s,p}}$.

For the rest of this section, $\| u \|_{s,p}$ and $[u]_{s,p}$ will denote the $H^{s,p}$ norm and semi-norm. The next theorem was proved in the special case $p = 2$ by Peetre [11].

Theorem 5.1. For $u \in H^{\sigma,p}(R)$

$$\| u, R_\pm \|_{\sigma,p} \sim \| Y_+(D \pm i)^\sigma u \|_{L^p} \tag{5.1}$$

where $(D \pm i)^\sigma$ *is the convolution operator whose Fourier transform is multiplication by* $(\xi \pm i)^\sigma$.

Proof. We prove (5.1) for R_+ (the proof for R_- is similar). Let

$$Pu = (D + i)^{-\sigma} Y_+(D + i)^\sigma u .$$

Then P is a bounded operator in $H^{\sigma,p}(R)$. Indeed:

$$(\xi + i)^\sigma u\hat{\ }(\xi) = \frac{(\xi + i)^\sigma}{(1 + |\xi|^2)^{\sigma/2}} [(1 + |\xi|^2)^{\sigma/2} u\hat{\ }(\xi)] .$$

The factor in brackets belongs to $\mathscr{F}L^p$, since $u \in H^{\sigma,p}(R)$. The other factor belongs to $\mathscr{M}_p$, by Michlin's theorem. Therefore, $(D + i)^\sigma$ maps $H^{\sigma,p}(R)$ boundedly into $L^p(R)$. Also, $v \to Y_+ \cdot v$ is bounded in $L^p(R)$, and finally $(D + i)^{-\sigma}$ maps back boundedly into $H^{\sigma,p}(R)$.

Since $P^2 = P$, P is a projection. The subspace annihilated by P is $H^{\sigma,p}_{\bar{R}_-}$ the set of all the functions in $H^{\sigma,p}$ which are supported in $\bar{R}_-$. This is easily derived from Paley-Wiener theorem. Indeed if u is supported in $\bar{R}_-$, then $u\hat{\ }(\xi)$ is analytic in $\operatorname{Im} \xi > 0$ and of arbitrarily small exponential type. Since $(\xi + i)^\sigma \cdot u\hat{\ }(\xi)$ has the same properties, the support of $(D + i)^\sigma u$ is in $\bar{R}_-$ and $Y_+(D + i)^\sigma u = 0$.

Conversely, if $Pu = 0$ then

$$Y_+(D + i)^\sigma u = (D + i)^\sigma Pu = 0$$

so that $(D + i)^\sigma u$ is supported in $\bar{R}_-$, and so is u since $(\xi + i)^{-\sigma}$ is analytic in $\operatorname{Im} \xi > 0$.

The restriction map $u \to u\,|_{x>0}$ of $H^{\sigma,p}(R)$ on $H^{\sigma,p}(R_+)$ is continuous, and its kernel is $H^{\sigma,p}_{\bar{R}_-}$. Hence

$$H^{\sigma,p}(R_+) \cong H^{\sigma,p}(R)/H^{\sigma,p}_{R_-} \cong PH^{\sigma,p}(R).$$

In other words, for every $u \in H^{\sigma,p}(R)$

$$\| u, R_+ \|_{\sigma,p} \sim \| Pu, R \|_{\sigma,p} .$$

If we denote $v = Y_+(D + i)^\sigma \cdot u$, we obtain

$$\| u, R_+ \|_{\sigma, p} \sim \| J^s(Pu), R \|_{L^p} = \| \{(1 + | \xi |^2)^{\sigma/2}(\xi + i)^{-\sigma} v^\wedge(\xi)\}^\vee, R \|_{L^p}$$

$$\sim \| v, R \|_{L^p} = \| Y_+ (D + i)^\sigma u \|_{L^p} ,$$

where we have used the fact that $(1 + | \xi |^2)^{\sigma/2}(\xi + i)^{-\sigma} \in \mathcal{M}_p$.

Corollary. *If $(D \pm i0)^\sigma$ is defined by $((D \pm i0)^\sigma)^\wedge = (\xi \pm i0)^\sigma$, then*

$$[u, R_\pm]_{\sigma,p} \sim \| Y_\pm(D \pm i0)^\sigma u \|_{L^p} . \tag{5.2}$$

Indeed, for $\lambda > 0$

$$J^\sigma u(\lambda x) = \lambda^\sigma[(\lambda^{-2} + \xi^2)^{\sigma/2} u^\wedge(\xi)]^\vee(\lambda x)$$

$$(D \pm i)^\sigma u(\lambda x) = \lambda^\sigma[(\xi \pm i/\lambda)^\sigma u^\wedge(\xi)]^\vee(\lambda x) .$$

Using (5.1) for $u(\lambda x)$ and letting $\lambda \to \infty$, we obtain (5.2).

Remark. The formulas of Peetre [11], [12] for the norms (and the semi-norms) $\| u, R^n_\pm \|_{s,2}$ can also be extended to any p. The proof is the same as for $n = 1$.

We turn now to prove the $H^{\sigma,p}$ version of the estimates of Section 3. However, the logical order of the arguments will be reversed. We first prove the analogue of Theotrem 3.4, and from it derive the other estimates.

Theorem 5.2. *If the eigenvalues of $- C^{-1} D$ are outside the ray $\arg \lambda = 2\pi(1/p - \sigma)$, then for $u \in H^{\sigma,p}(R)$*

$$[u, R]_{\sigma,p} \leqq K([u, R_-]_{\sigma,p} + [\mathscr{A} u, R_+]_{\sigma,p}) . \tag{5.3}$$

(Note that we do not require $\sigma \neq 1/p$).

Proof. By (5.2), we have to prove that

$$\| I^{\sigma}u \| \leqq K(\| Y_{-}(D - i0)^{\sigma}u \| + \| Y_{+}(D + i0)^{\sigma}\mathscr{A}u \|)$$

all the norms being $L^p(R)$ norms. Or, setting $v = (D - i0)^{\sigma}u$ and noting that $|i\xi|^{\sigma}(\xi - i0)^{-\sigma} \in \mathscr{M}_p$, we have to show that

(5.4) $\| v \| \sim \| I^{\sigma}(D - i0)^{-\sigma}v \| \leqq K(\| Y_{-}u \| + \| Y_{+}(D + i0)^{\sigma}\mathscr{A}(D - i0)^{-\sigma}v \|)$.

Let now $\mathscr{A}_{\sigma} = (D + i0)^{\sigma}\mathscr{A}(D - i0)^{-\sigma}$. Taking Fourier transform: $\mathscr{A}^{\wedge}(\xi) = - CY_{+}(\xi) + DY_{-}(\xi)$ and

$$\hat{\mathscr{A}_{\sigma}}(\xi) = (\xi + i0)^{\sigma}(\xi - i0)^{-\sigma}\mathscr{A}^{\wedge}(\xi) = - CY_{+}(\xi) + e^{2\pi i\sigma}DY_{-}(\xi)\,.$$

Comparing the two transforms, we see that $\mathscr{A}_{\sigma}$ itself has the same form as $\mathscr{A}$, except that $e^{2\pi i\sigma}\cdot D$ is substituted for D. Now the required estimate (5.4), which is

$$\| v \| \leqq K(\| Y_{-}v \| + \| Y_{+}\mathscr{A}_{\sigma}v \|) \tag{5.5}$$

corresponds to the case $\sigma = 0$ of Theorem 3.4. Hence (5.5) is true if and only if no eigenvalue of $- e^{2\pi i\sigma}C^{-1}D$ is on $\arg \lambda = 2\pi/p$, which is an equivalent way of stating the eigenvalues condition for $- C^{-1}D$.

Example. In the scalar case ($m = 1$) we take $C = - 1$, $D = 1$. Then $- C^{-1}D = 1$ and $\mathscr{A}u = u$. As a result we obtain the estimate

$$[u, R]_{\sigma, p} \leqq K([u, R_{-}]_{\sigma,p} + [u, R_{+}]_{\sigma,p}), \qquad \sigma \neq 1/p; \tag{5.6}$$

(and the analogous estimate in the norms). This result is the main tool we used in [16] to prove a conjecture of Lions-Magenes [7].

We shall consider now briefly the estimate for $H^{\pm}$ and A in $H^{\sigma,p}(R_{+})$. First of all we have, assuming the eigenvalues condition,

$$[\phi, R_{+}]_{\sigma,p} \leqq K[A\phi, R_{+}]_{\sigma,p}\,, \qquad \phi \in H^{\sigma,p}(R_{+}). \tag{5.7}$$

Indeed, for $\phi \in H^{\sigma,p}(R)$ and supported in R_{+}, (5.7) is exactly (5.3). Thus for $\sigma \leqq 1/p$ we are through. If $\sigma > 1/p$, we have the estimate in a 1-codimensional subspace and since A is $1-1$ operator, the estimate extends to the whole space. Next, to obtain the estimate

$$[H^{\pm}u, R_+]_{\sigma,p} \leqq K[u, R_+]_{\sigma,p}$$

we derive first the estimate for functions on R:

$$[\mathscr{H}u, R]_{\sigma,p} \leqq K[u, R]_{\sigma,p} \leqq K([u, R_-]_{\sigma,p} + [u, R_+]_{\sigma,p}), \qquad \sigma \neq 1/p,$$

and then argue as before. Alternatively, we can follow the arguments in the proof of Theorem 5.2. This will show that $\sigma \neq 1/p$ is indeed necessary here.

The estimates in the $H^{\sigma,p}$ norms are obtained as usual by combining the L^p with the $[\ ,\ ]_{\sigma,p}$ estimates. Thus A is bounded in $H^{\sigma,p}(R_+)$, if $\sigma \neq 1/p$. The properties of range (A) in $H^{\sigma,p}$ are exactly as in $W^{\sigma,p}$. We have seen that the same eigenvalues condition ensures the closeness of the range. Its codimension is also the same in both spaces—the results of Section 4 carry over since they were obtained by using diagonalization in $L^p = H^{0,p}$.

Finally, the extension of the results to $H^{s,p}$, $s \geqq 1$ is again performed as in the $W^{s,p}$ case.

§ 6. The case of a finite interval.

We consider now the Hilbert transform on a finite interval, which can be taken without loss of generality as $0 < x < 1$;

$$(I^{\pm}\phi)(x) = \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_0^1 \frac{\phi(y)}{x \pm i\varepsilon - y}\, dy, \qquad 0 < x < 1.$$

Again, $I^{\pm}$ is bounded in $L^p(0, 1)$. The corresponding system of singular integral equations is

$$CI^+\phi + DI^-\phi = \psi. \tag{6.1}$$

The L^2 theory was discussed in [13], where I^+ is diagonalized and its spectrum is computed. In the L^p case, although I^+ is not diagnoalizable, $x^{\sigma}I^+x^{-\sigma}$ and $(1-x)^{\sigma}I^+(1-x)^{-\sigma}$ are, where $\sigma = 1/q - 1/p$. This observation is sufficient for the complete resolution of (6.1) in $L^p(0, 1)$.

For a fixed p and $\phi(x)$ defined in $(0, 1)$, let

$$(U\phi)(t) = e^{t/p}(e^{-t} + 1)^{2/p}\,\phi\left(\frac{1}{e^{-t}+1}\right), \qquad -\infty < t < \infty\,;$$

$$(6.2)\quad (M\phi)(\tau) = (\mathscr{F}U\phi)(\tau) = \frac{1}{\sqrt{2\pi}}\int_0^1 x^{-1/q}(1-x)^{-1/q}\left(\frac{x}{1-x}\right)^{-i\tau}\phi(x)dx, \qquad 0 < \tau < 1.$$

Clearly U sets up an isometry between $L^p(0, 1)$ and $L^p(R)$. The inverse of M is a given by

$$(M^{-1}f)(x) = (U^{-1}\mathscr{F}^{-1})(x) = x^{-1/p}(1-x)^{-1/p}(\mathscr{F}^{-1}f)\left(\log\frac{x}{1-x}\right)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{x}{1-x}\right)^{i\tau} x^{-1/p}(1-x)^{-1/p}f(\tau)d\tau, \quad 0 < x < 1.$$

Note the similarity between the transformations M and U defined here and those of Section 1. In fact, near $x = 0$ we have the same behaviour in both cases Also our new M behaves relatively the same for $x \downarrow 0$ and $x \uparrow 1$.

Theorem 6.1. *For* $\sigma = 1/q - 1/p$ *we have*

$$(6.3)\qquad x^\sigma I^+ x^{-\sigma}\phi = M^{-1}\rho_1 M\phi,$$
$$(1-x)^\sigma I^+(1-x)^{-\sigma}\phi = M^{-1}\rho_2 M\phi, \quad \textit{where}$$
$$\rho_1(\tau) = [\exp(2\pi i(i\tau + 1/p)) - 1]^{-1}, \qquad -\infty < \tau < \infty$$
$$\rho_2(\tau) = [\exp(2\pi i(i\tau - 1/p)) - 1]^{-1} = \overline{\rho_1(\tau)}.$$

Proof. The proof is essentially similar to that of Theorem 1.1. We represent ϕ as $M^{-1}M\phi = M^{-1}f$, and then compute $(x^\sigma I^+ x^{-\sigma})M^{-1}f$. After changing the order of integration we obtain

$$\lim_{\varepsilon\downarrow 0}\frac{x^\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)d\tau\,\frac{1}{2\pi i}\int_0^1 \frac{\eta^{-1/q+i\tau}(1-\eta)^{-1/p-i\tau}}{x + i\varepsilon - \eta}d\eta.$$

To compute the inner integral, we integrate the function

$$-z^{-1/q+i\tau}(1-z)^{-1/p-i\tau}\,/(z - x - i\varepsilon)$$

over a closed path Γ which consists of two "small" circles of radius r around $z = 0$ and $z = 1$, connected above and below the positive unit interval, and of a great circle of radius R around $z = 0$. The integrand is holomorphic in the domain bounded by Γ, since after encircling both $z = 0$ and $z = 1$ we return with a factor

$$\exp(2\pi i(-1/q - 1/p + i\tau - i\tau)) = 1.$$

(This is the reason for choosing $\sigma = (1/q - 1/p)$.) Letting $r \to 0$, $R \to \infty$ and using Cauchy's formula we see that the value of our integral is

$$(x + i\varepsilon)^{-1/q+i\tau}[\exp(2\pi i(i\tau - 1/q)) - 1]^{-1}.$$

Letting $\varepsilon \to 0$, we obtain (6.3). The second formula is obtained in the same way. (The corresponding formulas for I^- can be written down immediately, using the relation $I^- = I^+ + 1$.)

Let now μ be a C^∞ function such that $\mu = 0$ for $x < 1/3$ and $\mu = 1$ for $x > 2/3$. Let

$$\psi_1 = \mu\psi,\ \psi_2 = (1 - \mu)\psi,\ \beta_1 = x^\sigma\psi_1,\ \beta_2 = (1 - x)^\sigma\psi_2. \tag{6.4}$$

Then $\psi = x^{-\sigma}\beta_1 + (1 - x)^{-\sigma}\beta_2$ and for $j = 1, 2$

$$\| \beta_j \| \sim \| \psi_j \| \leqq K \| \psi \| \sim \| \psi_1 \| + \| \psi_2 \| \tag{6.5}$$

where the norms are always in $L^p(0, 1)$. Assume for a moment that we can also write $\phi = x^{-\sigma}\alpha_1 + (1-x)^{-\sigma}\alpha_2$. Substituting these expressions for ϕ and ψ in (6.1) and equating the coefficients of $x^{-\sigma}$ and of $(1 - x)^{-\sigma}$, we obtain two systems, which we can solve by diagonalization:

$$C(x^\sigma I^+ x^{-\sigma})\alpha_1 + D(x^\sigma I^- x^{-\sigma})\alpha_1 = \beta_1$$

$$C(1 - x)^\sigma I^+(1 - x)^{-\sigma}\alpha_2 + D(1 - x)^\sigma I^-(1 - x)^{-\sigma}\alpha_2 = \beta_2.$$

We shall restrict our consideration to the scalar case, in other words, to the computation of the spectrum of I^+ in $L^p(0,1)$. But it will be clear that the vectorial case is completely analogous.

Theorem 6.2. *The essential spectrum of I^+ in $L^p(0,1)$ is the set of values assumed by $\rho_1(\tau)$ and $\rho_2(\tau)$, $-\infty \leqq \tau \leqq \infty$. (In particular the essential spectrum is the same for p and q where $1/p + 1/q = 1$.) This set consists of two symmetric circular arcs, with endpoints 0 and -1. In case $p = 2$, both arcs coincide with the real interval $[-1, 0]$. Every point λ of the open set S bounded by these arcs is an eigenvalue of multiplicity 2 of I^+ if $p < 2$, while $I^+ - \lambda$ is $1-1$ operator with closed range of codimension 2 if $p > 2$. Every other λ is in the resolvent set.*

Proof. If $p \leqq 2$, we have $\sigma = 1/q - 1/p \leqq 0$. We first show that $(I^+ - \lambda)\phi = \psi$ is solvable for every $\psi \in C_0^\infty$. To this end we use (6.4) and write $\psi = x^{-\sigma}\beta_1 + (1-x)^{-\sigma}\beta_2$. If we have also $\phi = x^{-\sigma}\alpha_1 + (1-x)^{-\sigma}\alpha_2$ then

$$(6.6)\quad (I^+ - \lambda)\phi = x^{\sigma}x^{-\sigma}(I^+ - \lambda)x^{\sigma}\alpha_1 + (1-x)^{\sigma}(1-x)^{-\sigma}(I^+ - \lambda)(1-x)^{\sigma}\alpha_2$$

$$= x^{\sigma}M^{-1}(\rho_1 - \lambda)M\alpha_1 + (1-x)^{\sigma}M^{-1}(\rho_2 - \lambda)M\alpha_2 .$$

Hence if we can solve in $L^p(0,1)$ the two equations:

$$(6.7)\qquad M^{-1}(\rho_j - \lambda)M\alpha_j = \beta_j, \qquad j = 1,2,$$

then since $\sigma \leqq 0$ if $p \leqq 2$, $\phi = (x^{-\sigma}\alpha_1 + (1-x)^{-\sigma}\alpha_2) \in L^p(0,1)$ is the required solution of $(I^+ - \lambda)\phi = \psi$. But the solution of (6.7) is immediate:

$$(6.8)\qquad \alpha_j = M^{-1}(\rho_j - \lambda)^{-1}M\beta_j, \qquad j = 1,2,$$

and in Section 2 it was proved that $(\rho_j - \lambda)^{-1} \in \mathscr{M}_p$ if $\lambda \neq \rho_j(\tau)$, $-\infty \leqq \tau \leqq \infty$. For such λ we have that $\alpha_j \in L^p$ and $\|\alpha_j\| \leqq K\|\beta_j\|$. If now $\psi^{(k)} \in C_0^\infty(0,1)$ and $\psi^{(k)} \to \psi$, then $\beta_j^{(k)} \to \beta_j$, hence also $\alpha_j^{(k)} \to \alpha_j$ and $\phi^{(k)} \to \phi = x^{-\sigma}\alpha_1 + (1-x)^{-\sigma}\alpha_2$. We clearly have now $(I^+ - \lambda)\phi = \psi$ in $L^p(0,1)$. This shows that for $p \leqq 2$ and $\lambda \neq \rho_j(\tau)$, the range of $I^+ - \lambda$ is the whole space.

The set of values assumed by $\rho_1(\tau)$ and $\rho_2(\tau)$ is clearly in the essential (and continuous) spectrum of I^+. Now

$$\rho_1(\tau) = \left[\exp\left(\frac{2\pi i}{p}\right)\exp(-2\pi\tau) - 1\right]^{-1},$$

and the set of values of ρ_1 is the circular arc obtained as the image of the positive axis under the Möbius transformation $w = \left[\left(\exp\left(\frac{2\pi i}{p}\right)\right)z - 1\right]^{-1}$. The endpoints are $\rho_1(-\infty) = 0$, $\rho_1(\infty) = -1$. Since $\rho_2(\tau) = \overline{\rho_1(\tau)}$, the arc $\lambda = \rho_2(\tau)$ is the complex conjugate of $\lambda = \rho_1(\tau)$. The same two arcs are obtained for the index q conjugate to p (but their order is reversed).

We notice now that

$$\int_0^1 (I^+ - \lambda)u\bar{v} = \int_0^1 u\,\overline{(I^+ - \bar{\lambda})v}$$

so that if $v \to \int u\bar{v}$ identifies L^q with $(L^p)^*$, we have $(I^+ - \lambda)^* = I^+ - \bar{\lambda}$. From the previous result for $p \leqq 2$ we obtain now for $p \geqq 2$ that for λ outside the two arcs, the range of $I^+ - \lambda$ is closed. We shall prove now that the codimension of the range is 2 if λ is in S and zero if $\lambda \notin S$. Dualizing again, we obtain that for $p < 2$, $\lambda \in S$ is a double eigenvalue and $\lambda \notin \bar{S}$ is in the resolvent (for any p in fact).

To accomplish this, we consider again $(I^+ - \lambda)\phi = \psi$, $\psi \in C_0^\infty$. Once more, we use (6.6), and we have (6.8), with $\alpha_j \in L^p$. But now $p > 2$, $\sigma = 1/q - 1/p > 0$ so that $\phi = x^{-\sigma}\alpha_1 + (1 - x)^{-\sigma}\alpha_2$ is not necessarily in L^p.

The reasoning now is exactly analogous to the proof of Theorem 4.1. In order that $x^{-\sigma}\alpha_1 \in L^p$ near $x = 0$, we have to extend the equation

$$M\alpha_1 = (\rho_1 - \lambda)^{-1}M\beta_1$$

to the complex plane, and using the definition of M, we see that $(\rho_1 - \lambda)^{-1}M\beta_1$ has to be regular analytic in $-\sigma \leqq \operatorname{Im}\tau \leqq 0$. In this case (remembering that near $x = 0$ M^{-1} here and of Section 1 behave alike) we have $\alpha_1 = O(x^{-\sigma+\epsilon})$ and $\alpha_1 \in L^p$ near $x = 0$. Since $M\beta_1$ is always regular in the strip and $(\rho_1 - \lambda)^{-1}$ has at most one pole τ_0 there, we have to require $(M\beta_1)(\tau_0) = 0$. This is a single continuous linear condition on α_1 (hence on ψ).

Analogously, $(1-x)^{-\sigma}\alpha_2 \in L^p$ near $x = 1$ if $(\rho_2 - \lambda)^{-1}M\beta_2$ has no poles in $0 \leqq \operatorname{Im}\tau \leqq \sigma$, so that $(M\beta_2)(\tau_0')$ should vanish if τ_0' is in this strip. This adds another condition.

The λ's for which we have to require the first condition are the values of $\rho_1(\tau)$ for $-\sigma \leqq \operatorname{Im}\tau \leqq 0$. For $\operatorname{Im}\tau = 0$ we obtain the first arc of the essential spectrum, and for $\operatorname{Im}\tau = -\sigma$, the second arc. For $-\sigma < \operatorname{Im}\tau < 0$ we obtain all the points of the domain S bounded by the arcs. The same set is covered by $\lambda = \rho_2(\tau)$, $0 < \operatorname{Im}\tau < \sigma$. Thus for every $\lambda \in S$, the range of $I^+ - \lambda$ is of codimension 2. Every $\lambda \notin \bar{S}$ is in the resolvent.

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