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CALDERON'S REPRODUCING FORMULA AND SINGULAR INTEGRAL OPERATORS ON A REAL LINE

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A generalization of Calderon's reproducing formula involving finite Borel measures is considered. This generalization gives rise to new representations of singular integral operators with constant coefficients on a real line and clarifies the conditions under which a reproducing formula holds. A convergence of new representations in L_p-norm and almost everywhere is studied.

INTRODUCTION

A reproducing formula due to Calderon which is well known in the theory of wavelet transforms and in some other areas of analysis has the form

$$\phi = \frac{1}{c_{u,v}} \int_0^\infty \frac{\phi * u_t * v_t}{t} dt. \tag{0.1}$$

Here $u_t(x) := \frac{1}{t}u(\frac{x}{t})$, $v_t(x) := \frac{1}{t}v(\frac{x}{t})$, $x \in \mathbf{R}$, $c_{u,v}$ is a constant depending on functions u, v and "$*$" denotes a convolution operation.

Formula (0.1) first appeared in the pioneering paper [Ca] by A.P. Calderon. It went on to play an important role as a tool in harmonic analysis; see e.g. [FS] or, more recently, [Dy]. It also occurs in wavelet theory [Da], [FJW], [HT], [Me] . Examination of this formula shows that the combination $u * v$ plays the key role in (0.1). So it is natural to replace $u * v$ by some finite measure μ and to investigate a bilinear operator

$$I(\mu, \phi) := \int_0^\infty \frac{\phi * \mu_t}{t} dt \tag{0.2}$$

where

$$(\phi * \mu_t)(x) = \int_{-\infty}^\infty \phi(x - ty) d\mu(y) \tag{0.3}$$

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is a convolution with a dilated measure.

In the present paper we show that under suitable conditions on the measure μ the integral (0.2) admits the following representation:

$$(0.4) \qquad I(\mu, \phi) = \alpha_+ \phi + \alpha_- H\phi$$

where

$$(0.5) \qquad \alpha_+ := \int_{-\infty}^{\infty} \log \frac{1}{|x|} d\mu(x), \quad \alpha_- := \frac{\pi i}{2} \int_{-\infty}^{\infty} sgn x \;\; d\mu(x)$$

and $H\phi$ is a Hilbert transform of ϕ. If $\phi \in L_p$, then the left-hand side of (0.4) is interpreted as a limit

$$(0.6) \qquad I(\mu, \phi) = \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \int_{\varepsilon}^{\rho} \frac{\phi * \mu_t}{t} dt$$

in L_p-norm or almost everywhere. Precise statements of these and related results are given below in Theorems 1.1, 1.2, 2.1, 2.2, 2.3 (see also Corollaries 1.1–1.3).

The formula (0.4) shows what assumptions on μ lead to a reproducing formula ($\alpha_+ = 1, \;\; \alpha_- = 0$) or to the Hilbert transform ($\alpha_+ = 0, \;\; \alpha_- = 1$). Moreover, one can readily see that any singular integral operator $c_1 I + c_2 H$ with constant coefficients c_1, c_2 may be represented by (0.2) with a suitable finite measure μ.

§1 is devoted to L^2-theory of operators (0.2). In §2 we develop L^p-theory of these operators and investigate convergence of (0.6) in a.e. sense.

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Notation. In the sequel $\mathcal{M}$ denotes the set of all complex-valued finite Borel measures μ on the real line **R**. For $\mu \in \mathcal{M}$ the values $\mu(\{\pm\infty\})$ are assumed to be zero. Define

$$\mu' := \mu - \mu(\{0\})\delta_o$$

where δ_o stands for the Dirac measure at the point $x = 0$. The Fourier and Hilbert transforms are here defined as follows:

$$\hat{\mu}(\xi) := \int_{-\infty}^{\infty} e^{ix\xi} d\mu(x), \quad (H\phi)(x) := p.v. \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\phi(y)}{x - y} dy.$$

The letter c will be used for constants which may assume different values at distinct occurrences.

§1 L^2-THEORY

Given a measure $\mu \in \mathcal{M}$, it will be convenient to divide $\hat{\mu}(\eta)$ into its even and odd parts. Denote

$$\hat{\mu}_{\pm}(\eta) := \frac{\hat{\mu}(\eta) \pm \hat{\mu}(-\eta)}{2}, \quad \alpha_{\pm} := \int_0^{\infty} \frac{\hat{\mu}_{\pm}(\eta)}{\eta} d\eta. \tag{1.1}$$

Throughout this paper the integrals above will be interpreted as improper ones.
Theorem 1.1. *Let $\phi \in L^2$ and let the integrals $\alpha_{\pm}$ be finite. Then*

$$I(\mu, \phi) := \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}}^{(L^2)} \int_{\varepsilon}^{\rho} \frac{\phi * \mu_t}{t} dt = \alpha_+ \phi + \alpha_- H\phi. \tag{1.2}$$

Proof. Denote

$$I_{\varepsilon,\rho}(\mu, \phi) = \int_{\varepsilon}^{\rho} \frac{\phi * \mu_t}{t} dt \tag{1.3}$$

and assume first that $\phi \in L^1 \cap L^2$. Then $I_{\varepsilon,\rho}(\mu,\phi) \in L^1 \cap L^2$ and $[I_{\varepsilon,\rho}(\mu,\phi)]^{\wedge}(\xi) = \hat{k}_{\varepsilon,\rho}(\xi)\hat{\phi}(\xi)$ where

$$\hat{k}_{\varepsilon,\rho}(\xi) := \int_{\varepsilon}^{\rho} \frac{\hat{\mu}(t\xi)}{t} dt = \int_{\varepsilon}^{\rho} \frac{\hat{\mu}_+(t\xi)}{t} dt + \int_{\varepsilon}^{\rho} \frac{\hat{\mu}_-(t\xi)}{t} dt =: \hat{k}^+_{\varepsilon,\rho}(\xi) + \hat{k}^-_{\varepsilon,\rho}(\xi).$$

For $\xi \neq 0$ we have

$$\hat{k}^+_{\varepsilon,\rho}(\xi) = \int_{\varepsilon|\xi|}^{\rho|\xi|} \frac{\hat{\mu}_+(\eta)}{\eta} d\eta, \quad \hat{k}^-_{\varepsilon,\rho}(\xi) = sgn\xi \int_{\varepsilon|\xi|}^{\rho|\xi|} \frac{\hat{\mu}_-(\eta)}{\eta} d\eta$$

Since the integrals $\alpha_{\pm}$ from (1.1) are finite, the functions

$$\psi_{\pm}(t) = \int_0^t \frac{\hat{\mu}_{\pm}(\eta)}{\eta} d\eta$$

are continuous on $[0, \infty]$. It follows that there exist constants $A_{\pm} > 0$ for which

$$|\hat{k}^{\pm}_{\varepsilon,\rho}(\xi| \leq A_{\pm} \quad \forall \rho > \varepsilon > 0, \quad \xi \in \mathbb{R} \tag{1.4}$$

and

$$\|I_{\varepsilon,\rho}(\mu,\phi)\|_2 = \|\hat{\phi}(\xi)\hat{k}_{\varepsilon,\rho}(\xi)\|_2 \le A\|\phi\|_2, \quad A = A_+ + A_-. \tag{1.5}$$

Now the Lebesgue theorem on dominated convergence yields

$$\|I_{\varepsilon,\rho}(\mu,\phi) - \alpha_+\phi - \alpha_- H\phi\|_2 \to 0 \text{ as } \varepsilon \to 0,\ \rho \to \infty. \tag{1.6}$$

The result for arbitrary $\phi \in L^2$ may be easily derived from (1.6), (1.5) by using the inequality

$$\|I_{\varepsilon,\rho}(\mu,\phi)\|_2 \le V_\mu \|\phi\|_2 \log\frac{\rho}{\varepsilon}$$

where V_μ is a total variation of μ. □

Our next purpose is to represent the coefficients $\alpha_\pm$ in (1.2) just in terms of μ without Fourier transform. This will clarify for which measure $I(\mu,\phi)$ will be a reproducing operator, and when $I(\mu,\phi)$ will give the Hilbert transform. In addition, it is interesting to know under what conditions on μ the integrals $\alpha_\pm$ in (1.2) are finite. The lemma below will be helpful to answer these questions.

Lemma 1.1. (i) *Let $\mu \in \mathcal{M}$ satisfy the following conditions:*

$$\mu(\{0\}) = 0, \quad \mu(\mathbf{R}) = 0, \quad \int_{-\infty}^{\infty} |\log|x||\ d|\mu|(x) < \infty. \tag{1.7}$$

Then the integral α_+ is finite and represented as

$$\alpha_+ = \int_{-\infty}^{\infty} \log\frac{1}{|x|} d\mu(x). \tag{1.8}$$

(ii) *The integral α_- is finite for any $\mu \in \mathcal{M}$ and*

$$\alpha_- = \frac{\pi i}{2}\int_{-\infty}^{\infty} sgn x\ \ d\mu'(x). \tag{1.9}$$

where $\mu' = \mu - \mu(\{0\})\delta_o$, δ_o being a Dirac measure at the point $x = 0$.

Remark 1.1.

1. The condition $\mu(\mathbf{R}) = 0$ which may be written as $\hat{\mu}(0) = 0$ is necessary for the convergence of α_+ (see (1.1)). The equality $\mu(\{0\}) = 0$ follows in fact from the third condition in (1.7). It was convenient to write down this equality separately because of its great importance for the sequel.

Proof of Lemma 1.1. Consider α_+. For $0 < \varepsilon < \rho < \infty$ we have

$$A^+_{\varepsilon,\rho} := \int_{\varepsilon}^{\rho} \frac{\hat{\mu}_+(\eta)}{\eta} d\eta = \int_{\varepsilon}^{\rho} \frac{d\eta}{\eta} \int_{-\infty}^{\infty} \cos(y\eta) d\mu(y).$$

Denote by $g(\eta)$ the function which is equal to 1 for $0 < \eta \le 1$ and to zero for $\eta > 1$. Since $\mu(\mathbf{R}) = 0$,

$$A^{+}_{\varepsilon,\rho} = \int_{-\infty}^{\infty} \lambda_{\varepsilon,\rho}(y)d\mu(y), \quad \lambda_{\varepsilon,\rho}(y) := \int_{\varepsilon}^{\rho} \frac{\cos(y\eta) - g(\eta)}{\eta} d\eta.$$

Let us prove the following relations:

$$\left|\lambda_{\varepsilon,\rho}(y)\right| \le c_1 \left|\log|y|\right| + c_2, \quad y \ne 0, \tag{1.10}$$

(c_1, c_2 being independent of ε, ρ),

$$\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \lambda_{\varepsilon,\rho}(y) = \log\frac{1}{|y|} + C, \quad y \ne 0, \tag{1.11}$$

with some constant C which will not play any role because $\mu(\mathbf{R}) = 0$.

To prove (1.10) consider the following three cases:

1) $\rho < 1$, 2) $\varepsilon < 1 < \rho$, 3) $\varepsilon > 1$.

In the first case we have:

$$\lambda_{\varepsilon,\rho} = -\int_{\varepsilon}^{\rho} \frac{1-\cos(y\eta)}{\eta} d\eta = \left(\int_{\rho}^{1} - \int_{\varepsilon}^{1}\right) \frac{1-\cos(y\eta)}{\eta} d\eta =: \lambda_{\rho} - \lambda_{\varepsilon}.$$

If $|y| \ge 1$, then for $\varepsilon < \frac{1}{|y|}$ we obtain (for $\varepsilon \ge \frac{1}{|y|}$ the changes are obvious).

$$\lambda_{\varepsilon} = \left(\int_{\varepsilon}^{1/|y|} + \int_{1/|y|}^{1}\right) \frac{1-\cos(y\eta)}{\eta} d\eta = \left(\int_{\varepsilon|y|}^{1} + \int_{1}^{|y|}\right) \frac{1-\cos t}{t} dt =: \tag{1.12}$$

$= \lambda_{\varepsilon}^{(1)} + \lambda_{\varepsilon}^{(2)}$, where

$$\left|\lambda_{\varepsilon}^{(1)}\right| \le \int_{0}^{1} \frac{1-\cos t}{t} dt, \tag{1.13}$$

$$\lambda_{\varepsilon}^{(2)} = \log|y| - \int_{1}^{|y|} \frac{\cos t}{t} dt = \log|y| + O(1). \tag{1.14}$$

It follows that

$$|\lambda_{\varepsilon,\rho}| \le 2\log|y| + C.$$

If $|y| < 1$, then

$$\lambda_\varepsilon = \left(\int\limits_\varepsilon^{1/|y|} - \int\limits_1^{1/|y|} \right) \frac{1-\cos(y\eta)}{\eta} d\eta = \left(\int\limits_{\varepsilon|y|}^{1} - \int\limits_{|y|}^{1} \right) \frac{1-\cos t}{t} dt$$

and hence $|\lambda_{\varepsilon,\rho}| \le C$.

Consider the case 2):

$$\lambda_{\varepsilon,\rho} = \int\limits_\varepsilon^1 \frac{\cos(y\eta)-1}{\eta} d\eta + \int\limits_1^\rho \frac{\cos(y\eta)}{\eta} d\eta =: \Lambda_\varepsilon + \Lambda_\rho. \tag{1.15}$$

The estimate of the first term is already obtained. For the second term in the case $|y| \ge 1$ we have

$$\Lambda_\rho = \int\limits_{|y|}^{\rho|y|} \frac{\cos t}{t} dt.$$

This integral is obviously bounded uniformly in ρ and y for $|y| \ge 1$. If $|y| < 1$ then

$$\left| \int\limits_{|y|}^{\rho|y|} \frac{\cos t}{t} dt \right| \le \left| \int\limits_{|y|}^{1} \frac{\cos t}{t} dt \right| + \left| \int\limits_{1}^{\rho|y|} \frac{\cos t}{t} dt \right| \le \log \frac{1}{|y|} + C.$$

In the case 3) we have

$$\lambda_{\varepsilon,\rho} = \left(\int\limits_1^\rho - \int\limits_1^\varepsilon \right) \frac{\cos(y\eta)}{\eta} d\eta$$

and the required estimate follows from the argument above. The inequality (1.10) is proved.

Let us verify (1.11). One can use (1.15). If $|y| \ge 1$ then for $\varepsilon < 1/|y|$ we have (see (1.12))

$$\lambda_{\varepsilon,\rho} = \int\limits_{\varepsilon|y|}^{1} \frac{\cos t - 1}{t} dt + \int\limits_{1}^{|y|} \frac{\cos t - 1}{t} dt + \int\limits_{|y|}^{\rho|y|} \frac{\cos t}{t} dt =$$

$$= \int\limits_{\varepsilon|y|}^{1} \frac{\cos t - 1}{t} dt - \log|y| + \int_1^{\rho|y|} \frac{\cos t}{t} dt \to C + \log \frac{1}{|y|}$$

as $\varepsilon \to 0$, $\rho \to \infty$. In the case $|y| < 1$ by assuming $\rho > \frac{1}{|y|}$ we have

$$\lambda_{\varepsilon,\rho} = \int\limits_\varepsilon^{1/|y|} \frac{\cos(y\eta)-1}{\eta} d\eta - \int_1^{1/|y|} \frac{\cos(y\eta)-1}{\eta} d\eta + \int\limits_1^\rho \frac{\cos(y\eta)}{\eta} d\eta =$$

$$= \int_{\varepsilon|y|}^{1} \frac{\cos t - 1}{t} dt - \int_{|y|}^{1} \frac{\cos t - 1}{t} dt + \int_{|y|}^{\rho|y|} \frac{\cos t}{t} dt =$$

$$= \int_{\varepsilon|y|}^{1} \frac{\cos t - 1}{t} dt + \log \frac{1}{|y|} + \int_{1}^{\rho|y|} \frac{\cos t}{t} dt \to C + \log \frac{1}{|y|}$$

as $\varepsilon \to 0$, $\rho \to \infty$.

Thus (1.11) is true, and formula (1.8) now follows by using the Lebesgue dominated convergence theorem together with (1.10) and the inequality in (1.7).

A proof of the second statement (ii) is simple:

$$\alpha_- = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \frac{1}{2} \int_{\varepsilon}^{\rho} \frac{\hat{\mu}(\eta) - \hat{\mu}(-\eta)}{\eta} d\eta = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} i \int_{\varepsilon}^{\rho} \frac{d\eta}{\eta} \int_{-\infty}^{\infty} \sin(y\eta) d\mu(y) =$$

$$= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} i \int_{-\infty}^{\infty} sgny\ d\mu'(y) \int_{\varepsilon|y|}^{\rho|y|} \frac{\sin \eta}{\eta} d\eta = \frac{\pi i}{2} \int_{-\infty}^{\infty} sgny\ d\mu'(y)$$

where $\mu' = \mu - \mu(\{0\})\delta_0$, δ_o being the Dirac measure at the point $x = 0$. □

Lemma 1.1 combined with Theorem 1.1. enables us to know when the integral $I(\mu, \phi)$ defined by (0.2) reproduces ϕ (up to some constant multiplier). Namely, we have

Theorem 1.2. *Let $\mu(\mathbb{R}) = 0$. If the integral α_+ from (1.1) is finite, then the equality*

$$I(\mu, \phi) := \overset{(L^2)}{\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} I_{\varepsilon,\rho}(\mu, \phi) = c\phi \tag{1.16}$$

holds for some constant $c = c(\mu)$ and for all $\phi \in L^2$ if and only if $c = \alpha_+$ (see (1.1)) and

$$\mu((0, +\infty)) = \mu((-\infty, 0)) = -\frac{1}{2}\mu(\{0\}). \tag{1.17}$$

If (1.17) is valid and $\int_{-\infty}^{\infty} |\log|x||\, d|\mu|(x) < \infty$, then (1.16) holds with $c = \alpha_+ = \int_{-\infty}^{\infty} \log \frac{1}{|x|} d\mu(x$

Proof. Let $I(\mu, \phi) = c\phi \quad \forall \phi \in L^2, \quad c = c(\mu)$. Then, by Lemma 1.1 (ii) and by Theorem 1.1, $c\phi = \alpha_+\phi + \alpha_- H\phi$. For $\hat{\phi}(\xi) = exp(-\xi^2)$ the last equality yields $c - \alpha_+ = \alpha_- sgn\xi$. This implies $\alpha_+ = c$, and $\alpha_- = 0$. Since $\mu(\mathbf{R}) = \mu((-\infty, 0)) + \mu((0, +\infty)) + \mu(\{0\}) = 0$, then according to (1.9)

$$\alpha_- = \frac{\pi i}{2}[\mu((0, +\infty)) - \mu((-\infty, 0))] = \pi i[\mu((0, +\infty)) + \frac{1}{2}\mu(\{0\})] = 0$$

and

$$\alpha_- = -\pi i[\mu((-\infty, 0)) + \frac{1}{2}\mu(\{0\})] = 0.$$

Hence $\mu((0,+\infty)) = \mu((-\infty,0)) = -\frac{1}{2}\mu(\{0\})$. All other statements follow immediately from Theorem 1.1 and Lemma 1.1. □

The following corollaries seem to be useful.

Corollary 1.1. *Let $\psi \in L^1$ satisfy the following conditions:*

(1.18) $$\int_{-\infty}^{\infty} \psi(x)dx = 0, \quad \int_{-\infty}^{\infty} |\psi(x)\log|x||\, dx < \infty.$$

Then

(1.19) $$\int_0^\infty \frac{\phi * \psi_t}{t} dt = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}^{(L^2)} \int_\varepsilon^\rho \frac{\phi * \psi_t}{t} dt = c\phi, \quad c = c(\psi),$$

for any $\phi \in L^2$ iff

(1.20) $$\int_{-\infty}^{0} \psi(x)dx = \int_0^\infty \psi(x)dx = 0.$$

If (1.19) holds, then

$$c = \int_{-\infty}^{\infty} \psi(x)\log\frac{1}{|x|}dx.$$

Corollary 1.2. *Let $\mu \in \mathcal{M}$ be such that*

(1.21) $$supp\ \mu \subset (0,+\infty), \quad \mu((0,+\infty)) = 0, \quad \int_0^\infty |\log|x||\, d|\mu|(x) < \infty.$$

Then (1.16) holds with $c = \int_0^\infty \log\frac{1}{|x|}d\mu(x)$.

Corollary 1.3. *Let $\phi \in L^2$, μ be an arbitrary finite measure for which $\hat{\mu}(\eta)$ is odd. Then*

(1.22) $$I(\eta,\phi) := \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}^{(L^2)} I_{\varepsilon,\rho}(\mu,\phi) = \alpha_- H\phi, \quad \alpha_- = \frac{\pi i}{2}\int_{-\infty}^{\infty} sgnx\ d\mu(x).$$

§2. L^p-THEORY AND ALMOST EVERYWHERE CONVERGENCE

Let us transform $I_{\varepsilon,\rho}(\mu,\phi)$ assuming $\phi \in L^p$, $1 \le p \le \infty$. Put $\mu = \mu' + \mu(\{0\})\delta_0$. Then

$$I_{\varepsilon,\rho}(\mu,\phi)(x) = \int_\varepsilon^\rho \frac{dt}{t}\int_{-\infty}^{\infty} \phi(x-ty)d\mu(y) =$$

$$= \int_{-\infty}^{\infty} d\mu'(y) \int_{\varepsilon}^{\rho} \phi(x-ty)\frac{dt}{t} + \mu(\{0\})\phi(x)\log\frac{\rho}{\varepsilon}.$$

In the sequel we put

$$\mu(\{0\}) = 0, \tag{2.1}$$

otherwise the last term above tends to infinity when $\varepsilon \to 0$, $\rho \to \infty$. Under this condition one can easily obtain

$$I_{\varepsilon,\rho}(\mu,\phi)(x) = \int_{-\infty}^{\infty} \phi(x-t)k_{\varepsilon,\rho}(t)dt \tag{2.2}$$

where $k_{\varepsilon,\rho} \in L^1$ and has the form

$$k_{\varepsilon,\rho}(t) = \frac{1}{|t|}\begin{cases} \mu([t/\rho, t/\varepsilon)) & \text{if } t > 0, \\ \mu((t/\varepsilon, t/\rho]) & \text{if } t < 0. \end{cases} \tag{2.3}$$

Denote

$$k(t) := \frac{1}{t}\int_0^t d\mu(y) = \frac{1}{|t|}\begin{cases} \mu((0,t)) & \text{if } t > 0, \\ \mu((t,0)) & \text{if } t < 0, \end{cases}$$

Then

$$k_{\varepsilon,\rho}(t) = k_\varepsilon(t) - k_\rho(t), \quad k_\varepsilon(t) := \frac{1}{\varepsilon}k\left(\frac{t}{\varepsilon}\right). \tag{2.5}$$

In parallel with $k(t)$ we shall use a "dual" function

$$\tilde{k}(t) := \frac{1}{|t|}\begin{cases} \mu([t,+\infty)) & \text{if } t > 0, \\ \mu((-\infty,t]) & \text{if } t < 0. \end{cases}$$

The functions $k(t)$, $\tilde{k}(t)$ will play a key role throughout this section.

In the sequel it will be convenient to investigate separately the case when $\hat{\mu}$ is even and the case when $\hat{\mu}$ is odd. As a consequence one can obtain the corresponding results for the general case.

2.1. The case of $\hat{\mu}$ even

We first establish the connection between the kernel $k(t)$ (2.4) and the expressions for α_+ presented in (1.1), (1.8).

Lemma 2.1. *Let $\hat{\mu}$ be even and assume that the following relations hold:*

$$\mu(\mathbb{R}) = 0, \quad \mu(\{0\}) = 0, \tag{2.7}$$

$$(2.8) \qquad \int_0^\infty |\log x|\, d|\mu|(x) < \infty.$$

Then

$$(2.9) \qquad k(t) \in L^1, \quad \hat{k}(\xi) = \int_{|\xi|}^\infty \frac{\hat{\mu}(\eta)}{\eta} d\eta, \quad \int_{-\infty}^\infty k(t)dt = \alpha_+.$$

Proof. Since $\hat{\mu}$ is even, then $\mu((-|t|, 0)) = \mu((0, |t|))$ for any $t \neq 0$ (see Lemma 0.1), and therefore by (2.7), (2.4) we obtain

$$(2.10) \qquad \mu((0, +\infty)) = 0, \quad k(t) = \frac{\mu((0, |t|))}{|t|} = -\frac{\mu([|t|, +\infty))}{|t|}.$$

It follows that

$$(2.11) \qquad \int_0^\infty |k(t)|dt \leq \int_0^1 \frac{dt}{t} \int_0^t d|\mu|(x) + \int_1^\infty \frac{dt}{t} \int_t^\infty d|\mu|(x) = \int_0^\infty |\log x|\, d|\mu|(x) < \infty$$

and thus, $k(t) \in L^1$.

Let us calculate $\hat{k}(\xi)$. For $\xi \neq 0$, by (2.10), we have

$$(2.12) \qquad \hat{k}(\xi) = 2\int_0^\infty \cos(t\xi)\frac{dt}{t} \int_0^t d\mu(y) = 2\int_0^\infty d\mu(y) \int_y^\infty \cos(t\xi)\frac{dt}{t}.$$

The second equality in (2.12) is justified as follows.

$$\int_0^\infty \cos(t\xi)\frac{dt}{t} \int_0^t d\mu(y) =$$

$$= \int_0^{1/|\xi|} \cos(t\xi)\frac{dt}{t} \int_0^t d\mu(y) - \int_{1/|\xi|}^\infty \cos(t\xi)\frac{dt}{t} \int_t^\infty d\mu(y) =$$

$$\int_0^{1/|\xi|} d\mu(y) \int_y^{1/|\xi|} \cos(t\xi)\frac{dt}{t} - \int_{1/|\xi|}^\infty d\mu(y) \int_{1/|\xi|}^y \cos(t\xi)\frac{dt}{t} =$$

(due to (2.8) the interchange of integrals is possible.)

$$= \int_0^{1/|\xi|} d\mu(y) \left[\int_y^\infty - \int_{1/|\xi|}^\infty\right] \cos(t\xi)\frac{dt}{t} - \int_{1/|\xi|}^\infty d\mu(y) \left[\int_{1/|\xi|}^\infty - \int_y^\infty\right] \cos(t\xi)\frac{dt}{t} =$$

$$= \int_0^\infty d\mu(y) \int_y^\infty \cos(t\xi)\frac{dt}{t}.$$

The relation (2.12) yields

$$\hat{k}(\xi) = 2\int_0^\infty d\mu(y)\left[\int_1^A + \int_A^\infty\right]\cos(y\eta\xi)\frac{d\eta}{\eta} =: 2(I_1 + I_2). \tag{2.13}$$

Let us prove that

$$\lim_{A\to\infty} I_2 = 0. \tag{2.14}$$

We have

$$I_2 = \int_0^\infty d\mu(y) \int_{Ay|\xi|}^\infty \frac{\cos\tau}{\tau}d\tau = \left(\int_0^{1/(2A|\xi|)} + \int_{1/(2A|\xi|)}^\infty\right) d\mu(y) \int_{Ay|\xi|}^\infty \frac{\cos\tau}{\tau}d\tau =$$

$$=: I_{21} + I_{22}$$

In order to estimate I_{21} we note that in this case $Ay|\xi| < \frac{1}{2}$, and hence (one may take $A|\xi| > 1$)

$$\left|\int_{Ay|\xi|}^\infty \frac{\cos\tau}{\tau}d\tau\right| \le c\log\frac{1}{Ay|\xi|} \le c\log\frac{1}{y}$$

with some constant c. It follows that

$$|I_{21}| \le c \int_0^{1/(2A|\xi|)} \log\frac{1}{y}\, d|\mu|(y) \to 0, \quad A\to\infty,$$

by virtue of (2.8). In order to estimate I_{22} introduce an auxiliary function $r(t)$ such that

$$r(t) = \int_t^\infty \frac{\cos\tau}{\tau}d\tau \text{ for } t \ge \frac{1}{2} \text{ and } r(t) = 0 \text{ for } t < \frac{1}{2},$$

which is obviously bounded. Then

$$\lim_{A\to\infty} I_{22} = \lim_{A\to\infty}\int_0^\infty r(Ay|\xi|)d\mu(y) = \int_0^\infty \lim_{A\to\infty} r(Ay|\xi|)d\mu(y) = 0,$$

and (2.14) is proved.

Consider I_1 (see 2.13)). We have

$$2I_1 = 2\int_1^A \frac{d\eta}{\eta}\int_0^\infty \cos(y\eta\xi)d\mu(y) = \int_1^A \frac{\hat{\mu}(\eta\xi)}{\eta}d\eta.$$

Since $\hat{k}(\xi)$ is well-defined and $\lim\limits_{A\to\infty} I_2 = 0$, then the limit

$$\lim_{A\to\infty}\int_1^A \frac{\hat{\mu}(\eta\xi)}{\eta}d\eta$$

exists, and thus, the second formula in (2.9) is proved. The last formula in (2.9) follows from the previous one by the continuity of $\hat{k}(\xi)$, taking into account (1.1). □

Theorem 2.1. *Let μ satisfy the conditions of Lemma 2.1, $\phi \in L^p$, $1 \le p < \infty$. Then*

(i) $\overset{(L^p)}{\lim\limits_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} I_{\varepsilon,\rho}(\mu,\phi) = \alpha_+\phi$ *for* $1 < p < \infty$.

(ii) *If $1 \le p < \infty$ and $k(t)$ (see (2.10)) has an even decreasing summable majorant, then $I_{\varepsilon,\rho}(\mu,\phi) \to \alpha_+\phi$ as $\varepsilon \to 0$, $\rho \to \infty$ almost everywhere.*

Proof. The first assertion follows from the equality $I_{\varepsilon,\rho}(\mu,\phi) = \phi * k_\varepsilon - \phi * k_\rho$ since

$$\overset{(L^p)}{\lim_{\varepsilon\to 0}} \phi * k_\varepsilon = \alpha_+\phi \quad \text{for } k \in L^1,\ \ \phi \in L^p,\ \ 1 \le p < \infty,$$

and $\phi * k_\rho \overset{(L^p)}{\longrightarrow} 0$ as $\rho \to \infty$, for $1 < p < \infty$ (concerning the last relation for arbitrary $k \in L^1$ see, e.g., [Sa], p. 22). In order to prove (ii) we note that $\phi * k_\varepsilon \overset{a.e.}{\longrightarrow} \alpha_+\phi$ as $\varepsilon \to 0$, according to the well-known property of approximate identities (see e.g. [St.]).

Let us check that $\phi * k_\rho \overset{a.e.}{\longrightarrow} 0$, $\rho \to \infty$. For any $\delta > 0$ we have

$$|(\phi * k_\rho)(x)| \le \int_{|t|<\delta} |\phi(x-t)|\frac{|\mu|((0,|t|/\rho))}{|t|}dt + \int_{|t|>\delta} |\phi(x-t)|\frac{|\mu|((0,|t|/\rho))}{|t|}dt$$

$$=: A_\delta(x,\rho) + B_\delta(x,\rho).$$

If $\rho \ge 1$, then

$$A_\delta(x,\rho) \le \int_{|t|<\delta} |\phi(x-t)|\frac{|\mu|((0,|t|))}{|t|}dt.$$

As a consequence of (2.8) (cf. (2.11)) we have $\frac{1}{|t|}|\mu|((0,|t|)) \in L^1_{\text{loc}}$. It follows that the function

$$\psi_x(t) := \begin{cases} |\phi(x-t)||\mu|((0,|t|))/|t| & \text{if } |t| < 1, \\ 0, & \text{if } |t| \ge 1, \end{cases}$$

is summable for almost all x, and hence, $A_\delta(x,\rho)$ may be made arbitarily small for sufficiently small $\delta = \delta(x)$ uniformly with respect to $\rho \geq 1$.

Consider $B_\delta(x,\rho)$ for any fixed $x \in \mathbf{R}$ and $\delta > 0$. If $1 \leq p < \infty$, then $t \to |\phi(x-t)|/|t|$ is a summable function for $|t| > \delta$ and hence

$$\lim_{\rho\to\infty} B_\delta(x,\rho) = \int\limits_{|t|>\delta} \frac{|\phi(x-t)|}{|t|} \lim_{\rho\to\infty} |\mu((0,|t|/\rho))|\, dt = 0.$$

□

2.2. The case of $\hat{\mu}$ odd.

As before we assume that $\mu(\mathrm{R}) = \mu(\{0) = 0$. Then by (1.1), (1.9),

$$\alpha_+ = 0, \quad \alpha_- = \pi i \mu((0,\infty)). \tag{2.15}$$

Introduce a conjugate Poisson kernel

$$q_\varepsilon(t) = \frac{1}{\varepsilon} q(\frac{t}{\varepsilon}), \quad q(t) = \frac{1}{\pi i}\frac{t}{1+t^2},$$

and denote

$$h(t) := d(t) - \alpha_- q(t), \quad h_\varepsilon(t) := \frac{1}{\varepsilon}h(\frac{t}{\varepsilon}). \tag{2.16}$$

We remind that $k(t)$, $\tilde{k}(t)$ are the functions defined in (2.4), (2.6).
Lemma 2.2. *Let $\mu(\in \mathcal{M})$ be odd and satisfy (2.7), (2.8). Then*

$$h(t) \in L^1, \quad \int\limits_{-\infty}^{\infty} h(t)dt = 0; \tag{2.17}$$

$$h(t) = \begin{cases} O(t) + k(t) & \text{if } |t| < 1, \\ O(t^{-3}) - \tilde{k}(t) & \text{if } |t| > 1. \end{cases} \tag{2.18}$$

Proof. The summability of $h(t)$ is a consequence of (2.18) because $k(t)$ and $\tilde{k}(t)$ are summable in the neighbourhoods of the origin and of infinity respectively (this follows from (2.8)). The second relation in (2.17) is obviuos because $h(t)$ is odd. Let us prove (2.18). For $|t| < 1$ this estimate follows immediately from the definition of $h(t)$.

For $t > 1$ we have

$$h(t) = \frac{\mu((0,t))}{t} - \frac{t\mu((0,+\infty))}{1+t^2} = \frac{\mu((0,+\infty))}{t(1+t^2)} - \tilde{k}(t) = O(t^{-3}) - \tilde{k}(t).$$

A similar equality holds for $t < -1$. □

Corollary 2.1. *If $k(t)$ has a summable decreasing majorant in a neighbourhood of the origin and $\tilde{k}(t)$ has a summable decreasing majorant in a neighbourhood of infinity, then $h(t)$ has an even majorant belonging to L^1 and decreasing for $|t| > 0$.*

Theorem 2.2. *Let μ be odd and satisfy* (2.7), (2.8). *Then for any $\phi \in L^p$, $1 < p < \infty$, the following statements hold.*

(i) $\overset{(L^p)}{\lim\limits_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} I_{\varepsilon,\rho}(\mu,\phi) = \alpha_- H\phi.$

(ii) *If $k(t)$ has a summable decreasing majorant in a neighbourhood of the origin and $\tilde{k}(t)$ has a summable decreasing majorant in a neighbourhood of infinity, then the limit in* (i) *can be interpreted in a.e.-sense.*

Proof. By (2.16) we have

$$I_{\varepsilon,\rho}(\mu,\phi) = \alpha_- \phi * q_\varepsilon + \phi * h_\varepsilon - \phi * k_\rho. \tag{2.19}$$

It is well known (see, e.g., [Ne]), that $\phi * q_\varepsilon \to H\phi$, in L^p-norm and almost everywhere when $\varepsilon \to 0$. For the second term in (2.19) we have $\lim\limits_{\varepsilon\to 0} \|\phi * h_\varepsilon\|_p = 0$ due to (2.17). If μ satisfies the conditions of (ii), then Corollary 2.1 implies that $\overset{a.e.}{\lim\limits_{\varepsilon\to 0}} \phi * h_\varepsilon = 0$. Let us prove that $\lim\limits_{\rho\to\infty} \|\phi * k\rho\|_p = 0$. According to (2.16) one should check that

$$\lim_{\rho\to\infty} \|\phi * h_\rho\|_p = 0 \ \text{ and } \ \lim_{\rho\to\infty} \|\phi * q_\rho\|_p = 0.$$

The first relation is true because $h \in L^1$ (one can apply, e.g., Lemma 3.2 from [Sa]). In order to prove the second relation we take the sequence of compactly supported smooth functions ω_m such that $\lim\limits_{m\to\infty} \|\phi - \omega_m\|_p = 0$ and make use of the uniform estimate $\|\phi * q_\rho\|_p \le A_p\|\phi\|_p$ (see [Ne]). Then

$$\|\phi * q_\rho\|_p \le \|(\phi - \omega_m) * q_\rho\|_p + \|\omega_m * q_\rho\|_p \le A_p\|\phi - \omega_m\|_p + \rho^{-1/p'}\|\omega_m\|_1\|1\|_p$$

and the desired assertion follows.

The relation $\overset{a.e.}{\lim\limits_{\rho\to\infty}} \phi * k_\rho = 0$ have been actually checked when proving Theorem 2.1.

□

2.3. The case of arbitrary measures.

In the general case according to (1.1) we put $\mu = \mu_+ + \mu_-$ where $\hat{\mu}_+$ is even and $\hat{\mu}_-$ is odd. Using the equality

$$I_{\varepsilon,\rho}(\mu,\phi) = I_{\varepsilon,\rho}(\mu_+,\phi) + I_{\varepsilon,\rho}(\mu_-,\phi)$$

we can combine the results of sections 2.1 and 2.2.

Denote

$$k_\mu^+(t) := \frac{\mu((0,|t|)) + \mu((-|t|,0))}{2|t|} = \frac{\mu((-|t|,|t|))}{2|t|}, \quad k_\mu^-(t) := \frac{\mu((0,|t|)) - \mu((-|t|,0))}{2t},$$

$$\tilde{k}_\mu^+(t) := \frac{\mu((-\infty,-|t|)) + \mu((|t|,+\infty))}{2|t|}, \qquad \tilde{k}_\mu^-(t) := \frac{\mu((|t|,+\infty)) - \mu((-\infty,-|t|))}{2t}.$$

As before, $\alpha_\pm$ are the coefficients defined in (1.1) (see also (1.8), (1.9)).

Theorem 2.3. *Let*

$$\mu(\mathbb{R}) = 0, \quad \mu(\{0\}) = 0, \quad \int_{-\infty}^{\infty} |\log|x||\, d|\mu|(x) < \infty. \tag{2.20}$$

Then for any $\phi \in L^p$, $1 < p < \infty$, *the following statements hold.*

(i) $\lim\limits^{(L^p)}_{\substack{\varepsilon\to 0\\ \rho\to\infty}} I_{\varepsilon,\rho}(\mu,\phi) = \alpha_+\phi + \alpha_- H\phi.$

(ii) *If the functions* $k_\mu^\pm(t)$ *have summable decreasing majorants in the neighbourhood of the origin and* $\tilde{k}_\mu^\pm(t)$ *have summable decreasing majorants in the neighbourhood of infinity, then the limit in* (i) *can be interpreted in a.e.-sense.*

(iii) *If* $\mu((0,\infty)) = 0$ *(and hence the singular term in* (i) *disappears), then the statements* (i), (ii) *also hold for* $p = 1$.

Concluding Remarks

I. The investigation above leads to the following questions which seems to be of the theoretical interest.

(i) Is it possible to describe the set of all measures μ for which the integral α_+ exists (see (1.1)) and the equality

$$\alpha_+ := \frac{1}{2}\int_0^\infty \frac{\hat{\mu}(\eta) + \hat{\mu}(-\eta)}{\eta}\, d\eta = \int_{-\infty}^{\infty} \log\frac{1}{|x|}\, d\mu(x)$$

is true?

(ii) Under what <u>necessary and sufficient</u> conditions on μ the relations

$$\text{a)} \lim^{(L^p)}_{\substack{\varepsilon\to 0\\ \rho\to\infty}} I_{\varepsilon,\rho}(\mu,\phi) = \alpha_+\phi, \qquad \text{b)} \lim^{(L^p)}_{\substack{\varepsilon\to 0\\ \rho\to\infty}} I_{\varepsilon,\rho}(\mu,\phi) = \alpha_- H\phi$$

hold for any $\phi \in L^p$?

II. In future publications we shall study more general operators

$$I^\alpha(\mu,\phi) := \int_0^\infty \frac{\phi * \mu_t}{t^{1+\alpha}}\, dt$$

which give rise to various fractional and hypersingular integrals in one and many dimensions. In this connection see [Ru] where another approach based on A. Marchaud's ideas is developed.

III. After the paper had been submitted for publication it was proved (see [Ru1]) that the measure μ in Calderon's reproducing formula may be assumed to be absolutely continuous with the density belonging to the Hardy space H^1. This class of measures and that in the present paper do not contain each other. In [Ru1] the reader can also find alternative proofs of some statements from above and examination of an order of approximation of a function ϕ by truncated integrals of the form (1.3).

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