

## In a nutshell: The solution concept to prove the RH

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Dec 2, 2014  
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The Riemann Hypothesis states that the non-trivial zeros of the Zeta function all have real part one-half. The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self-adjoint operator. All attempts failed so far to represent the Riemann duality equation in the critical stripe as convergent Mellin transforms of an underlying self-adjoint integral operator equation. The solution concept to answer the RH is about the following:

1. *building two self-adjoint, bounded (singular integral) operators of  $(s-1)\zeta(s)$  on the critical line, which correspond to two convolution integral representation ([CaD]) with appropriate Hilbert space domains enabling proof P1 (Gaussian function based) and proof P2 (fractional part function based)*
2. *applying the special degenerated hypergeometric (Kummer) function ([GrI] 3.952, 7.612) with remarkable asymptotic of its zeros [SeA] (and, as option to model co-variant energy description of photons as  $(n+1/2)\hbar\omega$ )*

$$H[f](x) = 2 \int_0^{\infty} f(\xi) \sin(2\pi\xi x) d\xi = 2xf(x) {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\pi x^2\right) = 2x {}_1F_1\left(1; \frac{3}{2}; -\pi x^2\right)$$

to define a modified integral exponential function (with slower decrease to infinity as the integral exponential function) and a modified Gamma functions (with more appreciated properties for  $s \rightarrow 0$  and  $s \rightarrow 1$ ) by

$$E(-x) := -\int_x^{\infty} e^{-t} d \log t \quad \rightarrow \quad E^*(-x) := 2 {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x\right)$$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} x^{s/2} dE \quad \rightarrow \quad \Gamma^*\left(\frac{s}{2}\right) := \int_0^{\infty} x^{s/2} dE^* = \Gamma\left(\frac{s}{2}\right) \frac{s}{s-1} = 2 \frac{\Pi(1 + \frac{s}{2})}{s-1}$$

Let  $H$  denotes the singular integral convolution Hilbert transform operator with domain  $L_2(-\infty, \infty)$  and  $f_H := H[f]$  ([PeB]). Then for the Gaussian function  $f$  it holds in the neighborhood of  $s=1$

$$M[f_H](s) = \frac{1}{2} \pi^{-s/2} \frac{\Gamma\left(\frac{1+s}{2}\right) \Gamma\left(\frac{1-s}{2}\right)}{\Gamma(1-s)} \quad \text{with} \quad f_H(x) = 2xe^{-\pi x^2} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\pi x^2\right)$$

The functions  $f_H$  and  $f$  are identical in a weak  $L_2(-\infty, \infty) \cong l_2 \cong l_2^0$  ensuring appropriate relationship to the RH theory ([EdH], [IvA], [TiE]). A similar approach with respect to the Hilbert space  $L_2^{\#}(0,1) \cong l_2$  is valid for the fractional part function  $\varphi$  and its Hilbert transform  $\varphi_H$ :

$$\varphi(x) := 2\pi(x - [x]) = \pi - \sum_1^{\infty} \frac{2}{n} \sin 2\pi n x \quad , \quad \varphi_H(x) := -2 \log 2 \sin(\pi x) = \sum_1^{\infty} \frac{2}{n} \cos 2\pi n x$$

The special Kummer function enables the definition of an alternative Li-function with appreciated approximation properties in the form

$$Li^*(x) := 2 {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \log x\right) = \sum_0^{\infty} \frac{1}{n+1/2} \frac{\log^n x}{n!} = \int_0^1 x^t t^{-1/2} dt$$

In [BaB] a  $l_2^{\alpha}$  – Hilbert space reformulation of the Nyman-Beurling criterion is given ( $\alpha = -1$ ). The Hilbert scale factor  $\alpha = 1/2$  is related to the Dirichlet integral of harmonic function over the unit circle e.g. in the context of conformal mapping and minimal surfaces theory ([CoR] (1.4)). The Hilbert scale factor  $\alpha = -1/2$  is related to wavelet theory ([LoA]) on the unit disk (by Daubechies and Möbius wavelets). The reproducing property (Calderón's reproducing formula for the continuous wavelet transform) of Möbius wavelets is valid in a weak  $H_{-1/2}$  – sense. The zeros  $\sigma_n$  of the Kummer function  ${}_1F_1(2\pi n x)$  lie in the intervals  $(n-1/2, n)$  [SeA]. Those properties are proposed to leverage the Hardy-Littlewood circle method to prove the binary Goldbach conjecture.

## References

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