An alternative framework to answer the RH

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The Riemann Hypothesis states that the non-trivial zeros of the Zeta function all have real part one-half. The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self-adjoint operator ([BeM]). All attempts failed so far to represent the Riemann duality equation in the critical stripe as convergent Mellin transforms of an underlying self-adjoint integral operator equation. The solution concept to answer the RH is about the following:

1. *to replace the current RH theory framework (Banach space, Stieltjes integrals, Fourier series analysis) by an alternative one, which is about Hilbert scale, Lebesgue integrals, generalized Fourier transformation analysis and variational theory. This enables convergent Mellin transform integrals in the critical stripe in a weak sense building on appropriately defined Pseudo-Differential operators with corresponding Hilbert space domains. The new framework can be applied to basically all known RH criteria. This paper is concerned with spectral analysis based solutions (P1, P2), P3+ (Riemann function convergence behavior) options and P4+ (Polya criterion ([PoG]) options*
2. *P1 and P2 provide two self-adjoint, bounded (singular integral) operators based on the Hilbert transforms of the Gaussian function (P1) and the fractional part function (P2), building the Riemann duality equation*
3. *P1 (and P4+ option) also apply the special degenerated hypergeometric (Kummer) function*

$$ {}_1F_1(\frac{1}{2};\frac{3}{2},-x)=\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{(-x)^n}{n!}=\frac{1}{2}\int_0^1 t^{-1/2}e^{-xt}dt=\frac{1}{\sqrt{x}}\int_0^{\sqrt{x}}e^{-t^2}dt $$

to define a modified integral exponential and a modified Gamma function (in the critical stripe) by

$$ E(-x):=-\int_x^{\infty}e^{-t}d\log t \qquad \to \qquad E^*(-x):=2{}_1F_1'(\frac{1}{2};\frac{3}{2},-x) $$

$$ \Gamma(\frac{s}{2})=\int_0^{\infty}x^{s/2}dE \qquad \to \qquad \Gamma^*(\frac{s}{2}):=\int_0^{\infty}x^{s/2}dE^*=\Gamma(\frac{s}{2})\frac{s}{s-1}. $$

For the Hilbert transform f_H of the Gaussian function f it holds ([GrI] 3.952, 7.612)

$$ M[f_H](s)=\frac{1}{2}\pi^{-s/2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-s)} \quad \text{with} \quad f_H(x)=2x{}_1F_1(1;\frac{3}{2},-\pi x^2). $$

The functions f_H and f are identical in a weak $L_2(-\infty,\infty)$ ensuring appropriate relationship to the RH theory ([EdH]). The same correlation is valid for the fractional part function and its Hilbert transform

$$ \varphi(x):=2\pi(x-[x])=\pi-\sum_1^{\infty}\frac{2}{n}\sin 2\pi nx \ , \ \varphi_H(x):=-2\log 2\sin(\pi x)=\sum_1^{\infty}\frac{2}{n}\cos 2\pi nx $$

with respect to the Hilbert space $L_2^{\#}(0,1)$ ([TiE]). The Kummer function enables appropriate integral operator representations of $(s-1)\zeta(s)$ on the critical line and an alternative Li-function ((EdH) 1.15) with appreciated approximation properties in the form

$$ Li^*(x):=2{}_1F_1(\frac{1}{2};\frac{3}{2},\log x)=\sum_0^{\infty}\frac{1}{n+1/2}\frac{\log^n x}{n!}=\int_0^1 x^t t^{-1/2}dt \ . $$

In [BrK1] some conceptual ideas are presented to use the alternative framework to leverage the Hardy-Littlewood circle method. This methods seems to be not sufficient to prove the Goldbach conjecture, as it requires estimates on purely trigonometric series without any additional information about the concerned prime number distribution.

Proof 1

The Poisson summation formula applied to the Gauss-Weierstrass density function

$$f(x) := e^{-\pi x^2}$$

enables the proof of the Jacobi's $\vartheta -$relation $xG(x) = G(1/x)$ for the Theta function

$$G(x) := 1 + \psi(x^2) := \sum_{-\infty}^{\infty} f(nx) \ .$$

Let M denote the Mellin transform then it holds

$$M[f](s) = \frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \ .$$

The natural statement of the functional equation is the symmetric duality equation ([EdH], 1.6)

$$(*) \qquad \zeta(s)M[f](s) = \zeta(1-s)M[f](1-s) \ .$$

Multiplying by $s(s-1)$ and putting

$$\xi(s) := s(s-1)\zeta(s)\frac{1}{2}\Gamma(\frac{s}{2})\pi^{-s/2}$$

leads to the Riemann duality equation in the form

$$\xi(s) = \zeta(s)(s-1)M[xf'(x)](s) = \xi(1-s) \quad , \quad s \in C \ .$$

Due to the corresponding property of the Hilbert transform the functions f, f_H are identical in a weak $L_2(-\infty,\infty) -$sense, i.e. it holds

i) $\|f\|_0^2 = \|f_H\|_0^2 = \frac{1}{\sqrt{2}}$

ii) $(f, \chi) = (f_H, \chi) \quad \forall \chi \in L_2(-\infty,\infty)$

iii) $(f, f_H) = 0 \ , \ [xH - Hx]f_H = 0$.

From this it follows the identity

$$F_s := M[f](s) = M[f_H](s) =: F_s^H$$

in the sense of distributional complex-valued functions ([PeB] I.15).

Remark: It holds ([TiE] 2.14, [BeB] 6, example 1)

$$\zeta(s) - \frac{1}{s-1} = 1 - \frac{1}{s-1}\left[\sum_{2}^{\infty} \frac{1}{(n-1)^{s-1}} - \frac{1}{n^{s-1}} - \frac{s-1}{n^s}\right] \underset{s \to 1^+}{\rightarrow} \gamma \ .$$

There is an only formally valid representation of Riemann's duality equation as transform of an integral operator in the form ([EdH] 10.3):

$$(**)\quad \frac{\xi(s)}{s-1}=\frac{s}{2}\left[\int_0^1 x^{1-s}G(x)\frac{dx}{x}+\int_1^\infty x^{1-s}G(x)\frac{dx}{x}\right].$$

This operator has no transform at all, as the integral

$$\int_0^1 x^{1-s}\frac{dx}{x}+\int_1^\infty x^{1-s}\frac{dx}{x}= formally, only(!) = \frac{1}{1-s}-\frac{1}{1-s}=0$$

does not converge for any s. The integral (**) would converge if the constant term

$$f(0)=\hat{f}(0)=1$$

is absent. In order to overcome this issue to build series representation of $\xi(s)$ as an even function of $s-1/2$, Riemann introduced the auxiliary function ([EdH] 10.3)

$$H(x):=\frac{d}{dx}\left[x^2\frac{d}{dx}G(x)\right]$$

which fulfills

$$2\xi(s)=\int_0^\infty x^{1-s}H(x)\frac{dx}{x}.$$

The Hilbert transform "*spirits away*" the jeopardizing constant Fourier series term ([PeB] Example 9.11)], i.e. the related Theta function G_H has a vanishing constant term.

The Hilbert transform of the Gaussian function $f(x)$ is given by ([Grl] 3.952, 7.612, 9.21-9.23)

$$f_H(x)=2\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi)=2xf(x)_1F_1(1;\frac{3}{2},-\pi x^2)$$

and

$$f'_H(\frac{x}{2\sqrt{\pi}})=2\int_0^\infty 2\pi\xi f(\xi)\cos(\sqrt{\pi}\xi x)d\xi).$$

With respect to

$$e^{-x}F(-x)=e^{-x}\sum_{n=0}^\infty\frac{2}{2n+1}\frac{x^n}{n!}$$

we refer to the following auxiliary function of ([BeB] III, Entry 10):

$$\varphi_\infty(x):=e^{-x}\sum_{n=0}^\infty\frac{\varphi(n)}{n}\frac{x^n}{n!}$$

which fulfills the Borel summable condition. It holds

$$M[f_H](s)==\frac{1}{2}\pi^{-s/2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-s)}.$$

The key idea is to replace

$$\int_0^\infty x^s f(x)\frac{dx}{x} = \pi^{-s/2}\frac{1}{s}\Gamma(\frac{s}{2}) \to \int_0^\infty x^s f_H(x)\frac{dx}{x}$$

and

$$G(x) \to G_H(x)$$

to build a modified functional equation, being valid in a weak sense, but with same set of Zeta function zeros.

With density arguments in combination with the Hardy theorem ([EdH] 11.1: *the number of zeros of the Zeta function on the critical line is infinite*) the weak functional equation is then also valid in a strong sense.

The prize to be paid is a RH analysis in a weak $L_2(-\infty,\infty)-$Hilbert space framework and (weak) variational representations of affected functions and dual integral operator equations. As this is anyway the natural framework to deal with singular integral operators the prize seems to be more than adequate.

Let

$$E(x) := e^{-x} \quad , \quad \varphi(n) := \frac{2n}{2n+1}$$

$$\Phi(x) := xF'(x) := 2x\,{}_1F_1'(\frac{1}{2};\frac{3}{2},-x) \quad .$$

Lemma: It holds

i) $E(x) \approx \Phi(x) = \sum_0^\infty \varphi(n)\frac{(-x)^n}{n!} \quad , \quad \operatorname{Re}s_{s=-n}[\Gamma(s)] = \frac{(-1)^n}{n!}$

ii) $E(x)$ decreases faster than $\Phi(x)$ for all $x > 0$

iii) $Ei(-x) := -\int_x^\infty e^{-t}\frac{dt}{t} \approx F(x) = -\int_x^\infty dF$

iv) $Li(x) = Ei(\log x) \approx Li^*(x) := F(\log x) = \sum_0^\infty \frac{\varphi(n)}{n}\frac{\log^n x}{n!}$.

Lemma: ([GrI] 7.612) It holds

i) $M[E](s) = \Gamma(s)$

ii) $M[\Phi](s) = \Gamma(s)\frac{2s}{2s-1} = \Gamma(s)\varphi(-s)$ for $0 < \operatorname{Re}(s) < \frac{1}{2}$,

which corresponds to the Ramanujan Master Theorem representation ([BeB] IV, Entry 11) resp.

$$\int_0^\infty x^{s/2} d\,{}_1F_1(\frac{1}{2};\frac{3}{2},-x) = \frac{\Gamma(1+\frac{s}{2})}{s-1} = \frac{\Pi(\frac{s}{2})}{s-1} \quad \text{for} \quad 0 < \operatorname{Re}(s) < 1 \quad ,$$

which corresponds to the Gauss notation of the Gamma function ([EdH] 1.3).

Remark: From [SeA] we note that all zeros z_n of

$$ {}_1F_1(\frac{1}{2};\frac{3}{2},z)=\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{z^n}{n!}=\frac{1}{2}\int_0^1 e^{zt}t^{-1/2}dt=z^{-3/4}e^{z/2}M_{\frac{1}{4},\frac{1}{4}}(z) $$

lie in the half-plane $\mathrm{Re}(z)>1/2$ and in the horizontal stripe $(2n-1)\pi<\left|\mathrm{Im}(z)\right|<2n\pi$. It holds

$$ \int_0^{\infty}x^{s/2}{}_1F_1(\frac{1}{2};\frac{3}{2},-x)\frac{dx}{x}=\frac{\Gamma(\frac{s}{2})}{s-1} \quad . $$

Remark: With respect to [PoG1] we note that

$$ zf(z)_1F_1(\frac{1}{2};\frac{3}{2},\pi z^2)=cze^{-\pi z^2}\prod_1^{\infty}(1-\frac{z}{\alpha_n})e^{\frac{z}{\alpha_n}} $$

is a function of the Laguerre-Polya class, i.e. a function of increased genus 1 (see also [CsG]).

Remark: The theorem of Erdös-Kac ([ErP]) concerning the Gaussian law of errors in the theory of additive number theoretic functions gave a first linkage between probability theory and additive number theory. In [KaM] the concepts of trigonometric gap series from Rademacher, Banach and Kolmogoroff are put into relationship to number theory and central limit theorem of probability theory.

The relationship to the Kummer function and the Hilbert transform of the Gaussian function is given by ([GrI] 3.761, 9.236, ([AbM] 7.16)

$$ \frac{1}{\sqrt{\pi}}f_H(\frac{x}{\sqrt{\pi}})=\frac{2x}{\sqrt{\pi}}e^{-x^2}{}_1F_1(\frac{1}{2};\frac{3}{2},x^2) $$

$$ 2{}_1F_1(\frac{1}{2};\frac{3}{2},x)=\sum_{n=0}^{\infty}\frac{2}{2n+1}\frac{x^n}{n!}=\int_0^1(\cosh(xt)+\sinh(xt))\frac{dt}{\sqrt{t}} \qquad \text{for } x>0 \; . $$

Remark: With respect to additive number theory problems and an alternative density definition we also refer to Schnirelmann density ([PeB]).

Remark (ball symmetric potential of linear oscillator): for the energy $E=\frac{1}{2}\hbar\omega$ the radial function $f(r)$ of the corresponding eigenfunction of the Schrödinger equation is given by ([FlS] p 99)

$$ f(r)=e^{\lambda\frac{r^2}{2}}\left[c_1\cdot{}_1F_1(\frac{1}{2};\frac{3}{2},\lambda r^2)+c_2\frac{1}{r}{}_1F_1(\frac{1}{2};\frac{3}{2},\lambda r^2)\right] \quad . $$

Remark (Yukawa potential): The $Ei(-x)$ – function is also related to the Yukawa potential of a point charge in the form ([DuR])

$$e^{-\mu r}/r \quad , \quad \mu > 0$$

in order to define a nuclear force potential which decays rapidly at infinity. Yukawa assumed that μ^{-1} was of the order of magnitude of a nuclear radius. It results that the potential u of a charge distribution satisfies the Yukawa equation

$$\Delta u = \mu^2 u$$

at points of free space. Thus the Yukawa potentials are invariant under the group of rotations and translations of space, like those of Newton potentials. As μ approaches zero the Yukawa potentials approach those of Newton. The function $e^{-\mu r}/r$ is a member of the Bessel family of functions. Bessel functions have certain advantages over Newtonian potentials in functional analysis (([DoW]).

Analog to the above we propose a modified Yukawa potential dY^* by the substitution

$$e^{-\mu r}/r \to \mu F'(\mu r) \ .$$

Proof 2

The same idea of proof 1 can also be applied to the fractional part function in an alternative Hilbert space framework ([KBr]):

Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β Fourier coefficients and norms are defined by

$$u_\nu := \frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx \quad \|u\|_\beta^2 := \sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 \ .$$

Then the Fourier coefficients of the convolution operator

$$(Au)(x) := -\oint \log 2\sin\frac{x-y}{2}u(y)dy =: \oint k(x-y)u(y)dy \quad \text{and} \quad D(A) \subseteq H_A = H_{-1/2}(\Gamma) \ .$$

are given by

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|}u_\nu \ .$$

The operator A enables characterization of the Hilbert spaces $H_{-1/2}$ and H_{-1} in the form

$$H_{-1/2} = \left\{\psi \middle| \|\psi\|_{-1/2}^2 = (A\psi,\psi)_0 < \infty\right\},\ H_{-1} = \left\{\psi \middle| \|\psi\|_{-1}^2 = (A\psi,A\psi)_0 < \infty\right\} \ .$$

From ([BeB] 8, Entry 17 (iv)) we quote

> "*Ramanujan informs us to note that*
>
> $$\cot(\pi x) = 2\sum_1^{\infty}\sin(2\pi\nu x) \ ,$$
>
> *which also is devoid of meaning, ... may be formally established by differentiating the equality*
>
> $$2\sum_1^{\infty}\frac{\cos 2\pi\nu x}{\nu} = -2\log 2\sin(\pi x) \ .$$

With respect to the Dirac function we note that building on the Dirichlet kernel there is a formal representation of $\delta(x)$ in the distribution sense in the form

$$\delta(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{inx} = \frac{1}{2\pi}\int_0^{2\pi}e^{ikx}dk = \frac{1}{\pi}\int_0^{\pi}\cos(kx)dk = \frac{1}{2}\operatorname{sgn}'(x) \in H_{-1/2-\varepsilon}(-\pi,\pi) \subset H_{-1}(-\pi,\pi) \ .$$

For

$$\varphi(x) := 2\pi(x - [x]) = \pi - \sum_1^\infty \frac{2}{n} \sin 2\pi nx$$

$$\varphi_H(x) := -2\log 2\sin(\pi x) = \sum_1^\infty \frac{2}{n} \cos 2\pi nx$$

$$\sigma(x) = \frac{1}{2}\cot(\pi\circ) = \sum_1^\infty \sin(2\pi n\circ)$$

it holds (see also [ZyA] XIII, (11-3))

$$(A\sigma)(x) = \sum_{-\infty}^\infty \frac{1}{2|n|}\sigma_n(x) = \sum_1^\infty \frac{\sin(2\pi nx)}{n} \in H_0^\# \; .$$

From literature (e.g. [GaD] pp.63, [GrI] 1.441) we recall

$$\frac{1}{2\pi}\oint_{0\to 2\pi} e^{in\varphi} \ln \frac{1}{2\sin\frac{\varphi-\vartheta}{2}} d\vartheta = \begin{cases} -\frac{1}{2n}e^{in\varphi} & n = 1,2,3,... \\ 0 & n = 0 \\ \frac{1}{2n}e^{in\varphi} & n = -1,-2,.. \end{cases}$$

$$\frac{1}{\pi}\oint_{0\to 2\pi} e^{in\varphi} \frac{1}{2}\cot\frac{\varphi-\vartheta}{2} d\vartheta = \begin{cases} -ie^{in\varphi} & n = 1,2,3,... \\ 0 & n = 0 \\ ie^{in\varphi} & n = -1,-2,.. \end{cases}$$

$$\frac{1}{\pi}\oint_{0\to 2\pi} e^{in\varphi} \frac{1}{4\sin^2\frac{\varphi-\vartheta}{2}} d\vartheta = \begin{cases} -ne^{in\varphi} & n = 1,2,3,... \\ 0 & n = 0 \\ ne^{in\varphi} & n = -1,-2,.. \end{cases} \; .$$

Due to the corresponding property of the Hilbert transform the functions φ, φ_H are identical in a weak $L_2^\#(0,1)-$sense, i.e. it holds

i) $\|\varphi\|_0^2 = \|\varphi_H\|_0^2$

ii) $(\varphi, \chi) = (\varphi_H, \chi) \quad \forall \chi \in L_2^\#(0,1)$

iii) $(\varphi'_H, \chi)_{-1/2} = (\sigma, \chi)_{-1/2} \quad \forall \chi \in L_2^\#(0,1)$

because of $(\varphi'_H, \chi)_{-1/2} = (A\varphi'_H, \chi)_0 = (\varphi_H, \chi)_0 = (A\sigma, \chi)_0 = (\sigma, \chi)_{-1/2}$.

In the same way as for the P1 Hilbert space framework the prize to be paid is a RH analysis in a weak $H_{-1/2}^\#(0,1)-$Hilbert space framework ([PeB] I, §15, [BeJ]) and (weak) variational representations of affected functions and dual integral operator equations (which is anyway the natural framework to deal with singular integral operators and spectral theory):

In a weak $H_{-1/2}^\#(0,1)-$ sense it holds

$$\zeta(s)\Gamma(s)\cos(\frac{\pi}{2}s) = M[-\log 2\sin(\pi\circ)](s) = \zeta(s)M[\cos](s) \qquad \text{for} \quad 0 < \mathrm{Re}(s) < 1 \; .$$

The relationship to the Riemann duality equation is given by ([TiE] 2.1)

$$M[\cos](s) =: \pi(\frac{2}{\pi})^{1-s}\chi(1-s)$$

with

$$\chi(1-s)\chi(s)=1$$

and

$$\zeta(s)=\chi(s)\zeta(1-s)\,.$$

The distributional P2 Hilbert scale framework is proposed to be applied to other areas, as well. Based on the P2 Hilbert space framework in [Br1] an alternative framework for the Hardy-Littlewood circle method is given to analyze additive prime number problems.

Remark: There is a still open characterization question raised by Riemann ([RiB] p. 18) about the representation of a function f(x) as convergent Fourier and/or trigonometric series ([ZyA] XII, 6, 7, 11):

> "*Wenn eine Function durch eine trigonometrische Reihe darstellbar ist, was folgt daraus über ihren Gang, über die Aenderung ihres Werthes bei stetiger Aenderung des Arguments*?"

Remark: The dual space of $H_{-1/2}^{*}=H_{1/2}\subset L_2$ is isometric to the classical Hardy space $\mathbf{H_2}$ of analytical functions in the unit disc with norm

$$\|f(re^{i\varphi})\|_{H_2} := \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\varphi})|^2 d\varphi\,.$$

It holds

i) If $f\in\mathbf{H_2}$, then there exists boundary values $f(e^{i\varphi})=\lim_{r\to 1} f(re^{i\varphi})\in L_2(-\pi,\pi)$ with

.

$$\|f\|_{H_2}=\|f(e^{i\varphi})\|_{L_2}$$

ii) If $f(e^{i\varphi})=\sum_{-\infty}^{\infty}u_\nu e^{i\nu\varphi}\in H_{1/2}$, then its Dirichlet extension into the disc is given by ($z=re^{i\varphi}$):

.

$$F(z)=\sum_{-\infty}^{\infty}u_\nu r^{|\nu|}e^{i\nu\varphi}=(\sum_{1}^{\infty}u_\nu z^\nu)+(\sum_{1}^{\infty}u_{-\nu}z^{-\nu})$$

with

$$\|\nabla F\|_0^2=\sum_{-\infty}^{\infty}|\nu||u_\nu|^2=\|f\|_{1/2}^2\quad.$$

Remark: The fundamental principle of the Hardy-Littlewood method is the fact, that for N being an integer it holds

$$\int_0^1 e^{2\pi iN\alpha} d\alpha = \begin{cases} 1 & if \quad N=0 \\ 0 & otherwise \end{cases} .$$

They used the principle in the form of the following the

Lemma: Let $f(x)=\sum_0^\infty a_n x^n$ with $|x|<1$, then for $0<r<1$ it holds

$$r^N a_N = \frac{1}{2\pi}\int_0^1 f(re^{it})e^{-iN\alpha} d\alpha \cdot$$

Remark: The Voronoi summation formula is related to the Dirichlet divisor problem. The Euler function $\varphi(n)$ is defined as product of all prime divisors of n ([LaE])

$$\frac{\varphi(n)}{n} := \frac{1}{n}\sum_{\substack{m\le n \\ (m,n)=1}} 1 = \prod_{p|n}(1-\frac{1}{p})$$

The error function in the divisor problem is given by ([IvA] chapter 3)

$$\Delta(x) := \sum_{n\le\sqrt{x}}{}' d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4} = \sum_{n\le\sqrt{x}} \psi(\frac{x}{n}) + O(x^{\varepsilon})$$

with

$$\psi(x) := \frac{1}{2} - (x - [x]) = \frac{1}{2} - \varphi(\frac{x}{2\pi}) = 2\sum_{n=1} \frac{\sin(nx)}{n}$$

where by $\sum_{a\le n\le b}{}'$ means that if a or b is an integer than $d(a)/2$ or $d(b)/2$.

In [NaC] a symmetric and simplified formula of the Voronoi summation formula is given. It requires that the Fourier cosine transforms are elements of a certain subspace of $L_2(0,\infty)$

$$G_1^2(0,\infty) := \{h | xh'(x) \in L_2(0,\infty)\} \subset L_2(0,\infty) .$$

The definition of $G_1^2(0,\infty)$ with $xh' \in L_2$ is related to the framework of P1. An analogue definition in the Hilbert space framework of P2 leads to the requirement $xh' \in H_{-1/2}$. In this context we note that

$$M[xh'](s) = -sM[h](s) = \int_0^\infty x^s dh .$$

Remark: With respect to the Kummer function (P 1) we recall from [LoA] 1.1, [SeA]:

1. The zeros σ_n of the Kummer function ${}_1F_1(2\pi ix)$ lie in the intervals $(n-1/2,n)$

2. If ${}_1F_1 \in L_1(R)$ continuous and differentiable and $\psi := {}_1F_1' \in L_2(R)$, then ψ is a wavelet.

For the expansion of Kummer functions in terms of Laguerre polynomials and Fourier transforms we refer to [PiA]. Putting ([AbM] 7.1.6)

$$K(x) := {}_1F_1(\frac{1}{2},\frac{3}{2};-\pi x^2) = \frac{1}{x}\int_0^x e^{-\pi t^2}dt = e^{-\pi x^2}\sum_0^\infty \frac{(2\pi)^n x^{2n}}{1\cdot 3\cdot\cdot(2n+1)} \quad \text{and} \quad \tilde{G}(x) := \int_{-\infty}^x e^{-\pi t^2}dt$$

the link between the polynomial function z^n and the Hermite polynomials $H_n(z)$ is given by ([CaD]):

$$H_n(x) = \int_{-\infty}^{\infty}(z-i\frac{x}{2})^n d\tilde{G}(x) \ .$$

Alternatively we propose a corresponding polynomial system defined by

$$K_n(x) = \int_{-\infty}^{\infty}(z-i\frac{x}{2})^n dK(x) \ .$$

Remark: The Kummer function is related to the Fresnel integrals ([AbM] 7.3)

$$C(z) := \int_0^z \cos(\frac{\pi}{2}t^2)dt = \sum_0^\infty \frac{(-1)^n(\pi/2)^{2n}}{(2n)!(4n+1)}z^{4n+1} = -C(-z) \underset{z\to\infty}{\to} \frac{1}{2}$$
$$S(z) := \int_0^z \sin(\frac{\pi}{2}t^2)dt = \sum_0^\infty \frac{(-1)^n(\pi/2)^{2n+1}}{(2n+1)!(4n+3)}z^{4n+3} = -S(-z) \underset{z\to\infty}{\to} \frac{1}{2}$$

by

$$C(z) + iS(z) = x\,{}_1F_1(\frac{1}{2},\frac{3}{2};i\frac{\pi}{2}x^2) \ .$$

Remark: In [NaS] a relationship between the Hilbert space $H_{1/2}$, the Teichmüller theory and the universal period mapping via quantum calculus is given. The Teichmüller spaces are also related to Riemann surfaces and the geometrization of 3-manifolds.

Remark: A function, which is in a sense a generalization of $\zeta(s)$ is the Hurwitz Zeta function, defined by ([TiE] 2.17:

$$\cdot\ \zeta(s,a) := \sum_{n=0}^\infty \frac{1}{(n+a)^s} \quad \text{for} \quad \mathrm{Re}(s) > 1 \ , \ 0 < a \le 1 \ .$$

For $a=1$ resp. $a=1/2$ this reduces to

$$\zeta(s) \ , \quad (2^s-1)\zeta(s) \ .$$

There are also other generalized Zeta function, e.g. Lerch or Epstein or Dedekind Zeta function, as well as Zeta functions associated with cusp forms ([IvA] 11.8). Related to the Hurwitz Zeta function is the Dirichlet series ([IvA] 1.8), defined by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1-\chi(p)p^{-s})^{-1} \quad \text{for} \quad \mathrm{Re}(s) > 1$$

where for a fixed $q \geq 0$ $\chi(n)$ is the arithmetical function known as a character modulo q (for $q = 1, L(s,\chi) = \zeta(s)$, if $(a,q) > 1$ then $\chi((a) = 0$). For $q = 1$ it holds

$$L(s,\chi_1) = \zeta(s)\prod_p (1-p^{-s})^{-1} = \prod_p (1-p^{-s})^{-1} \prod_{p|q} (1-p^{-s})^{-1} .$$

Thus $L(s,\chi_1)$ has a first-order pole at $s = 1$ just like $\zeta(s)$ and it behaves similar to $\zeta(s)$ in many other ways, while $L(s,\chi)$ for $\chi \neq \chi_1$ is regular for $\mathrm{Re}(s) > 0$.

The Generalized Riemann Hypothesis (GRH) states that all non-trivial zeros of all Dirichlet L-functions have real part equal to ½.

Remark: Let $f, g : N \to C$ be multiplicative number theoretic functions with a related number theoretic product defined by

$$f * g := \sum_{d|n} f(d) g(\frac{n}{d})$$

and let

$$F(s) := \sum \frac{f(n)}{n^s} \quad , \quad G(s) := \sum \frac{g(n)}{n^s}$$

Dirichlet series, which are absolute convergent for $\sigma > \sigma_0$. Then

$$F(s)G(s) = \sum \frac{h(n)}{n^s}$$

is absolute convergent for $\sigma > \sigma_0$ with

$$h(n) := \sum_{d|n} f(d) g(\frac{n}{d}) \quad .$$

In other words: the number theoretic product corresponds to the product of the related Dirichlet series. The corresponding most famous examples with respect to the Zeta function are built on the formulas

i) $\mu * \underline{1} = \varepsilon$ with $\varepsilon(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$

ii) $\Lambda * \underline{1} = \log$ resp. $\Lambda = \log * \mu$

leading to the Dirichlet series representations

i) $\zeta(s) \sum \frac{\mu(n)}{n^s} = 1$

ii) $\sum \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$.

Recalling the distributional functions from the above

$$\sigma(x) = \frac{1}{2}\cot(\pi x) = \sum_1^\infty \sin(2\pi\nu x) \in H^{\#}_{-1/2} \quad , \quad (A\sigma)(x) = \sum_1^\infty \frac{\sin(2\pi n x)}{n} \in H^{\#}_0$$
$$\varphi_H(x) := -2\log 2\sin(\pi x) = \sum_1^\infty \frac{2}{n}\cos 2\pi n x \in H^{\#}_{-1/2}$$

the link of the “convolution” multiplicative product, the Zeta function for $s = 1$ (i.e. $\zeta(1) = \infty$) and the distributional Hilbert scale framework is given by the

Lemma: In a weak $H^{\#}_{-1/2}$ – sense (as counterpart to the definition of the Dirac function in a $L^{\#}_2$ – sense) it holds

$$\zeta(1) = (\varphi_H, \sigma)_{-1/2} = (\varphi_H, A\sigma)_0 = (\varphi_H, \varphi'_H)_{-1/2} = \frac{1}{2}\left\| \frac{d}{dx}(\varphi_H^2) \right\|_{-1/2} \quad , \quad \Big(\sum_1 \frac{1}{n} \cong \sum_1 \frac{1}{n}\int_0^1 2\cos(2\pi n x)\sin(2\pi n x)dx\Big).$$

Proof 3+ proposals based on alternative Li-function

We recall from ([BeM]), [EdH]) the following: the density of the primes is the distribution

$$\pi'(x) = \sum_{p} \delta(x-p) \quad .$$

A consequence of the prime number theorem is (e.g. [ViJ])

$$\pi(x) \approx \frac{x}{\log x} \quad .$$

Riemann's exact formula for $\pi(x)$ relates to the non-trivial zeros $z_n = \frac{1}{2} + it_n$ of the zeta function. With the notations

$$M(x) := \frac{1}{(x^2-1)} \frac{1}{x}$$

$$N(x) := 2e^{-x/2} \sum_{\mathrm{Re}(t_n)>0} e^{-\mathrm{Im}(t_n)x} \cos\{\mathrm{Re}(t_n)x\}$$

$$J'(x) := \frac{1}{\log x} - \frac{1}{\log x}\left[M(x) + N(\log x)\right]$$

$$\Psi(x) := xJ'(x), \quad \Psi_k(x) := \Psi(x^{1/k}) \quad .$$

the Riemann formula is given by

$$x\pi'(x) = \sum_{k=1}^{\infty} \frac{\mu_k}{k^2} \Psi_k(x)$$

whereby μ_k denote the Möbius numbers.

Each term in the sum of $N(x)$ describes an oscillatory contribution to the fluctuations of the density of primes, with larger $\mathrm{Re}(t_n)$, corresponding to higher "frequencies". As $E(x)$ decreases faster than $\Phi(x)$ a slower decrease of a infinite series of oscillations $N^*(x)$ enables a faster convergence of corresponding $M^*(x)$, which supports the RH convergence criterion.

If the RH is true, then $\mathrm{Im}(t_n) = 0$ for all $n \in N$. Then the support of the Fourier transform of $N(x)$ is discrete, i.e. $N(x)$ has a discrete spectrum, which are the frequencies of a sophisticated vibration system. Mathematically, these are discrete eigenvalues of a self-adjoint (hermitian) operator.

Oscillatory integrals are the main subject of Pseudo Differential Operators [PeB]. Singular integral operators are Pseudo Differential Operators of negative order with corresponding domain. We claim that the music of the primes is related to the eigenvalues of such a singular integral operator. Finite element methods provide the appropriate approximation tools to calculate approximation eigenvalues of corresponding singular integral equations [BrK].

With respect to the Mellin and the Hilbert transforms of the Gaussian and Kummer function

$$G(x) := e^{-\pi x^2} \quad , \quad K(x) := {}_1F_1(\frac{1}{2};\frac{3}{2},-\pi x^2)$$

we recall from P1:

Lemma: It holds

i.) $G_H(x) = 4\pi x G(x) K(ix)$

ii.) $K(x) = \frac{1}{x}\int_0^x G(t)dt$ resp. $dK = \frac{G(x)-K(x)}{x}dx$

iii.) $\int_0^\infty x^{s/2} dK = \frac{\Gamma(1+s/2)}{s-1}$ in the critical stripe

iv.) $M[G_H](s) = \pi^{-(1-s)/2}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s) = 2\pi^{1/2}\cot(\frac{\pi}{2}(1-s))M[G](s)$.

The alternative entire Zeta function representation in the form

$$\xi_*(s) :== \pi^{-(1-s)/2}\Gamma(\frac{s}{2})\cot(\frac{\pi}{2}(1-s))\zeta(s) = \xi_*(0)\cdot\prod_\rho(1-\frac{s}{\rho})$$

leads to a modified representation of the Riemann density function, e.g. with the principle term and the error function term in the form ([EdH] 1.14, 1.16)

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(\cot\frac{\pi}{2}(1-s))}{s}\right]x^s ds \quad , \quad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(\Gamma\frac{s}{2})}{s}\right]x^s ds \ .$$

As in the neighborhood of $x \approx 1$ it holds

$$\frac{\pi}{2}\tan\frac{\pi}{2}x = \frac{1}{1-x} + O(|1-x|)$$

this indicates an alternative prime number density function in the form

$$\log\frac{1}{x} \quad \rightarrow \quad \log\frac{\pi}{2}(\tan\frac{\pi}{2}(1-x)) = \log\frac{\pi}{2}\cot(\frac{\pi}{2}x) \quad ,$$

with an obvious fit into the proposed (distributional) Hilbert space framework of P2.

Proofs 4+ proposals based on P1/P2 Hilbert space frameworks

The degenerated hypergeometric function allows the definition of an alternative series representation of the Zeta function, alternatively with respect to Riemann's Zeta function and to Polya's Zeta fake function ([EdH] 1.8, 12.5):

$$\xi(z) = 2\int_0^\infty \Phi(u)\cos((zu)du \ .$$

It enables the application of corresponding Polya criterion ([PoG]) for other proofs of the RH.

In the context of the odd (!) function

$$xe^{-\pi x^2}{}_1F_1(\frac{1}{2};\frac{3}{2},\pi x^2)$$

we refer to the approach in [CaD] (based on [PoG1]) in the framework of Laguerre-Polya class functions with special functions of genus >1 and its Weierstrass factorization form [CsG]). The results of [SeA] provides the corresponding nonzero real numbers condition that

$$\sum_1^\infty \frac{1}{\alpha_\nu^2} < \infty \ .$$

The Hilbert transforms of P1 and P2 define appropriate kernels of integral operators with corresponding domains and ranges defined in appropriate Hilbert spaces. The corresponding orthogonal systems of the related Hilbert spaces enable the definition of alternative Fourier integral representations of the Zeta function itself. For the integral operators of P1 and P2 the following polynomial systems are proposed:

P1: The Hilbert transformed Hermite polynomials

P2: The Lommel polynomials ([DDi], [WaG] 9-7).

With respect to the Bessel function and in relationship to

$$\int_0^\infty x^{s/2}{}_1F_1(\frac{1}{2};\frac{3}{2},-x)\frac{dx}{x} = \frac{\Gamma(\frac{s}{2})}{s-1}$$

and to Fourier transforms of positive definite kernels and the Riemann Zeta function ([PoG1]) we note ([WaG] 15-53)

$$\frac{\Gamma(s)}{s-1} = \frac{\pi}{2}\int_0^\infty x^{s-1}e^{-x}\left[x(J_0^2(x)+Y_0^2(x))\right]d\arctan\frac{Y_0(x)}{J_0(x)} \ .$$

References

[AbM] Abramowitz M., Stegen I. A., Handbook of Mathematical Functions, Dover Publications, Inc., New York, 1965

[BaB] B. Bagchi, On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann Hypothesis, Indian Statistical Institute, Bangalore Centre, (2005)

[BeB] Berndt B. C., Andrews G. E., Ramanujan´s Notebooks Part I, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1985

[BeM] Berry M. V., Keating J. P., $H = xp$ and the Riemann zeros, in Super symmetry and Trace Formulae: Chaos and Disorder, Ed. I.V. Lerner, J.P. Keating, D.E. Khmelnitski, Kluver, New York (1999) pp. 355–367

[BeJ] Bertrand J., Bertrand P., Ovarlez J.-P., The Mellin Transform, in Transforms and Applications Handbook, ed. Alexander Poularikas, CRC Press, Boca Raton, Florida, 1996

[BrK] Braun K., Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations, Jap. Journal of Applied Mathematics, 3, 1, 59-72, (1986)

[BrK1] Braun K., A distributional way to prove the Goldbach conjecture leveraging the circle method, www.riemann-hypothesis.de

[CaD] Cardon D. A., Convolution operators and zeros of entire functions, Proc. Amer. Math. Soc., 130, 6 (2002) 1725-1734

[CsG] Csordas G., Fourier Transforms of Positive Definite Kernels and the Riemann Zeta-Function

[DDi] D. Dickinson, On Lommel and Bessel polynomials, Proc. Amer. Math. Soc. 5, 946-956 (1954)

[DeJ] Derbyshine J., Prime Obsession, Joseph Henry Press, Washington D.C., 2003

[DoW] Donoghue W. F., Distributions and Fourier Transforms, Academic Press, New York, 1969

[DuR] Duffin R. J., Yukawan potential theory, 1970, Carnegie Mellon University, Department of mathematical sciences, Paper 161, http://repository.cmu.edu/math

[EdH] Edwards H. M., Riemann´s Zeta Function, Dover Publications, Inc., Mineola, New York, 1974

[ErP] Erdös P., Kac M., The Gaussian Law of Errors in the Theory of Additive Number Theoretic Functions, Amer. Journal of Math. 62 (1940) 738-742

[EsT] Estermann T., On Goldbach's problem: Proof that almost all even positive integers are sums of two primes, Proc. Lond. Soc. 44, 307-314 (1938)

[FlS] Flügge S., Rechenmethoden der Quantentheorie, Springer-Verlag, Berlin, Heidelberg, New York,1976

[GaD] Gaier D., Vorlesungen über Approximation im Komplexen, Birkhäuser Verlag, Basel, Boston Stuttgart, 1980

[GrI] Gradshteyn I. S., Ryzhik I. M., Table of Integrals Series and Products, Fourth Edition, Academic Press, New York, San Francisco, London, 1965

[HaG] Hardy G. H., Littlewood J. E., Some problems of "Partitio Numerorum"; III, On the expression of a number as a sum of primes, Acta. Math., 44 (1923) 1-70

[HaH] Hamburger H., Über einige Beziehungen, die mit der Funktionalgleichung der Riemannschen Zeta-Funktion äquivalent sind, Math. Ann. 85 (1922) pp. 129-140

[IvA] Ivic A., The Riemann Zeta-Function, Theory and Applications, Dover Publications, Inc., Mineola, NewYork, 1985

[KaM] Kac M., Probability Methods In Some Problems Of Analysis And Number Theory, Bull. Amer. Math. Soc. 55 (1949), 641-665

[LaE] Landau E., Ueber die zahlentheoretische Function $\varphi(n)$ und ihre Beziehung zum Goldbachschen Satz, Gött. Nachr. 1900, 177-186

[NaC] Nasim C., On the summation formula of Voronoi, Trans. Amer. Math. Soc. 163 (1972) pp. 35-45

[NaS] Nag S., Sullivan D., Teichmüller Theory and the Universal Period Mapping via Quantum Calculus and the Space on the Circle, Osaka J. Math. 32 (1995) p. 1-34

[PeB] Petersen B. E., Introduction to the Fourier Transform and the Pseudo-Differential Operators, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1983

[PoG] Polya G., Über die Nullstellen gewisser ganzer Funktionen, Math. Zeit. 2 (1918), 352-383, also, Collected Papers, Vol II, 166-197

[PiA] Pichler A., Converging Series For The Riemann Zeta Function, arXix:1201.6538v1 [math.NT] 31 Jan 2012

[PoG1] Polya G., Über die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen, Mathematisk-fysiske Meddelelsev VII 17 (1927), 1-33

[RiB] Riemann, Bernhard. Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe. Göttingen: Dieterich, 1867

[PeB] Schnirelmann L., Über additive Eigenschaften von Zahlen, Math. Anal. 107 (1933)

[ScW] Schwarz W., Einführung in die Methoden und Ergebnisse der Primzahltheorie, BI, Hochschultaschenbücher, Mannheim, Wien, Zürich, 1969

[SeA] Sedletskii A. M., Asymptotics of the Zeros of Degenerate Hypergeometric Functions, Mathematical Notes, Vol. 82, No. 2 (2007) 229-237

[TiE] Titchmarsh E. C., The Theory of the Riemann Zeta-function, Clarendon Press, Oxford, 1951

[ViJ] Vindas J., Estrada R., A quick distributional way to the prime number theorem, Indag. Mathem., N.S. 20 (1) (2009) 159-165

[ViI] Vinogradov, I. M., Representation of an odd number as the sum of three primes, Dokl. Akad. Nauk SSSR 15, 291-294 (1937)

[ViI1] Vinogradov, I. M., The Method of Trigonometrical Sums in the Theory of Numbers, Dover Publications, Inc., Mineola, New York, 2004

[WaG] Watson G. N., A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, Second Edition first published 1944, reprinted 1996, 2003, 2004, 2006

[ZyA] Zygmund A., Trigonometric series, Vol. I, Cambridge University Press, 1968

Appendix

A. $$\int_0^\infty y^{\mu-1}e^{-\pi y^2}\sin(2\pi xy)dy=\pi^{\frac{1-\mu}{2}}xe^{-\pi x^2}\Gamma(\frac{\mu+1}{2})_1F_1(1-\frac{\mu}{2};\frac{3}{2};\pi x^2)$$, $\mathrm{Re}(\mu)>-1$, [GrI], 3.952, 7.

$$\int_0^\infty y^{\mu-1}e^{-\pi y^2}\cos(2\pi xy)dy=\frac{1}{2}\pi^{-\frac{\mu}{2}}\Gamma(\frac{\mu}{2})_1F_1(\frac{\mu}{2};\frac{1}{2};-\pi x^2)$$, $\mathrm{Re}(\mu)>0$, [GrI], 3.952, 8.

$M[\sin](s)=\Gamma(s)\sin(\frac{\pi}{2}s)$, $M[\cos](s)=\Gamma(s)\cos(\frac{\pi}{2}s)$ in the critical stripe ([GrI] 3.952).

,

$\Gamma(1\text{-}s)=\frac{1}{1\text{-}s}-\gamma+....$, $\zeta(s)\approx\frac{1}{s\text{-}1}+\frac{1}{2}+....$

$$\lim_{s\to1}(s-1)\zeta(s)=1 \qquad \lim_{s\to1}\left[\zeta(s)-\frac{1}{s-1}\right]=\gamma$$

B. The functions $$\rho(x)=\{x\}=x-[x]=\frac{1}{2}-\sum_1^\infty\frac{\sin 2\pi\nu x}{2\pi|\nu|}=\frac{1}{2\pi}\varphi(x)$$

$$\rho_H(x)=-\log 2\sin(\pi x)=\sum_1^\infty\frac{\cos 2\pi\nu x}{\nu}$$

are identical in a weak $L_2^{\#}(0,1)-$sense. It holds

i) $$-\frac{\zeta(s)}{s}=\int_0^\infty x^{-s}\rho(x)\frac{dx}{x}$$ for $0<\mathrm{Re}(s)<1$, [TiE] 2.1

ii) $$\zeta(s)\Gamma(s)\cos(\frac{\pi}{2}s)=\int_0^\infty x^s\rho_H(x)\frac{dx}{x}$$ for $0<\mathrm{Re}(s)<1$

iii) $$\|\varphi\|_0^2=\|\varphi_H\|_0^2=\frac{\pi^2}{3}$$ ([AmT], [GrI] (4.224).

C. [EdH] 12.5, [PoG]: *Let* $f(x)\to\int_{1/a}^a f(ux)F(u)du$ *define a real self-adjoint operator (where* $F(u)$ *is real and satisfy* $uF(u)=F(1/u)$*) which has the property that* $\sqrt{u}F(u)$ *is nondecreasing on the interval* $[1,a]$*, then its Mellin transform has all its zeros on the critical line.*

D. In a $H_{-1/2}-$sense ([PeB] I, §15) it holds for $0<\mathrm{Re}(s)<1$

$$\int_0^\infty x^s\rho_H(x)\frac{1}{x}=\sum_1^\infty\frac{1}{n}\int_0^\infty x^s\cos(nx)\frac{dx}{x}=\sum_1^\infty\frac{1}{n^s}\int_0^\infty\frac{1}{n^1}y^s\cos(y)\frac{dy}{y}=\zeta(s)\int_0^\infty y^s\cos y\frac{dy}{y}=\zeta(s)\Gamma(s)\cos(\frac{\pi}{2}s)$$.

The Riemann duality equation is given by

$$2\xi(s):=\zeta(s)s(1-s)\left[\int_0^\infty x^s(xf'(x))d\log x\right]=2\xi(1-s)$$
.

The idea for the proofs P1 and P2 are to replace

$P1$: $f(x)\to f_H(x)$, $M[f_H](s)==\frac{1}{2}\pi^{-s/2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-s)}$

$P2$: $\varphi(x)\to\varphi_H(x)$, $M[\varphi](1-s)=2\pi\frac{\zeta(s-1)}{s-1}\to M[\varphi_H](1-s)=\zeta(s)$ in the critical stripe.

Note:

1. $M[h'](s)=(1-s)M[h](s-1)$, $M[xh'](s)=-sM[h](s)$, $M[(xh)'](s)=(1-s)M[h](s)$

2. Hardy 's theorem, that the number of zeros of the Zeta function on the critical line is infinite, provides the "transfer" density argument from weak to strong solution propositions.

3. $M[\sin](s)=\Gamma(s)\sin(\frac{\pi}{2}s)$, $M[\cos](s)=\Gamma(s)\cos(\frac{\pi}{2}s)$ in the critical stripe.

4. $\|f\|_0^2=\|f_H\|_0^2=\frac{1}{\sqrt{2}}$, $\|\varphi\|_0^2=\|\varphi_H\|_0^2=\frac{\pi^2}{3}$

5 . [SeA]: All zeros z_n of the Kummer function lie in the half-plane $\mathrm{Re}(z)>1/2$ and in the horizontal stripe

$$(2n-1)\pi<|\mathrm{Im}(z)|<2n\pi\ .$$

6. $\Gamma(1-s)=\frac{1}{1-s}-\gamma+....$, $\zeta(s)\approx\frac{1}{s-1}+\frac{1}{2}+....$.

E. For the linkage of variational theory to holomorphic function in the distribution sense we recall from [BPe] chapter I §15 the

Definition: Let $z\to g_z$ be a function defined on a open subset $U\subset C$ with values in the distribution space. Then g_z is called a holomorphic in $U\subset C$ (or $g(z):=g_z$ is called holomorphic in $U\subset C$ in the distribution sense), if for each $\varphi\in C_c^\infty$ the function $z\to(g_s,\varphi)$ is holomorphic in $U\subset C$ in the usual sense.

F. In the context of proof P2 we recall from [BaB]:

Proposition Let H denote the weighted l^2 – space consisting of all sequences $a=\{a_n | n\in N\}$ of complex numbers such that

$$\sum_{n=1}^{\infty}\omega_n|a_n|^2<\infty \quad \text{with} \quad \frac{c_1}{n^2}\le\omega_n\le\frac{c_2}{n^2} \quad .$$

Let $\gamma:=\{1,1,1,1,......\}$, $\gamma_k:=\left\{\rho(\frac{n}{k}) \middle| n=1,2,3,....\right\}\in H$ for $k=1,2,3,...$

and Γ_k be the closed linear span of γ_k . Then the Nyman criterion states that the following statements are equivalent:

i) The Riemann Hypothesis is true

ii) $\gamma\in\overline{\Gamma}_k$.

G. [EdH] 1.8: The constant not vanishing Fourier term of

$$G(y):=\sum_{-\infty}^{\infty}f(ny)=1+2\sum_{1}^{\infty}f(ny)=:1+\psi(y^2)=y^{-1}G(y^{-1})$$

requires the definition of an auxiliary function ([EdH] 12.5)

$$H(y):=2yG'(y)+y^2G''(y)=4\pi\sum_{1}^{\infty}(ny)^2\left[2\pi(ny)^2-3\right]f(ny)=y^{-1}H(y^{-1}) \cdot$$

This leads to the representation

$$\xi(s)=\int_0^{\infty}2\Phi(u)\cos(su)du$$

with

$$2\Phi(\log x):=\sqrt{x}H(x) \quad ,$$

respectively

$$\xi(s)=\sum_{0}^{\infty}a_{2n}(s-\frac{1}{2})^{2n}$$

with

$$a_{2n}:=4\int_1^{\infty}x^{3/4}\frac{(\frac{1}{2}\log x)^{2n}}{(2n)!}\frac{d\left[x^{3/2}\psi'(x)\right]}{dx}\frac{dx}{x} \quad .$$

H. Putting

$$F(x):={}_1F_1(\frac{1}{2},\frac{3}{2};-x)=\sum_{1}^{\infty}\frac{1}{2n+1}\frac{(-x)^n}{n!}$$

It holds

$$-2xF'(x)\approx e^{-x}$$

and therefore for $0<\mathrm{Re}(s)<1/2$

$$\Gamma(s)=\int_0^\infty x^s e^{-x}\frac{dx}{x}\approx\Gamma^*(s):=-2\int_0^\infty x^s dF=2\frac{\Gamma(1+s)}{1-2s}=\Gamma(s)\frac{s}{\frac{1}{2}-s}$$

respectively

$$\frac{s}{(s-1)}\frac{\Gamma(s)}{\Gamma(1-s)}=\frac{\Gamma^*(s)}{\Gamma^*(1-s)}\quad .$$