FOURIER TRANSFORM IN NONSTANDARD ANALYSIS

E. I. Gordon

Izvestiya VUZ. Matematika,
Vol. 33, No. 2, pp. 17-25, 1989
UDC 517.518

This paper utilizes nonstandard analysis to provide a foundation for the common employed approximation of the integral Fourier transform by the discrete transform.

The terminology and notation related to nonstandard analysis are basically those in [1]. We will review some of this material. The classical natural, real, and complex numbers are referred to as standard. Each of these number sets is assumed to have nonstandard elements. A number whose modulus is greater than any standard real number is said to be infinitely large. A number x whose modulus is less than any standard positive number is said to be infinitely small (x = 0). If a number is not infinitely large, then it is said to be finite. It will have been proved that any finite number x is infinitely close to some (unique) standard number, which is termed the standard part of x and denoted by *x.*

Here we will study the conditions under which the nonstandard discrete Fourier transform (DFT), i.e., the DFT in a space whose dimension is an infinitely large natural number, approximates the Fourier transform in space L^1(R).

It should be noted that, since there is a fast algorithm for computing the DFT, approximation of the continuous Fourier transform (FT) by the discrete transform is widely employed in practice, i.e., in order to compute the FT of some function f onto R, a table is compiled for it over a sufficiently large interval [-T, T] with a sufficiently small step Δ and the DFT is applied to this table. Proceeding from some heuristic considerations based on a theorem of Kotelnikov (see, e.g., [2]), the vector obtained is assumed to be an approximation of the table representing the Fourier image of function f computed with step Δ = 1/(2T). The results in the present paper (which also admit standard formulation) provide a foundation for the method (this topic is discussed in greater detail in the remark following theorem 5).

1. NONSTANDARD FINITE-DIMENSIONAL APPROXIMATIONS OF BOUNDED OPERATORS

This section defines what is meant when a nonstandard operator approximates a standard bounded operator acting into spaces of the L_p(R) (p < ∞) type and gives sufficient
conditions for such an approximation. In order to do so, it is first necessary to clarify the question of when an improper integral is the standard part of a Riemann integral sum taken with infinitely small step on an infinitely large interval.

(A) Let \( \Delta \) be an infinitely small and \( N \) an infinitely large natural number such that \( N \cdot \Delta \) is infinitely large.

**Theorem 1.** Let \( f \) be a standard function Riemann-integrable on \( [0, \infty) \). In order for the equality

\[
\int f(x) \, dx = \lim_{n \to \infty} \left( \Delta \sum_{i=1}^{N} f(x_i) \right),
\]

(1)
to hold for any \( N, \Delta \) satisfying conditions (A), it is necessary and sufficient that the condition

\[
\lim_{\Delta \to 0} \left( \Delta \sum_{i=1}^{N} f(x_i) \right) = 0.
\]

(2)
be satisfied.

Proof. If (2) is satisfied, then ([3], pp. 617-720)

\[
\int f(x) \, dx = \lim_{\Delta \to 0} \left( \Delta \sum_{i=1}^{N} f(x_i) \right),
\]

i.e.,

\[
\int f(x) \, dx = \lim_{\Delta \to 0} \left( \Delta \sum_{i=1}^{N} f(x_i) \right).
\]

Further,

\[
\Delta \sum_{i=1}^{N} f(x_i) = \Delta \sum_{i=1}^{N} f(x_i) + \Delta \sum_{i=1}^{N} f(x_i).
\]

Assuming \( c = N \cdot \Delta \) and utilizing the nonstandard definition of a limit, we find from condition (2) that the second term in the latter equality is infinitely small, which also proves (1).

In order to prove necessity, we remark that the equality

\[
\int f(x) \, dx = \lim_{\Delta \to 0} \left( \Delta \sum_{i=1}^{N} f(x_i) \right)
\]

follows from (1) for any infinitely large \( c \) and infinitely small \( h \), i.e.,

\[
\int f(x) \, dx = \lim_{\Delta \to 0} \left( \Delta \sum_{i=1}^{N} f(x_i) \right).
\]

Equality (3) obviously implies (2).

It should be noted that the class of functions satisfying condition (2) is very broad. It includes, e.g., all Riemann-integrable functions \( f \) representable in the form \( f = \sigma \cdot \psi \), where \( \sigma \) is bounded and \( \psi \) is monotone decreasing at infinity ([3], pp. 617-620).

(A') In order to facilitate our exposition, we will consider everywhere below (except in the concluding part of the second section) \( N \) and \( \Delta \) satisfying (A) and such that
\[ N = 2M + 1 \] for some infinitely large natural \( M \).

If \((A')\) is satisfied, then \( M - \Delta \) is obviously infinitely large.

Corollary. Let \( f \) be a standard function Riemann-integrable on \( R \) in the sense of principal value. Then, in order for the equality

\[
\int_{-\Lambda} f(x) \, dx = -i \int_{-\Lambda} f(z) \, dz
\]

to hold for any \( N \) and \( \Delta \) satisfying \((A')\), it is necessary and sufficient that the condition

\[
\lim_{\Lambda \to \infty} \left( \frac{1}{\Lambda} \sum_{k=-\Lambda}^{\Lambda} f(k) \right) = 0
\]

be fulfilled.

The above corollary is a significant strengthening of theorem 2 in [4].

The class of Riemann-integrable functions satisfying condition (4) is denoted by

\[ \Theta \]

in the sequel.

Consider the interior complex \( N \)-dimensional norming of the space \( E_{\Delta}^{P} \) of sequences of the type \( \bar{x} = (x_{-M}, x_{-M+1}, \ldots, x_{-1}, x_{0}, x_{1}, \ldots, x_{M}) \) with norm

\[
\| x \|_{\Delta}^{p} = \left( \sum_{k=-\Delta}^{\Delta} |x_{k}|^{p} \right)^{1/p}
\]

(in the case where \( p = 2 \), this norm is generated by the scalar product \( \langle x, y \rangle_{\Delta} = \sum_{k=-\Delta}^{\Delta} x_{k} \overline{y_{k}} \) where \( \overline{y_{k}} \) is the complex conjugate of \( y_{k} \)). We will assume \( p \) and \( q \) to be standard in the sequel.

Denote by \( \Theta^{(p)} \) the space of functions \( f \) onto \( R \) for which \( |f|^p \in \Theta \), and define the operator \( \Phi_{A} : \Theta^{(p)} \to \Theta \), setting \( \Phi_{A}(f) = \langle f(M\Delta), \ldots, f(-2\Delta), f(-\Delta), f(0), f(\Delta), f(2\Delta), \ldots, f(M\Delta) \rangle. \) By the corollary to theorem 1,

\[
\Phi_{A} : L^{p} \to E_{\Delta}^{P}
\]

Denote \( A_{\Delta} \) be such that the pair \((N, A)\) satisfies the conditions \((A')\), \( A : E_{\Delta}^{P} \to E_{\Delta}^{P} \) be an interior linear operator with finite norm, \( A : L_{p}(R) \to L_{p}(R) \) is a bounded linear operator such that the space \( S_{A} = \{ f \in \Theta^{(p)} : A(f) \in \Theta^{(p)} \} \) is dense in \( L_{p}(R) \). We will say that \( A \) approximates \( A_{\Delta} \) if, for any \( f \in S_{A} \),

\[
\| A_{\Delta} f - A f \|_{\Delta}^{p} = 0
\]

Theorem 2. Let there exist under the conditions of the definition a set \( \mathfrak{S} \subset S_{A} \) such that its linear hull \( L(\mathfrak{S}) \) is dense in \( L_{p}(R) \), and let relation (5) be satisfied for any function \( f \in \Theta \). Then \( A \) approximates \( A \).

Proof. Proceeding in the same manner as in [4], we construct for space \( E_{\Delta}^{P} \) the exterior normed space \( E_{\Delta}^{P} \), which is a factor space of the exterior space of elements \( E_{\Delta}^{P} \) with finite norm of subspace of elements \( E_{\Delta}^{P} \) and denote by \( \pi : E_{\Delta}^{P} \to E_{\Delta}^{P} \) a unique projection

\[
\mathfrak{S}_{\Delta} = \bigcup_{f \in L_{p}(R)} \mathfrak{S}_{\Delta}(f) = D(A)
\]
(more precisely, domain of definition $s_\Lambda$ is the subspace of elements $E_\Lambda^\sigma$ of finite norm). It follows from [5] that $E_\Lambda^\sigma$ is a dense nonseparable normed space (compare with theorem 1 in [4]). Putting $\Phi_\Lambda^n = \Phi_\Lambda \cdot \Phi_\omega$, we find by (5) that $\Phi_\Lambda^n : E_\omega^\sigma \rightarrow E_\omega^\sigma$ is a norm-preserving linear operator, which is continued to the norm-preserving linear operator $\Phi_\Lambda^n : L_p^\omega(R) \rightarrow E_\omega^\sigma$. The operator $\Phi_\Lambda^n : L_p^\omega(R) \rightarrow E_\omega^\sigma$ is constructed similarly. It follows from the preservation of norm that subspaces $\Phi_\Lambda(L_p^\omega(R))$ and $\Phi_\Lambda(L_p^\omega(R))$ are closed in spaces $E_\omega^\sigma$ and $E_\omega^\sigma$ respectively.

Now construct operator $A^n : E_\omega^\sigma \rightarrow E_\omega^\sigma$ such that $s_\Lambda \cdot A = A^n \cdot s_\Lambda$. It was established in [6] that $A^n$ is a bounded operator (its norm equals the standard part of norm $A$). It is obvious that, for any function $f \in S_A$, relation (6) is equivalent to

$$\Phi_\Lambda^n(A(f) - A^n \cdot \Phi_\Lambda(f)). \quad (7)$$

Since (7) is satisfied for any function $f \in L_p^\omega(R)$, $L_p^\omega(R)$ is compact in $L_p^\omega(R)$, and both operators $\Phi_\Lambda^n \cdot A$ and $A^n \cdot \Phi_\Lambda$ are bounded, so that it is true for any function $f \in L_p^\omega(R)$ that

$$\Phi_\Lambda^n(f) = A^n \cdot \Phi_\Lambda(f). \quad \text{for} \ f \in S_A,$$

the latter equality is (7) and therefore (6) holds for $f \in S_A$.

We will now give a standard version of this theorem (for standard $N$ and $\Lambda$, spaces $E_\Lambda^\sigma$ and operators $A$ are defined in precisely the same manner as for nonstandard entities).

Theorem 3. Let $A : L_p^\omega(R) \rightarrow L_p^\omega(R)$ be a bounded linear operator, $S_A$ be dense in $L_p^\omega(R)$, $N_\omega = -\infty, N_\Lambda = 0$ so that $N_\Lambda \cdot A = -\infty, N_\Lambda A = -\infty$ ($N_\omega$ be a sequence of odd natural numbers), and $A_0 : E_\omega^\sigma \rightarrow E_\omega^\sigma$ be a sequence of linear operators whose norms are bounded in aggregate. If there exists a $M \in S_A$, such that the linear hull $M_\omega \subset L_p^\omega(R)$ is dense in $L_p^\omega(R)$ and

$$\lim_{n \rightarrow \infty} \|A_0(f) - A(f)\|_{E_\omega^\sigma} = 0, \quad \text{(8)}$$

is valid for any function $f \in E_\omega^\sigma$, then this equality also holds for any function $f \in S_A$.

Proof. We will first make an obvious comment. Let a sequence $x_\omega$ be specified. If, for any $c > 0$ and any nonprincipal ultrafilter $G$ on $\omega$, $|\{n \mid x_\omega < c\} \in G$, then $x_\omega + c$.

Theorem 2 is valid in any nonstandard universal set in which any countable type is realized, particularly that which is an ultradegree of a standard universal set with respect to some arbitrary nonprincipal ultrafilter $G$ on $\omega$.

Fix an arbitrary $c > 0$ and consider precisely such a universal set for arbitrary $A$. Denote by $N_\omega, A_\omega, E_\omega, E_\omega^\sigma, A_\omega$, the classes of sequences $N_\omega$, $A_\omega$, $E_\omega$, $E_\omega^\sigma$, $A_\omega$, respectively. It follows from the conditions of theorem 3 that these elements of the nonstandard universal set satisfy the conditions of theorem 2. For any function $f \in E_\omega^\sigma$, $A(f)$ is obviously the class of sequence $\Phi_\Lambda(f)$. Then, by (8), Eq. (6) holds for any function $f \in E_\omega^\sigma$, so that (6) also holds for any function $f \in S_A$. Then, since $c$ is standard,

$$\lim_{n \rightarrow \infty} \|A_0(f) - A(f)\|_{E_\omega^\sigma} = 0, \quad \text{for} \ f \in S_A.$$
2. NONSTANDARD FINITE-DIMENSIONAL APPROXIMATION OF FOURIER TRANSFORM

This section deals with the conditions under which the nonstandard DFT is an approximation of the FT in $L_2(\mathbb{R})$.

Let $N, A$ satisfy conditions (A'). Consider the group of smallest absolute residues for $\mod N - Z_N = \{-M, ..., M\}$. If we set the measure of each element $Z_N$ equal to $A$, then the FT on $Z_N$ is specified by the formula

$$\hat{f}_N = A \sum_{n=-M}^{M} f_x \exp \left( -\frac{2\pi ink}{N} \right).$$

(9)

The inversion formula here has the form

$$f_x = \frac{1}{NA} \sum_{n=-M}^{M} \hat{f}_N \exp \left( \frac{2\pi ink}{N} \right).$$

(10)

Put $\Delta = 1/(NA)$; then the pair $N, A$ satisfies conditions (A'), and Eq. (9) specifies norm-preserving (and bounded) operator $E_1 : E^1 \rightarrow E^1$, which is inverse to that specified by Eq. (10). This operator is also a DFT. Denote by $F$ the FT such that

$$\hat{f}(\omega) = F(f)(\omega) = \int_{\mathbb{R}} f(x) \exp (-2\pi i \omega x) dx.$$

Then $S_F$ is dense in $L_2(\mathbb{R})$, since $S_F$ contains, e.g., the Schwartz space $S(\mathbb{R})$.

Theorem 4. Operator $F$ approximates $F_1$.

Proof. We will utilize theorem 2. Take as $M$ the set of characteristic functions of intervals of the type $[0, a]$ and $[-a, 0]$. Let $f = \chi_{[-a, a]}$ (the proof for $\chi_{[-a, a]}$ is analogous). Take an infinitely natural $T$ such that $(T - 1)A < a < TA$. Since

$$\frac{1}{\Delta} \sum_{n=-M}^{M} \int_{0}^{\infty} \exp \left( -\frac{2\pi ink}{N} \right) dx = 0,$$

relation (5) for this case is equivalent to

$$\frac{1}{\Delta} \sum_{n=-M}^{M} \int_{0}^{\infty} \exp \left( -\frac{2\pi in \omega}{N} \right) \left( \int_{0}^{\infty} \exp \left( -\frac{2\pi it \omega}{N} \right) dt \right) dx = 0.$$

(11)

It is obviously sufficient to prove (11) only for $k = 0$. Transforming the expression on the left side of (11), we then arrive at the equivalent relation

$$\frac{\Delta}{N} \sum_{n=-M}^{M} \left| \frac{1}{1 - \exp \left( -\frac{2\pi ink}{N} \right)} \right|^{2} \left( 1 - \exp \left( -\frac{2\pi ink}{N} \right) \right) \left( \frac{1}{1 - \exp \left( -\frac{2\pi ink}{N} \right)} \right) = 0.$$

(12)

Since the function $\left( 1 - \exp \left( -\frac{2\pi ink}{N} \right) \right)^{-1} - (\delta_k)^{-1}$ is bounded on $[0, a]$ and for any $k \in [1, M]$ and $0 < 2\pi kN < a$, all the terms in the latter sum are bounded by a single standard constant. This makes relation (12) obvious.

It should be noted that $\delta(NA)$ is the length of the interval on which the table for function $f$ is compiled. The relation $\Delta = 1/(NA)$ therefore agrees with the practical
algorithm for computing the Fourier image mentioned in the introduction. Having these practical applications in mind, we will refine the last theorem, addressing the question of what happens to relation (6) if the equality \( \Delta_1 = 1/(NA) \) is not satisfied.

First remark that, if \( \forall \epsilon \in E_1^l, \mathcal{F}_{\epsilon}: E_1^l \rightarrow E_2^l \), then \( |\mathcal{F}_{\epsilon}\mathcal{A}_1|^2 = NA\mathcal{A}_1 |\mathcal{E}_1^l|^2 \) (assume as before that \( \mathcal{A}_1 \) is specified by Eq. (9)). In order for operator \( \mathcal{F}_{\epsilon}: E_1^l \rightarrow E_2^l \) to be a bounded operator different from zero, it is therefore necessary and sufficient that \( NA \mathcal{A}_1 \) be a finite number such that \( * (NA \mathcal{A}_1) > 0 \). The latter condition is assumed to be fulfilled in this and subsequent sections.

Denote by \( \mathcal{F}_{\eta} \) the FT such that
\[
\mathcal{F}_{\eta}(\omega) = \int_{E_1^l} e^{-i\omega x} f(x) \, dx.
\]
The following theorem significantly strengthens theorem 5 in [7].

**Theorem 5.** If the pairs \((N, \Delta)\) and \((N, \Delta_1)\) satisfy conditions \((\Delta')\) and \( NA_1 = 2h \), where \( h > 0 \) is a standard constant, then operator \( \mathcal{F}_{\eta} \) approximates \( \mathcal{F}_{\eta} \).

Remark that, when \( h = 1/(2\pi) \), \( \mathcal{F}_{\eta} = \mathcal{F} \), i.e., in order for \( \mathcal{F} \) to approximate \( \mathcal{F} \), it is necessary and sufficient that the relation \( NA_1 = 1 \) be satisfied.

**Proof.** It is sufficient to show that, when \( NA_1 = 1 \), \( \mathcal{F} \) approximates \( \mathcal{F} \). The general case is obtained from this assertion if it is applied to the function \( \psi(t) \rightarrow f(2\pi t) \) and \( \Delta \) is replaced by \( \Delta' = 2\pi \Delta \).

Consider the same class of functions \( \mathcal{B}_\text{II} \) as in theorem 4.

Arguing as in the proof of theorem 4, we arrive at the conclusion that it is sufficient to prove the following relation:
\[
\Delta_1 \sum_{l=1}^{N} \frac{1}{2n(N)} \left( \frac{1 - \exp(-2n(iI)T)}{1 - \exp(-2n(iI)/N)} \right) = 0.
\]  

(13)

If we replace \( \Delta_1 \) by \( 1/(NA) \) in the latter relation, we obtain a relation equivalent to (11), i.e., the one already proved in theorem 4. It is then sufficient for proof of (13) to show that
\[
\Delta_1 \sum_{l=1}^{N} \frac{1}{2n(N)} \left( \frac{1 - \exp(-2n(iI)(\Delta_1))}{1 - \exp(-2n(iI)/N)} \right) = 0.
\]  

(14)

(Here we have also made use of the fact that \( \Delta_1 = 1/(NA) \)). The latter relation follows from the two inequalities:
\[
\frac{1}{2n} \sum_{l=1}^{N} \left| (NA_1 - 1)(1 - \exp(-2n(iI)/N)) \right|^2 = 0.
\]
\[
\frac{1}{2n} \sum_{l=1}^{N} \left| \exp(-2n(iI)(\Delta_1)) - \exp(-2n(iI)/N) \right|^2 = 0.
\]

We will prove the first of these, as the second is proved analogously. Put \( NA_1 = 1 -
\[ s = 0, T = \infty \] (see the proof of theorem 4). Then (15) is equivalent to
\[ \frac{1}{2\pi} \int_0^{2\pi} \sin^{2N_\lambda} \frac{\pi t}{2N_\lambda} dt = 0. \]  
(16)

If \( \lambda^{2M_\lambda} = 0 \), then (16) immediately follows from the fact that \( \lambda \) is finite and from the inequality \( \sin^2 \alpha = \alpha^2 \). If \( \lambda^{2M_\lambda} \neq 0 \), then put \( S = \{ \lambda^{2M_\lambda} \} \). Then \( M_\lambda \) is infinitely large and \( \lambda^{2M_\lambda} = 0 \), i.e., \( S < M \). It holds that
\[ \frac{1}{\lambda^{2M_\lambda}} \sum_{j=1}^N \frac{1}{2\pi} \sin^2 \frac{\pi x}{N_\lambda} = \frac{1}{\lambda^{2M_\lambda}} \sum_{j=1}^N \frac{1}{2\pi} \sin^2 \frac{\pi x}{N_\lambda} + \frac{1}{\lambda^{2M_\lambda}} \sum_{j=2j-1}^N \frac{1}{2\pi} \sin^2 \frac{\pi x}{N_\lambda}. \]  
(17)
The first term on the right side of (17) is infinitely small because \( \lambda^{2M_\lambda} = 0 \), and the second term because
\[ \frac{1}{\lambda^{2M_\lambda}} \sum_{j=1}^N \frac{1}{2\pi} < \frac{1}{\lambda^{2M_\lambda}} \left( \frac{1}{S} \right). \]
while \( M_\lambda \) and \( N_\lambda \) are infinitely large.

Remark. The relation between the function table step and the length of the interval on which the Fourier image is considered, which was mentioned in the introduction and derived in theorems 4 and 5, has long been used in practice for the following heuristic reasons.

According to a theorem of Kotelnikov (see, e.g., §3.1 in [2]), if the spectrum of a bounded function \( f \) is concentrated in an interval \([-F, F]\), then \( f \) is completely determined by its values on the set \( \{ \alpha \} - \infty < \alpha < +\infty \) where \( \Delta = 1/(2F) \), in accordance with the following formula
\[ f(t) = \sum \frac{1}{2\pi} \sin 2\pi \alpha \frac{e^{-j\alpha t}} \left( \frac{\sin \pi \alpha}{\pi \alpha} \right) \quad \forall t \in \mathbb{R}. \]

It is obvious that it in no way follows from this theorem, which is valid for functions with finite spectrum, that there is mean-square similarity between the function Fourier image table and the discrete Fourier transform established for the function table in theorems 4 and 5.

We will conclude by briefly considering approximation of Fourier series by the nonstandard DFT. We will assume through the end of this section that infinitely large natural \( N = 2M + 1 \) and infinitesimal \( \lambda \) are such that \( \lambda \) is finite and \( \#(N_\lambda) > 0 \). Here it is sufficient to consider the case where \( \#(N_\lambda) = 1 \).

Put \( \lambda_1 = \lambda(N_\lambda) \); then the DFT \( \mathcal{F}_1 : \mathcal{E}^2_\lambda \rightarrow \mathcal{E}^2_\lambda \) is specified by Eq. (9) and the inverse transform \( \mathcal{F}_\lambda^{-1} \) by Eq. (10). Denote by \( R \) the space of functions Riemann-integrable in the interval \([-0.5; 0.5]\) with scalar product \( (f, g) = \int_{-0.5}^{0.5} f(t)g(t) \). By virtue of familiar facts from nonstandard analysis,
where operator \( \Phi_1: \mathbb{R} \to \mathbb{C}^n \) is defined as in \( \S 1 \).

Here the FT is the operator \( F_\delta: L_2[0, 1] \to L_2 \) such that
\[
(F_\delta(f))(x) = \int_{-\delta}^{\delta} \exp(-2\pi i k x) f(x) \, dx, \quad -\infty < k < \infty.
\]

We define the operator \( \Phi_1: L_2 \to \mathbb{C}^n \) putting \( t = (\omega) \in \mathbb{C}^n \) \( \Phi_1(t) = (\omega) \) for any \( \omega \in (-M, M) \). Since \( \lambda_1 = 1 \), then \( \| \Phi_1(t) \| = |t|_1 \).

We will say that \( \tilde{F}_\delta \) approximates \( \tilde{F}_0 \) if, for any \( f \in \mathbb{R} \), it is true that \( \Phi_1(F_\delta(f)) = F_\delta(\Phi_1(f)) \).

As in \( \S 1 \), it is sufficient to make certain that the latter relation is fulfilled on that set \( \mathbb{W} \subset \mathbb{R} \), whose linear hull is dense in \( \mathbb{R} \). Selecting as \( \mathbb{W} \) the set of characteristic functions of intervals of the type \([0, a]\), where \( a < 0.5 \), and repeating the arguments applied to theorem \( \S 4 \), we can ascertain that this is actually the case. The general case, where \( \lambda = \| \mathbb{W} \| > 0 \), is obtained from it by converting to the function \( \varphi(t) = f(2t) \) and replacing \( \lambda \) by \( \lambda^2 = \| \mathbb{W} \| \).

As a result, we arrive at the theorem.

Theorem 6. If infinitely large natural \( N \) and infinitely small \( \Delta \) are such that \( \lambda = \| \mathbb{W} \| > 0 \), then the relation
\[
\sum_{a_1} \left| \sum_{a} f(a) \exp \left( -\frac{2\pi a \Delta}{N} \right) \right|^2 = 0
\]
holds for any function \( f \) Riemann-integrable in the interval \([-1, 1]\).

Remark. The standard versions of theorems \( \S 4-6 \) can be obtained in precisely the same manner as theorem \( \S 3 \) was derived from theorem \( \S 2 \).

3. NONSTANDARD FINITE-DIMENSIONAL APPROXIMATIONS OF SOME OTHER OPERATORS IN \( L_2(\mathbb{R}) \)

We will make some comments on nonstandard finite-dimensional approximations of other operators at this point. It was shown in [7] that, if \( B: L_2(\mathbb{R}) \to L_2(\mathbb{R}) \) is a Hilbert-Schmidt operator with kernel \( K \) satisfying the condition
\[
\left( \int_{\mathbb{R}^2} |K(x, y)|^2 \, dx \, dy \right)^{1/2} = \sqrt{\sum_{a, b} |K(a, b)|^2},
\]
then the operator \( B: \mathbb{R} \to \mathbb{C}^n \), whose matrix is specified by the formula
\[
\delta_{ab} = \Delta K(a, b),
\]
approximates \( B \).

Theorem 5 shows that, if \( \lambda = 1 \), then the operator whose matrix is specified by Eq. (15), where \( K(x, y) = \exp(-2\pi i xy) \), approximates \( B \). The following proposition shows that if \( \lambda^2 \neq 1 \), then the corresponding assertion generally does not hold. Thus, the above
assertion from [7] does not generalize even to the case of operators having a smooth bounded kernel.

Proposition 1. If $N\lambda^2 = 2$, the operator $B$ whose matrix is specified by Eq. (16), where $K(x, y) = \exp(-2\pi i x y)$, does not approximate $F$.

Proof. Given the conditions of the proposition, operator $B$ is specified by the matrix $\Delta = 1 - \exp(-4\sin a/N)$. Put $f = \chi_{3/4} \psi_{3/4}$; we will show that
\[
\left\langle \left( I - \Phi_{\lambda}(f) \right) \Phi_{\lambda}(f) \right\rangle > 0.
\]
As in theorem 4, we choose an infinitely large natural $T$ under the condition $|T - 1/\lambda| < \sqrt{2}/2 < 74$. Then elementary computations show that (19) follows from the inequality
\[
\left\langle \left( \sum_{k=-T}^{T} \left( 1 - \exp\left(-\frac{2\pi ikT}{N}\right) \right) ^{2} \left( 1 - \exp\left(-\frac{4\pi \lambda k}{N}\right) \right) \right)^{-1} - \frac{N}{4\lambda} \right\rangle > 0.
\]
Assuming that $T/N = 0$, we readily find that the sum in (20) is greater than or equal to
\[
\left( \sum_{k=-T}^{T} \frac{\sin^{2}2\pi kT}{N} \right) / \sin^{2}2\pi T
\]

It is obvious that, when $M - T < k < M - \sin^2(2\pi kT/N)$ diminishes. Choose an infinitely large natural $S$ such that $S = 265T + 1$, where $0 < s < 1$. Then, when $T < k < M - S = T - 0.75 - 1 < 2\pi kT/N < \pi T - W - 1$, where $\gamma, \delta = 0$, i.e., $\sin^{2}(2\pi kT/N)$ increases in this interval. It is easily seen that the term corresponding to $k = M - T$ in sum (21) is greater than or equal to $D/k^2$, where $D$ is a standard constant. Since the terms increase when $M - T \leq k < M - S$, (21) is greater than or equal to $D(T - 3)/L$, which is obviously not infinitely small.

We will give two additional examples of nonstandard finite-dimensional approximations. Consider the one-parameter groups $U(u) = \exp(-i\lambda P)$ and $V(v) = \exp(-ivQ)$, where $Q$ and $P$ are the coordinate and momentum operators respectively, i.e., $Q$ is the operator for multiplication by an independent variable, and $P = (k\lambda) d/dx$, where $k > 0$ is a standard constant. It is well known that $U(u) \psi(x) = \psi(x - \lambda u)$, $V(v) \psi(x) = \exp(-ivx) \psi(x)$.

We introduce the nonstandard finite-dimensional operators $U_d, V_d: E^{l}_d \rightarrow E_{d}$ such that $(U_d f)_d = \Phi_{\lambda} f_d$, and $V_d$ is specified by the diagonal matrix $\exp(-2\pi i k/N) \delta_{k}$, where $k = 0$ is. It was shown in [7] that, for any infinitely large natural $r$ satisfying the condition $r \lambda = \pi \delta$, the operator $U_d$ approximates $U(u)$. It is sufficient for proof to apply theorem 2, taking the space of finite functions as $\mathbb{M}$.

Proposition 2. If an infinitely large natural $m$ is such that $m2\pi(\lambda m) = 0$, then $V_d$ approximates $V(v)$.

Proof. Put $\Delta = (2\pi)(\lambda m)$. Then, by theorem 5, operator $F_{r}: E_{r} \rightarrow E_{d}$ approximates $F_{d}$.
It is well known that \( F_t V(x) F_t^{-1} \) is \( \varphi(x + \theta) \). Consequently, if we consider an operator \( U_1 \colon L^2 \rightarrow L^2 \) such that \( (U_1 \varphi)_{\omega} = \varphi_{\omega} \), \( U_1 \varphi \) approximates \( F_t V(x) F_t^{-1} \) by the foregoing when \( m, n = \omega \) (i.e., when \( m = n/(N\lambda) = \omega \)). Consequently, \( E \varphi U_1 F_t F_t^{-1} \) approximates \( V(\varphi) \). Direct calculation shows that \( F_t U_1 F_t^{-1} - V(\varphi) \).

Operators \( U_1 \) and \( V(\varphi) \) satisfy the commutation relations \( U_1 V(\varphi) = \exp(2iN\lambda \varphi) V(\varphi) U_1 \), which, when \( \varphi, \omega \approx \theta \), \( m, n = \omega \), obviously become the familiar Weyl commutation relations for \( U(\varphi) \) and \( V(\varphi) \). \( U(\varphi) V(\varphi) = \exp(i\varphi) V(\varphi) U(\varphi) \).

Note that proposition 2 and the last remark were obtained in [7] under the additional assumption that \( |N\lambda - 2i\theta| < C\lambda \). where \( C \) is a standard constant.

Consider the operator \( P_\varphi = (\omega \xi) D_\varphi \), where \( (D_\varphi \xi) = (2\pi^{-1}(x_{\xi} - x_{\varphi})) \). It follows from proposition 1 in [7] that, for any finite function \( \varphi \) and any standard natural number \( n \), it is true that \( |P_\varphi \xi - P_\varphi \xi| = \varphi \).

It is known that \( \varphi \xi - Q_\varphi \xi \). Denote \( P_\varphi P_\varphi F_t^{-1} = Q_\varphi F_t^{-1} \). Direct verification shows that operator \( Q_\varphi \) is specified by the diagonal matrix \( (N\lambda)(2\pi) \sin(2\pi\theta/N) \).

Proposition 3. If \( \varphi \) is a finite function and \( \lambda > 0 \) is a standard natural number, then

\[
Q_\varphi \varphi_{\xi} = \varphi_{\xi} (Q_\varphi \xi)
\]

We will omit the simple proof of proposition 3.

The author wishes to express his deep gratitude to the reviewer for his helpful remarks.

REFERENCES


1 June 1987

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