THE OXFORD HANDBOOK OF

PHILOSOPHY OF TIME

As the study of time has flourished in the physical and human sciences, the philosophy of time has come into its own as a lively and diverse area of academic research. Philosophers investigate not just the metaphysics of time, and our experience and representation of time, but the role of time in ethics and action, and philosophical issues in the sciences of time, especially with regard to quantum mechanics and relativity theory. This Handbook presents twenty-three specially written essays by leading figures in their fields: it is the first comprehensive collaborative study of the philosophy of time, and will set the agenda for future work.

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CHAPTER 21

THE CPT THEOREM

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1. Introduction

The CPT theorem says that any (restricted) Lorentz invariant quantum field theory must also be invariant under the combined operation of charge conjugation C, parity P, and time reversal T, even though none of those individual invariances need hold. The CPT theorem is prima facie a perplexing theorem. Why would the combination of two space-time transformations plus a charge reversing transformation have to be a symmetry of any relativistic quantum field theory? What does charge conjugation have to do with space-time symmetries? Is there an analogous theorem for classical relativistic field theories? What, if anything does the CPT theorem tell us about space-time structure? I will try to answer these questions. The basic idea of my answer is that what standardly is called the CPT transformation really amounts to a PT transformation, that is, a pure space-time transformation.

In section 2, I will briefly clarify the notion PT invariance, and briefly explain why one might be interested in such an invariance. In section 3, I examine the PT invariance of classical tensor field theories, and suggest that this transformation includes charge conjugation. I then discuss the same with respect to quantum tensor field theories. After that I briefly discuss classical and quantum spinor field theories. I end with some tentative conclusions.

2. How do quantities transform under PT?

Suppose we describe a world (or part of a world) using some set of coordinates $\{x, y, z, t\}$. A passive PT transformation is what happens to this description when we describe the same world but instead use coordinates $\{x', y', z', t'\}$ where $x' = -x$, $y' = -y$, $z' = -z$, $t' = -t$. An active PT transformation is the following: keep using the same coordinates, but change the world in such a way that the description of the world in these coordinates changes exactly as it does in the corresponding passive PT transformation.[1] Suppose now that we have a theory which is stated in terms of coordinate dependent descriptions of the world, that is, a theory which says that only certain coordinate dependent descriptions describe physically possible worlds, that is, are solutions. Such a theory is said to be PT invariant iff PT turns solutions into solutions and non-solutions into non–solutions.[2] Why might one be interested in PT invariance or non-invariance of theories?

Failure of PT invariance tells us something about the structure of space-time: it tells us that space-time either has objective spatial handedness or has an objective temporal orientation, or both. Why? Well, suppose we start with a coordinate dependent description of a world which some theory allows. And suppose that after we do a passive coordinate transformation the theory says that the new (coordinate dependent) description of this world is no longer allowed. This seems odd: it's the same world after all, just described using one set of coordinates rather than another. How could the one be allowed by our theory and the other not? Indeed, this does not make much sense unless one supposes that the theory, as stated in coordinate dependent form, was true in the original coordinates but not in the new coordinates. And that means that, according to the theory, there is some objective difference between the $\{x, y, z, t\}$ coordinates and the $\{x', y', z', t'\}$ coordinates, that is, the $\{x, y, z, t\}$ coordinates do not stand in the same relation to the objective structure of the world as the $\{x', y', z', t'\}$ coordinates.

In the case of PT this could be because space-time has an objective temporal orientation, or an objective spatial handedness, or both. Note that failure of PT invariance does not indicate that space-time has a space-time handedness structure, for space-time handedness is invariant under PT.[3]

It is in fact more interesting to consider what one should infer if a theory is invariant under PT, but fails to be invariant both under P and under T, since it appears that our world is such.[4] What this, prima facie, would suggest is that space-time has a space-time handedness structure, while having neither a spatial handedness nor a

[1] Note that there are therefore many distinct PT transformations, one corresponding to each distinct inertial coordinate system. I will assume a flat space-time throughout this chapter, and am not here going to address how to talk about PT in General Relativity. For more on that issue see Malament 2004 and Arntzenius and Greaves 2007.

[2] If the theory is probabilistic, then the theory is PT invariant if its probabilities are invariant under PT.

[3] Space-time handedness is the 4-dimensional analogue of spatial handedness: when one mirrors a single coordinate of a 4-tuple of coordinates, one flips the space-time handedness of the coordinate system. So, if one mirrors all 4 coordinates, as one does in PT, one ends up in a coordinate system of the same handedness as the original coordinate system.

[4] That is to say, the world is PT-invariant on my understanding of what a PT-transformation amounts to, which, in standard terminology means that the world is CPT-invariant.

temporal orientation, since space-time handedness is the minimal natural structure which explains the symmetry properties in question.

So that is why we should be interested, but how do we go about investigating PT invariance; in particular, how do we know how the quantities occurring in our theories transform under PT? Well, though we often give our theories in coordinate dependent form, nature itself, of course, is coordinate independent. We can use (n-tuples of) numbers to denote locations in space-time, and we can use (m-tuples of) numbers to indicate the magnitudes and directions of various quantities in space-time, but nature itself does not come equipped with numbers. How our numerical representation of a quantity should transform under a change of coordinates depends on the structure of the quantity, and on the way in which we have designed our coordinate representation. Let me illustrate this for a very simple case.

Suppose a vector V at a point p in a three-dimensional Euclidean space is a coordinate independent quantity which has a magnitude and picks out a direction. We can put Cartesian coordinates $\{x, y, z\}$ on the Euclidean space, and then use these coordinates to numerically indicate the direction and magnitude of V. To be precise: the coordinates on the Euclidean space naturally induce corresponding coordinates on the space of tangent vectors at p. It follows from the vector nature of V that when we, say, switch to coordinates $\{x', y', z'\}$ where $x' = -x$, $y' = -y$, $z' = -z$, then the three numbers representing V will each flip their sign. My point here is simply that one is not free to choose how one's numerical representation of quantities transforms under certain transformations. In particular, one cannot make a theory invariant under some transformation simply by judiciously choosing how the quantities occurring in the theory transform. For how a quantity transforms is determined by the coordinate independent nature of the quantity in question (together with the way in which we manufacture coordinate representations of it).

3. PT IN CLASSICAL TENSOR FIELD THEORIES

Let's start with a simple case, namely, the classical real Klein Gordon field. The classical real Klein Gordon field is a real scalar field whose field values are invariant under the restricted Lorentz transformations.[5] (The restricted Lorentz transformations are the ones that are continuously connected to the identityy. They include spatial rotations and Lorentz boosts. They include neither P nor T nor PT.) We can then impose a law of evolution on our Klein Gordon field, namely the Klein Gordon equation: $(\partial^{\mu}\partial_{\mu} + m^2)\varphi = 0$, and check whether this law of evolution is invariant under the restricted Lorentz transformations. It is.

[5] For the sake of simplicity I am assuming that there are objective, path independent, facts as to whether the values of the real Klein Gordon field at two different locations in space-time are the same. That is to say, I am here ignoring the idea that the Klein Gordon field configurations correspond to sections on a fibre bundle with a connection on it.

How about PT invariance? Well, we first need to know how the Klein Gordon Field transforms under PT. The standard assumption is that the Klein Gordon field is invariant under PT: under PT the field $\varphi(t,r)$ transforms into $\varphi(-t, -r)$. It immediately follows that the Klein Gordon equation is invariant under PT. I will later return to the issue as to whether this is the only possible way in which a scalar field could transform under PT. In the meantime let us turn to another field: the classical electromagnetic field.

Maxwell's equations for the free electromagnetic field, written in terms of the four-potential A^μ : $\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$. In order to see whether this equation is invariant under PT, we need to know how the four-potential A^μ transforms when we move it from location (t,r) to location (−t, −r), or, equivalently, how it transforms when we switch from co-ordinates (t,r) to co-ordinates (t', r') where $t' = -t$ and $r' = -r$. This depends on what kind of co-ordinate independent quantity we take the four-potential to be. Let's make the simplest assumption, namely that it is the same kind of quantity as $\partial^\mu\varphi$ (t,r), namely a tangent vector in a four-dimensional space-time.[6] Now, we know how $\partial^\mu\varphi$ (t,r) transforms under active PT. For the scalar field φ just gets moved to its new location, and this means that its derivatives flip sign. So $\partial^\mu\varphi(t, r)$ transforms to $-\partial^\mu\varphi(-t, -r)$. Assuming that $A^\mu(t, r)$ is the same kind of quantity, it follows that $A^\mu(t, r)$ transforms to $-A^\mu(-t, -r)$ under active PT.

It is important to note that this is *not* the standard view of how $A^\mu(t, r)$ transforms under PT. On the standard view $A^\mu(t, r)$ transforms to $A^\mu(-t, -r)$ under PT. (For a more extensive discussion of why the standard view says this, and why I find it an unattractive view, see Arntzenius and Greaves 2007.) Since this assumption is controversial, and since it will turn out to be crucial in what follows, let me give two additional justifications for the claim that $A^\mu(t, r)$ transforms to $-A^\mu(-t, -r)$ under PT.

The ordinary, 'real', Lorentz transformations form a group of transformations that splits into four disconnected components. (The full group of Lorentz transformations is the group of transformations that leaves the Minkowski metric invariant.) Here is why. Parity (mirroring of all three spatial axes) is a Lorentz transformation. But in the space of all possible Lorentz transformations there is no continuous path that starts out at the Identity and ends up at Parity. (The pure spatial rotations are all continuously connected to the Identity, and so are the pure Lorentz boosts, but one cannot reach Parity by pure boosts or pure rotations or combinations of the two.) So the real Lorentz group splits up into at least two disconnected components: the Lorentz transformations that one can reach via a continuous path from the Identity (the 'restricted' Lorentz transformations), and the Lorentz transformations that one can reach via a continuous path from Parity. And there is another split, namely the split between the Lorentz transformations that include Time Reversal and the ones that do not. So the Lorentz group has at least four disconnected components. In fact it has exactly four disconnected components.

[6] Again, for simplicity, I am ignoring the idea that A^μ is a connection on a fibre bundle.

The Lorentz transformations of four-vectors can be represented as 4x4 matrices L with real entries acting on 'columns' of four real numbers representing the four-vectors, where these matrices have the property that they preserve the Minkowski inner product between the 'columns'. The demand that each L preserves the Minkowski inner product amounts to the demand that $L^{TR}GL = G$, where L^{TR} is the transpose of L, and G is the matrix whose diagonal entries equal $1, -1, -1, -1$, and whose off-diagonal entries equal 0. Now, while it is natural to suppose that the matrix L^{PT} representing PT (in some inertial frame of reference) should be the matrix whose diagonal entries each equal -1 and whose off-diagonal entries are 0, this is not the only possible representation of the real Lorentz group. Another possible one, for example, is one in which PT is represented as the identity matrix (though this is not a 'faithful' representation).[7] However, let us now turn to the complex Lorentz group.

One can introduce the complex Lorentz group abstractly as the so-called 'complexification' of the real Lorentz group. But it is easier to think of the complex Lorentz group in terms of its representation by means of 4x4 matrices with complex entries which preserve the Minkowski inner product, that is, the set of 4x4 complex matrices L such that $L^{TR}GL = G$. What is important for our purposes is that within the complex Lorentz group PT is connected to the Identity. Here is why. The following is a one parameter subset of the complex Lorentz matrices. where the parameter is t:

$$\begin{pmatrix} \cosh it & 0 & 0 & \sinh it \\ 0 & \cos t & -\sin t & 0 \\ 0 & \sin t & \cos t & 0 \\ \sinh it & 0 & 0 & \cosh it \end{pmatrix}$$

For t=0 we find that L_t is the Identity. Continuously increasing t to $t = \pi$ we arrive at minus the Identity, which is the representation of PT. So $A^{\mu}(t, r)$ transforms to $-A^{\mu}(-t, -r)$ if it transforms as an element of the standard (four-dimensional) representation of the complex Lorentz group.[8]

Note that this argument similarly establishes that a scalar field (which is invariant under the restricted Lorentz transformations) must be invariant under PT. Note also that this argument does not tell us how $A^{\mu}(t,r)$ transforms under P or T separately; for in the complex Lorentz group P and T are not connected to the Identity. There can be so-called 'pseudo-scalar' fields, 'pseudo-vector' fields, and 'pseudo-tensor' fields, which flip sign under P and under T.

Another argument for the claim that $A^{\mu}(t,r)$ must transform to $-A^{\mu}(-t, -r)$ under PT, and that a scalar field must be invariant under PT, can be given if one makes the assumption that the only types of quantities that can occur in our theories must be (restricted) Lorentz invariant tensor quantities. Let me start on this argument by

[7] A 'faithful' representation of a group (by matrices) is one whereby each distinct element of the group gets represented by a distinct matrix.

[8] I am not sure whether it must do so in every non-trivial 4-dimensional representation of the complex Lorentz group.

indicating how one can manufacture 'pseudo-tensors', tensors whose sign flips under P and under T, using tensors that are invariant under the restricted Lorentz transformations. Consider the totally anti-symmetric Levi-Civita tensor ε_{ABCD}. It is invariant under restricted Lorentz transformations. It can be taken to represent an objective space-time orientation (space-time handedness). One can also use it to manufacture a pseudo-scalar field from four distinct vector-fields. Suppose that vector fields V^A, W^B, X^C, and Y^D each flip their time component under time reversal T. Let's take it that space-time orientation is a geometric, invariant, object represented by ε_{ABCD}. Then the pseudo-scalar $\Sigma_{ABCD}\varepsilon_{ABCD}V^AW^BX^CY^D$ will transform to $-\Sigma_{ABCD}\varepsilon_{ABCD}V^AW^BX^CY^D$ under T. So in this manner one can manufacture pseudo-scalars whose signs flip under P and T. However, one cannot in this manner manufacture pseudo-scalars whose signs flip under PT.[9] So $A^\mu(t,r)$ transforms to $-A^\mu(-t,-r)$ under PT. It follows that Maxwell's equations for the free electromagnetic field are invariant under PT.

Let's do one more example: a complex Klein Gordon field φ interacting with the electromagnetic field. For ease of presentation let me represent the electromagnetic field using both the four-potential A^μ and the Maxwell-Faraday tensor $F^{\mu\upsilon}$. The following Lagrangian then gives a dynamics for the interacting fields:

$$L = \partial^\mu\varphi^*\partial_\mu\varphi - m^2\varphi^*\varphi - 1/4(F^{\mu\upsilon}F_{\mu\upsilon}) + e^2A_\mu A^\mu\varphi^*\varphi - ie(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*)A_\mu,$$

Is this theory invariant under PT? Well, suppose that under PT the four-potential $A^\mu(t,r)$ transforms to $-A^\mu(-t,-r)$. The Maxwell-Faraday tensor and the four-potential are related via the following equation: $F_{\mu\upsilon} = \partial_\mu A_\upsilon - \partial_\upsilon A_\mu$. It follows that the Maxwell-Faraday tensor is invariant under PT. The complex scalar field is invariant under PT, so $\partial^\mu\varphi$ flips sign under PT. All of this taken together implies that each of the five terms of the Lagrangian is separately invariant under PT. So the Lagrangian is invariant under PT. So our dynamics is invariant under PT.

It should be obvious from this last example that the PT-invariance of my sample theories is not a coincidence. If one's theory derives its dynamics from a local Lagrangian, where this Lagrangian is a Lorentz-scalar built (by contractions and summations) from tensors, each of which transform as indicated under PT, then this Lagrangian, and hence one's dynamics, will be invariant under PT.

In fact, J. S. Bell has proved a general classical PT theorem along these lines (see Bell 1955). Here is a statement of Bell's PT theorem. Let us suppose that the equations of a theory can be stated in the following form: $F^i(\varphi, A^\mu, \ldots, \partial^\nu\varphi, \partial^\lambda A^\mu, \ldots\ldots) = 0$, where φ, A^μ,...are fields which transform as tensors under the restricted real Lorentz transformations (the ones that are connected to the identity), that the $\partial^\nu\varphi$, $\partial^\lambda A^\mu$,.... are finite order derivatives of the tensor fields, and the F^i are finite polynomials in these terms. Then the equations are invariant under PT, when the representation of PT in some frame is

[9] For a proof, see Greaves (manuscript). Basic idea: any tensor representing a temporal orientation (just as ε_{ABCD} represents space-time handedness) would have to have an odd number of space-time indices, i.e. would have to have odd rank. But one can show that there are no tensor fields of odd rank that are invariant under the restricted Lorentz transformations.

$$L_{PT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

At this point one might very well ask: but what does all of this have to do with charge conjugation? Well, notice that under the above PT transformation all four-vectors flip. In particular therefore, any charge-current four vector, such as $ie(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*)$ in the case of the complex Klein Gordon field, will 'flip over' under PT. So any charge density, which is the first component of a charge-current four-vector, will flip sign under PT. So the PT transformation, when properly conceived, has as a consequence that the PT-transformed fields behave as if they have opposite charge. Indeed it is my contention that what standardly is called the CPT transformation should really have been called the PT transformation. This also provides an answer as to how it can be that what is allegedly a combination of two geometrical transformations (PT) and a non-geometrical transformation (C) has to be a symmetry of any quantum field theory. The answer I am suggesting is that what is standardly called the CPT transformation really is a geometric transformation, namely the PT transformation, and that invariance under PT, and lack of invariance under each of P and T, corresponds to the fact that our space-time has space-time handedness structure, but neither a spatial handedness nor a temporal orientation.

Note also that one can include classical particles (rather than just fields) in our considerations, by making the assumption that the four-velocities V^μ of particles flip over under PT. We can then, for example, consider an interaction between a charged particle and the electromagnetic field which is governed by the following Lagrangian:

$$L = -1/4(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - qV_\mu A^\mu$$

It is clear that this Lagrangian is invariant under our PT transformation. Moreover, note that if we switch the sign of q and switch the sign of the four-velocity V^μ, while keeping A^μ invariant, then the Lagrangian is invariant. That is to say: flipping over the four-velocity of a particle is equivalent to flipping the sign of the charge q. So, again, PT, when properly conceived, includes charge conjugation. (For a more extensive discussion of the classical charged particle case, see Arntzenius and Greaves 2007.)

4. PT IN QUANTUM TENSOR FIELD THEORIES

The basic idea of this section is very simple. In quantum field theory particle states correspond to 'positive frequency' solutions of the corresponding classical field theory, while anti-particle states correspond to 'negative frequency' solutions. Since PT turns positive frequency solutions into negative frequency solutions, PT in quantum field theory turns particles into anti-particles. Now for some details.

Let's start with the classical free complex Klein Gordon equation: $(\partial^\mu \partial_\mu + m^2)\varphi = 0$. This equation has plane wave solutions: $\varphi_k = \exp(ik_\mu x^\mu)$, where $k_\mu k^\mu = m^2$. Any solution $\varphi(x)$ is a unique superposition of these plane wave solutions: $\varphi(x) = \int f(k)\exp(ik_\mu x^\mu)dk$. Relative to a given direction of time, we will call plane wave φ_k with positive k^0 a 'positive frequency' plane wave, and a plane wave φ_k with negative k^0 a 'negative frequency' plane wave. More generally any solution that is a superposition only of positive frequency plane waves will be called a positive frequency solution, and any solution that is a superposition only of negative frequency plane waves will be called a negative frequency solution. The particle Hilbert space $\mathcal{H}$ of the quantized Klein Gordon field, and the operators on it, can be constructed from the positive frequency solutions to the classical Klein Gordon equation, and the anti-particle Hilbert space $\underline{\mathcal{H}}$ can be constructed from the negative frequency solutions to the classical Klein Gordon equation. Here is how. (In this section I am largely following Geroch 1971. The errors are all mine.)

Start by assuming that there is a 1–1 correspondence between single particle Hilbert space states $|\varphi>$ and positive frequency solutions $\varphi(x)$ to the classical Klein-Gordon equation. The superposition of two Hilbert space states is the Hilbert space state corresponding to the addition of the two corresponding positive frequency solutions, that is $|\varphi_1> + |\varphi_2>$ corresponds to positive frequency solution $\varphi_1(x) + \varphi_2(x)$. Scalar multiplication of a particle Hilbert space state corresponds to scalar multiplication of the corresponding positive frequency solution: $c|\varphi>$ corresponds to $c\varphi(x)$. The inner product between two particle Hilbert space states is defined as $(|\varphi_1>, |\varphi_2>) = \int f_1(k)f_2^*(k)dk$. This defines the particle Hilbert space.

The single anti-particle Hilbert space is constructed in analogous fashion, now using the negative frequency solutions, *except* that multiplication of an anti-particle state by a complex number c corresponds to multiplication of the corresponding classical solution with the complex conjugate number of that number: c^*. That is, if $|\varphi>$ corresponds to $\varphi(x)$ then $c|\varphi>$ corresponds to $c^*\varphi(x)$. The inner product between two anti-particle Hilbert space states is defined as $(|\varphi_1>, |\varphi_2>) = \int f_1^*(k)f_2(k)dk$. This defines the anti-particle Hilbert space.

Since it is important for what follows let me explain why multiplication by a complex number c of the anti-particle Hilbert space states corresponds to multiplication by c^* of the corresponding negative frequency solution. Associated with a constant time-like vector field r^a on Minkowski space-time is an energy operator E, where E acts on solutions of the complex Klein Gordon equation in the following way: $E\varphi(x) = -ir^a\nabla_a\varphi(x)$. Here multiplication by i means multiplication of the Hilbert space state $|\varphi>$, not (pointwise) multiplication of the (complex) classical field $\varphi(x)$. Now, suppose that multiplication of a Hilbert space state did correspond to (pointwise) multiplication of the corresponding solution $\varphi(x)$, both for negative frequency solutions and positive frequency solutions. Then the expectation value of E for $\varphi(x)$ would be: $\int r^a k_a f(k) f^*(k)dk$. This is positive for any k which points in the same direction of time as r and negative for any k which points in the opposite direction of time from r. That is to say, anti-particles and particles would then have opposite signs of energy. This would be a disaster. In the first place, it is experimentally known that particles and anti-particles have the same sign of energy: they can annihilate and thereby produce energy (particles with non-zero energy), rather than that their total energy is 0. Moreover, in an interacting theory, one would beget radical instability: decays into deeper and deeper negative energy states would be allowed, which would release unlimited amounts of positive energy. This is no good: we need the energies of particles and anti-particles to have the same sign (though it is perfectly all right if what sign that is, is a matter of convention.) So we choose the Hilbert space structure of the anti-particle state-space such that $c|\varphi>$ corresponds to $c^*\varphi(x)$ for negative frequency solutions $\varphi(x)$. For then the expectation value of E equals $\int_{M+} r^a k_a f(k) f^*(k) dk - \int_{M-} r^a k_a f(k) f^*(k) dk$, where M+ is the positive mass shell, and M− is the negative mass shell, that is, the first integration is over momenta that point in the same direction of time as r, and the second integration is over momenta that point in the opposite direction of time as r. This has as a consequence that the energies of particles and anti-particles have the same sign.

Let me clarify and emphasize one more point. At the beginning of this section I arbitrarily picked some direction of time, and called plane wave solutions 'positive frequency solutions' if their wave-vector pointed in that direction of time. Later on I associated particles with positive frequency solutions and anti-particles with negative frequency solutions. And then I said that multiplication of a Hilbert space particle state by a complex number c corresponded to multiplication of the corresponding positive frequency solution by c, while in the anti-particle case it corresponded to multiplication of the corresponding negative frequency solution by c^*. None of this implies that space-time has a temporal orientation. All I have made of this is that there is a fact of the matter as to whether two time-like vectors point in the same direction of time, or in opposite directions of time. Which direction of time gets called the future, which the past, which solutions get dubbed positive frequency, which negative frequency, which Hilbert space state multiplication corresponds to multiplication of the corresponding solution by c and which corresponds to multiplication by c^*: all of that can be taken to be a matter of convention. But that the two directions, the two frequency types, and the two particle types are not identical: that is not a matter convention. Let me now continue with the construction of the full particle and anti-particle Hilbert spaces. (So far we only have the single particle and single anti-particle Hilbert space.)

Given the single particle Hilbert space $\mathcal{H}$ we can build the corresponding Fock space $\mathcal{F} = C \oplus \mathcal{H}^\alpha \oplus (\mathcal{H}^{(\alpha} \oplus \mathcal{H}^{\beta)}) \oplus \cdots\cdots$. Here C is the space of complex numbers and the $\mathcal{H}^\alpha$, $\mathcal{H}^\beta$ etc. are simply copies of $\mathcal{H}$. The brackets around the indices indicates the restriction to symmetrical states in the tensor product Hilbert spaces. A typical element of this Fock space is $|\psi>= (\chi, |\chi>^\alpha, |\chi>^{\alpha\beta}, ...)$. Here χ is a complex number, representing the amplitude of the vacuum state, $|\chi>^\alpha$ is an element of $\mathcal{H}^\alpha$, that is, a one-particle state, $|\chi>^{\alpha\beta}$ is an element of $\mathcal{H}^{(\alpha} \oplus \mathcal{H}^{\beta)}$, that is a two-particle state, etc... Similarly, given the single anti-particle Hilbert space $\underline{\mathcal{H}}$ we can build the corresponding Fock space $\underline{\mathcal{F}} = C \oplus \mathcal{H}^\alpha(\mathcal{H}^{(\alpha}\mathcal{H}^{\beta)} \oplus \cdots\cdots$. (Note the underlining of the symbols associated with the anti-particle Hilbert spaces.)

We can then define the particle momentum creation operators C_k and C_{-k}which, when operating on the vacuum, create a particle in momentum eigenstate $|k>$, $|-k>$ respectively, that is, create a particle, and the corresponding anti-particle, respectively. Similarly one can define particle momentum annihilation operators A_k, and A_{-k} which when acting on momentum eigenstate $|k>$, $|-k>$ respectively, produce the vacuum. Then one can define a Klein Gordon field operator $\varphi(x) = \int A_k \exp(-ik_\mu x^\mu) + C_{-k} \exp(ik_\mu x^\mu))dk$, and its adjoint $\varphi^+(x) = \int C_k \exp(ik_\mu x^\mu) + A_{-k} \exp(-ik_\mu x^\mu))dk$. One can then show that these field operators, $\varphi(x)$ and $\varphi^+(x)$, satisfy the same equation that the corresponding classical fields, $\varphi(x)$ and $\varphi^*(x)$, satisfy, namely the complex Klein Gordon equation. (One has to be a bit careful. Strictly speaking the way I defined the field operators makes no sense: I should have smeared them out with test functions.) Now let us turn to PT. How do quantum states $|\varphi>$ transform under PT? Let's start with the single particle states. Well, under PT the corresponding classical solution $\varphi(x)$ should transform to $\varphi(PTx)$. If we choose our coordinate system x so that PT consists of mirroring in the origin of that coordinate system, then a classical solution $\varphi(x)$ transforms to $\varphi(-x)$ under this PT transformation. Given that multiplication in the anti-particle Hilbert space by c corresponds to multiplication of the corresponding solution by c^*, this is an anti-linear transformation on the Hilbert space.

How do the field operators transform? Well, a particle state $|k>$ transforms to the corresponding anti-particle state $|-k>$ (give a suitable choice of phases for the momentum states), and vice versa. So C_k transforms to C_{-k} and vice versa, and A_k transforms to A_{-k} and vice versa. This, together with the fact that the transformation is anti-unitary means that the field operators $\varphi(x) = \int A_k \exp(-ik_\mu x^\mu) + C_{-k} \exp(ik_\mu x^\mu))dk$ transform to $\int A_{-k} \exp(ik_\mu x^\mu) + C_k \exp(-ik_\mu x^\mu))dk = \varphi^+(-x)$. So $\varphi(x)$ transforms to $\varphi^+(-x)$ under PT. Similarly $\varphi^+(x)$ transforms to $\varphi(-x)$ under PT. But this is exactly how the standard view has it that the Klein-Gordon field operators transform under CPT! So I have argued that what standardly is called a CPT transformation in quantum field theory should really have been called a PT transformation. So standard proofs of the CPT theorem amount to proofs that relativistic quantum field theories must be invariant under what I have argued to be the PT transformation. (I should point out that such proofs can not be exactly the same as in the classical case. In the first place, as we have just seen, quantum fields get Hermitian conjugated under PT. Moreover, the fields are now operator fields, so we cannot re-order them as we please. Luckily, and somewhat mysteriously, these two effects manage to cancel each other, so that any quantum tensor Lagrangian must be invariant under the PT transformation.[10])

[10] Here is a very brief sketch of how such a proof goes. Other than the Hermitian conjugation and order worries, the PT transformation is the same as the classical one. Assuming that the Lagrangian is Hermitian, this means that corresponding to any non-Hermitian term in the Lagrangian there is the Hermitian conjugate term. The ordering problem is solved by specifying that the Lagrangian must be normal ordered (all creation operators in front of annihilation operators). Now consider an interaction

5. PT IN SPINOR FIELD THEORIES

Let me start by considering classical non-integer spin fields. I will here restrict attention to Dirac spinor fields. Does including Dirac spinor fields affect the argument for the PT invariance of classical local Lagrangian theories? In order to find that out we need to know how Dirac spinor fields transform under PT. In order to find that out we need to quickly review some spinor theory. (For more detail see, e.g. Wald 1984, Chapter 13, or Bogolubov, Logunov, and Todorov 1975, Chapter 7.)

Let W be a two-dimensional vector space over the complex numbers. The elements ξ^A of W are 'two-component spinors', or spinors for short. Given a choice of two basis vectors we can represent each spinor by a 'column' consisting of two complex numbers. The dual space W^D is the space of all linear maps from elements of W to complex numbers. I will call the elements η_A of W^D 'dual spinors'. We will assume that W comes equipped with an inner product: a bilinear map ε_{AB} from pairs of spinors to the complex numbers. (ε_{AB} thus maps elements of W to elements of W^D and vice versa). Next let us define the group of all SL(2,C) transformations on W to be the linear maps L from W into itself which have unit determinant. (The determinant of L is defined as $\det(L) = \varepsilon_{AB}\varepsilon^{CD}L^A_C L^B_D$.) Given a choice of basis vectors, one can represent each such transformation by a 2x2 complex matrix with unit determinant.

One can then show that the group of SL(2,C) transformations has the same structure as the group of 'restricted' Lorentz transformations. (The 'restricted' Lorentz transformations are the ones that are connected to the identity.) To be precise: one can show that there exists a two-to-one mapping h from the elements L of SL(2,C) to the elements Λ of the restricted Lorentz group such that if $h(L_1) = \Lambda_1$ and $h(L_2) = \Lambda_2$ then $h(L_2 \circ L_1) = \Lambda_2 \circ \Lambda_1$. It will be helpful to have an explicit representation. Here is how one can do that. Pick a Lorentzian frame of reference. Given such a frame of reference each 4-vector V has four components V_0, V_1, V_2, V_3. We can then use these components to define the following associated Hermitian matrix:

term such as the following:

$$W^\mu =: \varphi_a^+(x)\partial^\mu\varphi_b(x):+:\partial^\mu\varphi_b^+(x)\varphi_a(x):$$

(Here : .... : denotes normal ordering.) Under PT this will transform to:

$$-:\varphi_a(-x)\partial^\mu\varphi_b^+(-x):-:\partial^\mu\varphi_b(-x)\varphi_a^+(-x):$$

Given the commutation relations between scalar fields we can re-order this as:

$$-:\partial^\mu\varphi_b^+(-x)\varphi_a(-x):-:\varphi_a^+(-x)\partial^\mu\varphi_b(-x):$$

And that is just equal to $-W^\mu(-x)$. More generally, the commutation relations for integer spin fields combine with the Hermitian conjugation and normal ordering so as to always produce invariance under PT for relativistic tensor quantum field theories.

$$V = \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix}$$

Now, we can use any SL(2,C) matrix L to transform M to a new Hermitian matrix $M' = LML^+$, where L^+ is the complex conjugate transpose of L. The four-vector V' associated with the transformed Hermitian matrix M' will then be a Lorentz transformation of the original four-vector V. Thus we have associated a unique Lorentz transformation with each SL(2,C) matrix. Note that any SL(2,C) matrix L will induce the same transformation as −L. Indeed, it turns out that for every possible restricted Lorentz transformation there exist exactly two associated SL(2,C) matrices. Note, however, that there is no SL(2,C) matrix associated with PT. For PT should map V_0, V_1, V_2, V_3 into $-V_0$, $-V_1$, $-V_2$, $-V_3$, but there is no SL(2,C) matrix L such that $LML^+ = -M$.

In order to have a faithful representation of the full Lorentz group, we need to consider four-component spinors. (The full Lorentz group is the group of all transformations that preserves the Minkowski metric. It includes P, T and PT.) Let me start by defining the complex conjugate dual spinor space W^{*D} to be the space of all anti-linear maps from W to the complex numbers. Next let me define the complex conjugate spinor space W^* to be the space of all linear maps from W^{*D} to the complex numbers. I will denote elements of W^* as $\chi^{A'}$, that is, with a raised primed index, and call them conjugate spinors. I will denote elements of W^{*D} as $\mu_{A'}$, that is, with a lowered primed index, and call them conjugate dual spinors. There is a natural anti-linear 1–1 correspondence between spinors ξ^A and conjugate spinors $\xi^{A'}$ generated by the demand that $\mu_{A'}(\xi^A) = \xi^{A'}(\mu_{A'})$ for all $\mu_{A'}$. It follows that if under a restricted Lorentz transformation ξ^A transforms to $L\xi^A$ then $\xi^{A'}$ transforms to $L^*\xi^{A'}$. (L^* is the SL(2,C) matrix whose entries are the complex conjugates of L's entries.)

Dirac spinors are ordered pairs of spinors and complex conjugate dual spinors. Since we know how spinors and complex conjugate spinors transform under restricted Lorentz transformations, we know how Dirac spinors transform under restricted Lorentz transformations. (We can use ε_{AB} to transform conjugate spinors into conjugate dual spinors and vice versa.) But how about PT? In order to see how to do that let me introduce a relation between Dirac spinors and complex four-vectors V. Given a basis in spinor space, and the associated basis in conjugate dual spinor space, one can define a map from pairs of spinors and conjugate dual spinors to 2x2 complex matrices: $M_\alpha^\beta = \xi_\alpha \mu^{\beta'}$. (Here M_α^β denotes the components of matrix M, ξ_α denotes the components of spinor ξ^A, and $\mu^{\beta'}$ denotes the components of the complex conjugate dual spinor $\mu_{B'}$.) There is also a natural correspondence between such matrices M_α^β and the four components of a complex four-vector V (in some frame of reference), given by the following prescription: $2V_0 = M_1^1 + M_2^2$, $2V_1 = M_1^2 + M_2^1$, $2V_2 = -i(M_1^2 - M_2^1)$, $2V_3 = M_1^1 - M_2^2$. Now, suppose we take an SL(2,C) matrix L and transform spinor ξ^A to $L\xi^A$, and we transform $\mu_{A'}$ to $\varepsilon L^*(\varepsilon)^{-1}\mu_{A'}$. (Here * denotes complex conjugation of the matrix entries of L.) One can show that then the complex four-vector V that is associated with the Dirac spinor $< \xi^A, \mu_{A'} >$ will transform according to the restricted Lorentz transformation that corresponds to L. Moreover, one can generate all the complex Lorentz transformations on V by transforming ξ^A and $\mu_{A'}$ with two independent SL(2,C) matrices, that is, by transforming ξ^A to $L_1\xi^A$ and $\mu_{A'}$ to $\varepsilon L_2^*(\varepsilon)^{-1}\mu_{A'}$. Finally, by looking at the complex Lorentz transformations that correspond to the rotations in complex space, one finds that there are exactly two pairs of SL(2,C) matrices that correspond to PT, namely the pair (Identity, -Identity) and the pair (-Identity, Identity). Obviously, both of these pairs transform the associated matrix M to -M, and hence transform the associated four-vector V to $-V$. So now we know, up to a sign, how Dirac spinors transform under PT. And this is the standard view as to how Dirac spinors transform under CPT!

How about PT invariance? Let's consider an example: the Dirac equation for the free Dirac spinor field. The Dirac equation, written in terms of spinors and complex conjugate dual spinors corresponds to the following pair of equations (see e.g. Peskin and Schroeder 1995. page 44):

(1) $-m\xi^A + i(\partial_0 + \sigma \bullet \nabla)\mu_{A'} = 0$

(2) $i(\partial_0 + \sigma \bullet \nabla)\xi^A - m\mu_{A'} = 0$

(Here σ is the vector consisting of the three Pauli matrices, Δ is the spatial derivative operator, and $\bullet$ denotes the spatial inner product.) These equations are invariant under PT, since all the derivatives change sign under PT, and either $\mu_{A'}$ or ξ^A flips sign under PT. Good!

Can we more generally argue that any local Lagrangian that contains ordinary tensor fields as well as tensor fields constructed from Dirac spinor fields must be invariant under PT? No, we cannot. The problem is that one can construct four-vectors from Dirac spinors that are *invariant* under PT, rather than that they flip over under PT. (That is to say, one can construct 'PT-pseudo-vectors' from spinors, which are types of quantities which Bell excludes *by fiat*.) For instance, consider the 'probability current' $j^\mu = \psi^+\gamma_0\gamma^\mu\psi$, where ψ denotes a Dirac four-spinor, ψ^+ its complex conjugate transpose and the γ^μ denote Dirac's 'gamma matrices'. This probability current transforms like a four-vector under all restricted Lorentz transformations, but is invariant under PT. (Note e.g. that j^0 just equals $\psi^+\psi$ which is positive definite.) So, by using some four-vectors that are invariant under PT and some that are not, one can construct Lagrangians that are not invariant under PT.

Miraculously, this problem goes away when one goes to spinor *quantum* field theories: the anti-commutation relations between spinor quantum fields produce an extra sign flip under PT which makes it impossible to construct PT pseudo-tensors from quantum spinor fields. As yet I find it completely mysterious as to why this happens. But the fact remains that quantum spinor field theories must be invariant under what I have called the PT-transformation.

6. Tentative conclusions

Whether a particle has positive or negative charge is determined by the temporal direction in which the four-momentum of a particle points. What is standardly called the CPT-theorem should be called the PT-theorem. It holds for classical and quantum tensor field theories, fails for classical spinor field theories, but holds for quantum spinor fields. The fact that it holds for quantum field theories suggests that space-time has neither a temporal orientation nor a spatial handedness.

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