A Density Criterion for Frames of Complex Exponentials

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1. Introduction

The notion of a frame has been introduced by Duffin and Schaeffer in [1]. It can be defined in a general Hilbert space H as follows. A sequence (e_n) of vectors of H is a *frame* if there exist positive constants C_1 and C_2 such that, for all f in H,

$$C_1\|f\|^2 \le \sum |\langle f \mid e_n\rangle|^2 \le C_2\|f\|^2. \tag{1}$$

Frames are important in the study of complex exponentials (cf. [1] and the book of R. M. Young on nonharmonic Fourier series [3]).

The following problem will be studied in this paper. Let $\Lambda = (\lambda_n)_{n\in Z}$ be a sequence of distinct real numbers. What is the upper bound of all numbers R such that the sequence of functions $(e^{i\lambda_n t})$ is a frame of $L^2([-R,R])$? This number, denoted $R(\Lambda)$, will be called the *frame radius* of the sequence Λ. Partial results were found by Duffin and Schaeffer [1] and Landau [2]. They are summarized in Theorems 1 and 2. The goal of the present paper is to give a necessary and sufficient condition for Λ to have a strictly positive finite frame radius, and, when it does, to obtain a formula for that radius.

We shall consider only sequences with distinct λ_n's since the general case can be dealt with as follows. The frame radius of the sequence λ_n is not changed if we repeat some λ_n's a finite and uniformly bounded number of times. If the number of repetitions is not bounded, the functions $(e^{i\lambda_n t})$ can never be a frame on any interval. Note also that, if the sequence of functions $(e^{i\lambda_n t})$ is a frame for the interval I, it is also a frame for each subinterval of I.

The reference space is $L^2(I)$, where I is a finite interval, and the inner product is given by

$$\langle f \mid g\rangle = \frac{1}{|I|}\int_I f(t)\bar{g}(t)\,dt,$$

where $|I|$ denotes the length of the interval. We denote by C, C_1, and C_2 constants which can change from one line to the next.

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2. Some Definitions and Results

A sequence Λ is said to be *separated* if

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0.$$

If Λ is separated, it has a *uniform density* $d(\Lambda)$ if there exists a number L such that, for all integers n,

$$\left|\lambda_n - \frac{n}{d(\Lambda)}\right| \leq L.$$

Duffin and Schaeffer [1] proved the following theorem.

THEOREM 1. *If Λ is a sequence of uniform density $d(\Lambda)$, then the frame radius of Λ is at least $\pi d(\Lambda)$.*

Let Λ be a separated sequence and $n^-(r)$ the smallest number of λ_i in any interval of length r. The following result has been obtained by Landau [2].

THEOREM 2. *If Λ is a separated sequence, then the lower uniform density of Λ, defined as*

$$D^-(\Lambda) = \lim_{r\to\infty} \frac{n^-(r)}{r},$$

always exists, and the frame radius of Λ is at most $\pi D^-(\Lambda)$.

Let $U(\Lambda)$ be the set of all the subsequences of Λ with a uniform density. Then the "frame density" of Λ is defined by

$$D^f(\Lambda) = \sup_{\Theta\in U(\Lambda)} d(\Theta). \tag{2}$$

In this paper, the following result will be proved.

THEOREM 3. *Let Λ be a sequence of distinct real numbers. Then: If $U(\Lambda) = \emptyset$, or if the numbers A_n of elements of $\Lambda \cap [n, n+1]$ are not bounded (n taking all integer values), then there exists no interval over which $(e^{i\lambda_n t})$ is a frame. Otherwise, the frame radius of Λ is equal to $\pi D^f(\Lambda)$.*

The proof of Theorem 3 is divided into two parts. In the first part, we obtain a "qualitative" result on a convenient partitioning of the λ_n when the sequence of functions $e^{i\lambda_n t}$ is a frame over a certain interval, thus proving the first part of Theorem 3 and half of the last equality; in the second part, we complete the frame radius equality.

3. A Partitioning of Λ

The first lemma is a direct consequence of the definition of a frame.

LEMMA 1. *Let I be a finite interval. If $(e^{i\lambda_n t})$ is a frame of $L^2(I)$, then the number of points λ_n inside each interval of length 1 is uniformly bounded.*

Proof. Suppose that $(e^{i\lambda_n t})$ is a frame of $L^2([-a, a])$, since the position of the interval I is obviously of no importance. Let ϵ be chosen so that, for all η with $|\eta| \leq \epsilon$,

$$\left|\frac{1}{2a}\int_{-a}^{a} e^{i\eta t}\,dt\right|^2 \geq \frac{1}{2}. \tag{3}$$

Suppose now that the number of λ_n inside an interval of length 1 is not bounded; then neither is the number of λ_n inside an interval of length ϵ. Thus, there exists a sequence μ_k of real numbers with the following property: the number of λ_n inside $[\mu_k-\epsilon, \mu_k+\epsilon]$ is at least k. Let f_k be the function $e^{i\mu_k t}$. Then, an immediate consequence of (3) is that

$$\sum_n |\langle f_k | e^{i\lambda_n t}\rangle|^2 \geq \frac{k}{2}.$$

But $\|f_k\| = 1$, so that the second inequality of (1) cannot hold, and the contradiction proves the lemma. □

The following lemma gives the structure of all sequences $\Lambda = (\lambda_n)$ such that $(e^{i\lambda_n t})$ is a frame of $L^2(I)$, for a certain interval I.

LEMMA 2. *The following two assertions are equivalent.*

(a) *There exists I such that $(e^{i\lambda_n t})_{\lambda_n \in \Lambda}$ is a frame of $L^2(I)$.*

(b) *Λ is the disjoint union of a sequence with a uniform density (denoted by d_1) and a finite number of separated sequences.*

Furthermore, if (b) *holds, then $(e^{i\lambda_n t})$ is a frame of $L^2(I)$ for each I such that $|I| < 2\pi d_1$. Hence $R(\Lambda) \geq \pi d_1$.*

Proof. Let us prove (b)$\Rightarrow$(a). Let $\Lambda = \Lambda^1 \cup \cdots \cup \Lambda^n$, where Λ^1 has a positive uniform density d_1 and $\Lambda^2, \ldots, \Lambda^n$ are separated. By Theorem 1, $(e^{i\lambda t})_{\lambda \in \Lambda^1}$ is a frame for each interval of length less than $2\pi d_1$; denote one such interval by I. Then there exist positive constants C_1 and C_2 such that, for all f in $L^2(I)$,

$$C_1\|f\|^2 \leq \sum_{\lambda \in \Lambda^1} |\langle f | e^{i\lambda t}\rangle|^2 \leq C_2\|f\|^2. \tag{4}$$

A direct computation (performed in [1]) shows that, for each separated sequence Λ' and each interval I, there exists a constant C such that for all f in $L^2(I)$,

$$\sum_{\lambda\in\Lambda'} |\langle f\,|\,e^{i\lambda t}\rangle|^2 \le C\|f\|^2. \tag{5}$$

Hence, there are C_j' for $j=2,\dots,n$, such that

$$\sum_{\lambda\in\Lambda_j} |\langle f\,|\,e^{i\lambda t}\rangle|^2 \le C_j'\|f\|^2.$$

Adding inequalities, Λ satisfies the inequalities of (1). Hence (b)$\Rightarrow$(a) and the last statement of Lemma 2 is established.

We now prove (a)$\Rightarrow$(b). It is sufficient to prove that there exists an $N>0$ and a C_N larger than 1 such that, for each integer k, the number A_N^k of λ_i in each inteval $[kN,(k+1)N)$ lies between 1 and C_N. For, if it is so, we can define a subsequence μ_k of λ_i by picking one of the λ_i in each interval $[2kN,(2k+1)N)$. The μ_k satisfy $|\mu_{k+1}-\mu_k|>N$ and $|\mu_k-2kN|<N$; thus the μ_k will form a sequence having a uniform density. The remaining λ_i can then be divided into at most $2C_N-1$ separated sequences by picking at most one λ_i in each interval of the form $[2kN,(2k+1)N)$ (for C_N-1 sequences) or of the form $[(2k+1)N,(2k+2)N)$ for the remaining sequences.

We now proceed to show the existence of such an N. Because of Lemma 1, for each N, the number of λ_i in each interval $[kN,(k+1)N)$ is uniformly bounded. So it is sufficient to prove that each A_N^k is at least 1 for some N. If this were not the case, then for each N we could pick a half-open interval of length N such that no λ_k lies in this interval. Let μ_N be the center of this interval, and let $f_N(t)=e^{i\mu_N t}$. Then

$$|\langle f_N\,|\,e^{i\lambda_k t}\rangle|^2 = \left|\frac{2\sin((\lambda_k-\mu_N)|I|/2)}{|I|(\lambda_k-\mu_N)}\right|^2 \le \frac{4}{|I|^2|\lambda_k-\mu_N|^2}.$$

By Lemma 1, there are at most C_1 numbers λ_k in the interval $[n,n+1)$, and there are none if $|n-\mu_N|<N/4$ (for $N>4$). Thus

$$\begin{aligned}\sum_k |\langle f_N\,|\,e^{i\lambda_k t}\rangle|^2 &= \sum_{n\in Z}\ \sum_{\lambda_k\in[n,n+1)} |\langle f_n\,|\,e^{i\lambda_k t}\rangle|^2\\ &= \sum_{n,\,|n-\mu_N|>N/4} \frac{4C_1}{|I|^2(|\mu_N-n|-1)^2}\\ &\le \sum_{|n|>N/4} \frac{C'}{(|n|-2)^2}\\ &\le \frac{C''}{N} \quad (\text{for } N>4).\end{aligned}$$

Since $\|f_N\|=1$, if N is chosen large enough then a contradiction with the first inequality of (1) is obtained, and the first part of Lemma 2 follows. □

Because of Lemma 2, we shall assume from now on that all the sequences we consider are finite disjoint unions of separated sequences.

Some of the conclusions of Theorem 3 follow immediately from Lemmas 1 and 2. When $U(\Lambda)$ is empty, $(e^{i\lambda_n t})$ cannot be a frame by Lemma 2. When the cardinality of $\Lambda\cap[n,n+1]$ is unbounded, $(e^{i\lambda_n t})$ cannot be a frame by Lemma 1. If $(e^{i\lambda_n t})$ is a frame over some interval, by Lemma 2, Λ is a disjoint union of separated sequences. Consequently, each subsequence of Λ is such a union. If Θ is in $U(\Lambda)$ and has density $d(\Theta)$, then $\Lambda=\Theta\cup(\Lambda\backslash\Theta)$, where $\Lambda\backslash\Theta$ is a disjoint union of separated sequences. Again, using Lemma 2, we see that $R(\Lambda)\geq\pi d(\Theta)$. Hence

$$R(\Lambda)\geq\pi\sup_{\Theta\in U(\Lambda)} d(\Theta)=\pi D^f(\Lambda).$$

The purpose of the next section is to complete the proof of Theorem 3 by showing that $R(\Lambda)=\pi D^f(\Lambda)$.

It is perhaps worth noting that when Θ is a separated sequence of uniform density $d(\Theta)$, then $D^-(\Theta)=d(\Theta)$; this together with Theorem 2 establishes the conclusion of Theorem 3 in this case. Similarly, a slight improvement of this argument leads to the same conclusion if Θ is only separated. The main difficulty we shall have to deal with in the next part will come from the fact that Θ may not be separated.

4. A Determination of the Frame Radius

The key ingredient in this determination is given by the following proposition.

PROPOSITION 1. *Let $\Lambda^1=(\lambda_n^1)$ and $\Lambda^2=(\lambda_n^2)$ be two disjoint sequences of distinct real numbers such that*

$$|\lambda_n^1-\lambda_n^2|\to 0 \quad \textit{when} \quad |n|\to\infty;$$

let us also suppose that $R(\Lambda^1)$ exists. Then

$$R(\Lambda^1)=R(\Lambda^1\cup\Lambda^2).$$

The proof of Proposition 1 will use the two auxiliary lemmas that follow.

LEMMA 3. *If a sequence of vectors e_n is a frame of a Hilbert space H, then the mapping $T\colon l^2\to H$ defined by*

$$T((a_n))=\sum a_n e_n$$

is continuous and onto.

Proof. Let $g\in H$. Then

$$\begin{aligned}|\langle T((a_n))\,|\,g\rangle| &= |\textstyle\sum a_n\langle e_n\,|\,g\rangle|\\ &\leq \|(a_n)\|\,(\textstyle\sum\langle e_n\,|\,g\rangle^2)^{1/2}\\ &\leq C\|(a_n)\|\,\|g\|.\end{aligned}$$

Hence T is continuous. Thus, to prove that T is onto it is sufficient to prove that, for any f in H, if f is orthogonal to all the e_n then $f=0$. But this is a consequence of the first inequality of (1). □

LEMMA 4. *Suppose that a sequence of functions $(e_n)_{n\in Z}$ is a frame of $L^2(I)$. Then $(e_n)_{n\neq 0}$ is a frame on each interval $I'\subset I$ such that $|I'|<|I|$.*

Proof. The $(e_n)_{n\in Z}$ are a frame of $L^2(I')$. Then, either $(e_n)_{n\neq 0}$ is a frame of $L^2(I')$, and we have nothing to prove, or the $(e_n)_{n\in Z}$ are a Riesz basis of $L^2(I')$ (cf. [3, p. 186]). We now make this assumption.

Let f be a square integrable function defined on I, and vanishing on I' but not on I. By Lemma 3, $f=\sum a_n e_n$, with (a_n) in l^2. Since the $(e_n)_{n\in Z}$ are a Riesz basis of $L^2(I')$, and f vanishes on I', we obtain that $a_n=0$ for all n. Hence, f vanishes on I, and a contradiction is obtained. □

Proof of Proposition 1. The proposition will be proved in two steps. The first step is to prove it under the stronger assumption that

$$|\lambda_n^1-\lambda_n^2|\leq \frac{1}{n^2}.$$

If this assumption holds, then

$$\begin{aligned}|\langle f\,|\,e^{i\lambda_n^1 t}\rangle-\langle f\,|\,e^{i\lambda_n^2 t}\rangle| &\leq \|f\|\,\|e^{i\lambda_n^1 t}-e^{i\lambda_n^2 t}\|\\ &\leq C\|f\|\,|\lambda_n^1-\lambda_n^2|\\ &\leq C\frac{\|f\|}{n^2},\end{aligned}$$

so that

$$\big||\langle f\,|\,e^{i\lambda_n^1 t}\rangle|^2-|\langle f\,|\,e^{i\lambda_n^2 t}\rangle|^2\big|\leq C'\frac{\|f\|^2}{n^2}.$$

We saw that $R(\Lambda^1\cup\Lambda^2)\geq R(\Lambda^1)$. Let I be an interval over which $(e^{i\lambda t})_{\lambda\in\Lambda_1\cup\Lambda_2}$ is a frame. There exist C_1 and C_2 such that

$$C_1\|f\|^2\leq \sum_{\lambda\in\Lambda_1}|\langle f\,|\,e^{i\lambda_n^1 t}\rangle|^2+\sum_{\lambda\in\Lambda_2}|\langle f\,|\,e^{i\lambda_n^2 t}\rangle|^2\leq C_2\|f\|^2.$$

Let N be such that

$$C'\sum_{|n|\geq N}\frac{1}{n^2}\leq\frac{C_1}{2}.$$

Then

$$\begin{aligned}&\sum|\langle f\,|\,e^{i\lambda_n^1 t}\rangle|^2+\sum|\langle f\,|\,e^{i\lambda_n^2 t}\rangle|^2\\ &\quad\leq \sum_{|n|<N}|\langle f\,|\,e^{i\lambda_n^1 t}\rangle|^2+\sum_{|n|<N}|\langle f\,|\,e^{i\lambda_n^2 t}\rangle|^2+2\sum_{|n|\geq N}|\langle f\,|\,e^{i\lambda_n^1 t}\rangle|^2+\frac{C_1}{2}\|f\|^2,\end{aligned}$$

so that

$$\frac{C_1}{4}\|f\|^2\leq\sum|\langle f\,|\,e^{i\lambda_n^1 t}\rangle|^2+\sum_{|n|<N}|\langle f\,|\,e^{i\lambda_n^2 t}\rangle|^2\leq C_2\|f\|^2.$$

Lemma 4 means that the frame radius is not changed by deleting one element of a sequence (hence also by deleting a finite number), so that Proposition 1 holds under the assumption $|\lambda_n^1-\lambda_n^2|\leq 1/n^2$. The general case will be a consequence of the following lemma (proved in [1]).

LEMMA 5. *Let $(e^{i\lambda_n t})$ be a frame over I. There exists $\delta_1 > 0$ such that $(e^{i\mu_n t})$ is a frame over the same interval whenever (μ_n) is a real sequence such that $|\mu_n - \lambda_n| \le \delta_1$.*

We can now complete the proof of Proposition 1.

Suppose that $|\lambda_n^1 - \lambda_n^2| \to 0$, and let R be less than $R(\Lambda^1 \cup \Lambda^2)$. Then $(e^{i\lambda_n^1 t}) \cup (e^{i\lambda_n^2 t})$ is a frame over $[-R, R]$. Let δ_1 be as in Lemma 5. Change λ_n^2 into $\mu_n = \lambda_n^1 + 1/n^2$, if n is such that $1/n^2 \le \delta_1/2$ and $|\lambda_n^1 - \lambda_n^2| \le \delta_1/2$. By Lemma 5, the set of functions $(e^{i\lambda_n^1 t}) \cup (e^{i\mu_n t})$ is a frame over $[-R, R]$. But, since $|\lambda_n^1 - \mu_n| = 1/n^2$ for n large enough, $R \le R(\Lambda^1)$. Thus

$$R(\Lambda^1) \le R(\Lambda^1 \cup \Lambda^2),$$

and hence Proposition 1 is proved. □

Let us call $V(\Lambda)$ the set of all the subsequences of Λ that are separated. Then the following lemma holds.

LEMMA 6. *For any sequence Λ such that the cardinality of $\Lambda \cap [n, n+1]$ is bounded, the following equality holds:*

$$\sup_{\Theta \in V(\Lambda)} D^-(\Theta) = \sup_{\Theta \in U(\Lambda)} d(\Theta) = D^f(\Lambda).$$

Proof. A sequence of uniform density is separated, so that

$$\sup_{\Theta \in V(\Lambda)} D^-(\Theta) \ge \sup_{\Theta \in U(\Lambda)} d(\Theta)$$

because, for a sequence Θ with a uniform density, $d(\Theta) = D^-(\Theta)$. Suppose that μ_n is a separated subsequence of Λ, with a lower uniform density D^-. Let $\epsilon > 0$. Choose R large enough so that $R(D^- - \epsilon)$ is an integer and

$$\frac{n^-(R)}{R} \ge D^- - \epsilon.$$

In each interval $[kR, (k+1)R)$ there are at least $R(D^- - \epsilon)$ numbers μ_n. Extract a subsequence (γ_n) of (μ_n) that has exactly $R(D^- - \epsilon)$ elements in each of these intervals. The sequence (γ_n) is separated and

$$\left| \gamma_n - \frac{n}{D^- - \epsilon} \right| \le R$$

so that (γ_n) has a uniform density $D^- - \epsilon$, and

$$\sup_{\Theta \in U(\Lambda)} d(\Theta) \ge D^- - \epsilon.$$

This proves Lemma 6. □

The two following lemmas are a slight generalization of Theorem 2.

LEMMA 7. *If Λ is a finite union of separated sequences, then the limit*

$$D^-(\Lambda) = \lim_{r \to \infty} \frac{n^-(r)}{r}$$

exists; $D^-(\Lambda)$ is said to be the lower density of Λ.

Proof. Define, for $r \geq 1$, the function $Q(r) = n^-(r)/r$. Let us first establish three simple properties of the function Q. Because of Lemma 1, there exists a constant C such that $n^-(r) \leq Cr$. Thus Q is a bounded function.

Let p be an integer and I an interval of length pr. Let us write I as a disjoint union of p intervals $I_1, \ldots, I_p$, each of length r. For each k, the cardinality of $(\Lambda \cap I_k)$ is at least $n^-(r)$, so that the cardinality of $(\Lambda \cap I)$ is at least $pn^-(r)$; thus

$$Q(pr) \geq Q(r).$$

Let $\alpha > 1$. Obviously $n^-(\alpha r) \geq n^-(r)$, so that

$$Q(\alpha r) \geq \frac{1}{\alpha} Q(r).$$

Let

$$\bar{Q} = \sup_{r \geq 1} Q(r).$$

Given ϵ in $(0, 1/2)$, choose a positive and a positive integer n such that $Q(a) \geq \bar{Q} - \epsilon$ and

$$\frac{n+1}{n} \leq \frac{1}{1-\epsilon}.$$

Consider $x \geq na$. There exists an integer p at least equal to n such that

$$pa \leq x < (p+1)a.$$

Then

$$\begin{aligned} Q(x) = Q\left(\frac{x}{pa} pa\right) &\geq \frac{pa}{x} Q(pa) \geq \frac{pa}{x} Q(a) \geq \frac{pa}{x} (\bar{Q} - \epsilon) \\ &\geq \frac{pa}{(p+1)a} (\bar{Q} - \epsilon) \geq \frac{n}{n+1} (\bar{Q} - \epsilon) \geq (1-\epsilon)(\bar{Q} - \epsilon). \end{aligned}$$

Hence Lemma 7 is established. □

LEMMA 8. *If Λ is a finite union of separated sequences, the frame radius of Λ is at most $\pi D^-(\Lambda)$.*

Proof. Let Λ be a finite union of disjoint separated sequences. Then, for each δ, one can find a single separated sequence Λ' such that

$$|\lambda_n - \lambda'_n| < \delta.$$

From Lemma 5, if δ is small enough then the frame radius of Λ' will be at least the frame radius of Λ. But

$$D^-(\Lambda) = D^-(\Lambda').$$

Because of Theorem 2,

$$R(\Lambda') \leq \pi D^-(\Lambda'),$$

so that

$$R(\Lambda) \leq \pi D^{-}(\Lambda)$$

and Lemma 8 follows. □

We can now complete the proof of Theorem 3. It remains only to show that

$$R(\Lambda) \leq \pi D^{f}(\Lambda)$$

for a sequence Λ which is the union of k separated sequences.

The idea of the proof is to split Λ into a separated sequence Ω of uniform density at least $D^{f}(\Lambda)-\epsilon$, a finite union of sequences Γ^{i} each of which tends to Ω (or a subsequence of Ω), and a remaining sequence Θ of lower uniform density at most $3k\epsilon$.

Suppose that such a splitting is achieved. Then, by Proposition 1, we can disregard the Γ^{i} in the calculation of the frame radius, and by Lemma 8, the frame radius of $\Omega \cup \Theta$ is at most $\pi(D^{f}(\Lambda)+3k\epsilon)$. Thus Theorem 3 will be proved once this splitting of Λ is constructed.

Let ϵ be fixed. We can extract from Λ a sequence $\Omega=(\omega_n)$ which has a uniform density at least $D^{f}(\Lambda)-\epsilon$ and is separated. We now construct the sequence $\Theta=(\theta_n)$ by induction. We do this construction only for positive values of λ_n; it is the same for the negative values. Let

$$E_1 = \bigcup_{\omega_n \geq 0} [\omega_n - 1, \omega_n + 1].$$

Let $\Theta^{1}=(\theta_n^{1})$ be the subsequence of Λ composed of all the $\lambda_n > 0$ which are not in E_1. The sequence Θ^{1} is the union of at most k separated sequences, none of which has a density larger than 2ϵ; for, if such a subsequence Σ had a density larger than 2ϵ, then the union of Σ and Ω would be a separated subsequence of Λ with a density at least $D^{f}(\Lambda)+\epsilon$, which is impossible. Thus $D^{-}(\Theta^{1}) \leq 2k\epsilon$, and there exists an interval I_1 large enough such that the number of θ_n^{1} in I_1 is less than $3k\epsilon|I_1|$.

Let $A_1 = \sup I_1$ if I_1 does not intersect E_1; otherwise, let p be the largest integer such that

$$I_1 \cap [\omega_p - 1, \omega_p + 1] \neq 0;$$

then $A_1 = \omega_p + 1$. The beginning of the construction of Θ is as follows: $\theta_n = \theta_n^{1}$ if $\theta_n^{1} \leq A_1$.

The induction now works as follows. We suppose that Θ is constructed for $\theta_n \leq A_{m-1}$. We now define the set

$$E_m = \bigcup_{\omega_n \geq A_{m-1}} \left[\omega_n - \frac{1}{m}, \omega_n + \frac{1}{m}\right],$$

and the sequence Θ^{m} which is composed of the λ_n larger than A_{m-1} that are not in E_m. We can find by the same argument as above an interval I_m included in $[A_{m-1}, +\infty)$, of length at least m and such that the number of elements of the sequence Θ^{m} in I_m is less than $3k\epsilon|I_m|$. Let $A_m = \sup I_m$ if I_m does not intersect E_m; otherwise, let p be the largest integer such that

$$I_m \cap \left[\omega_p - \frac{1}{m}, \omega_p + \frac{1}{m}\right] \neq 0;$$

then $A_1 = \omega_p + 1/m$. The sequence Θ for $A_m < \theta_n \leq A_{m+1}$ is composed of the elements of Θ^m in the same interval.

Once the construction of Θ is achieved, we have finally split Λ into a sequence Ω of uniform density $A - \epsilon$, a sequence Θ of lower density less than $3k\epsilon$ (because the number of elements of Θ in I_N is at most $3k\epsilon |I_N|$), and a remaining sequence included in a set

$$E = \bigcup [\mu_n - \alpha_n, \mu_n + \alpha_n],$$

where the α_n are certain $1/m$, and are such that $\alpha_n \to 0$; this sequence can obviously be written as a finite union of sequences Γ_i, each of which tends to Ω or a subsequence of Ω. The requested splitting is thus performed and Theorem 3 is proved. □

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