A distributional Hilbert space framework to prove the Landau damping phenomenon

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Abstract:

In [BrK3] a common distributional Hilbert space framework is provided to answer the Riemann Hypothesis positively and to prove appropriate estimates of the non-linear term of the 3-D NSE ([BrK4]) to solve the related Millennium problems. Additionally, the distributional Hilbert space framework is used to define an alternative Schrödinger momentum operator ([BrK5]) (where the Hilbert transform resp. the Riesz transforms play a key role), addressing also the YME problem and the related quantum gravity modelling problem. In this note we build on the same framework providing an alternative approach to [MaC], [ViC] to prove the Landau damping. At the same point in time the usage of the Hilbert transform overcomes the problem of the incorrect Vlasov formula

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k} - v}$$

which is about the plasma dielectric for the longitudinal oscillation. It is divergent in case of the important physical phenomenon of electrons travelling with exactly the same material speed $\frac{\omega}{k}$ and the wave speed v ([ShF] p. 392).

A comprehensive overview of all related topics of this paper (variational principle /method, Green function, fluctuations and light scattering, Schauder / Hölder spaces, transport coefficients, Planck distribution, Hermite polynomials, equilibrium states & Maxwellian distributions, Navier-Stokes-Fourier fluids) are provided in ([CeG].

The considered distributional Hilbert space framework is enriched with an additional norm enjoying an "exponential decay" behavior in the form $(t > 0)$

$$(x, y)_{(t)} = \sum_k e^{-\sqrt{\sigma_k} t}(x, \varphi_k)(y, \varphi_k) \quad , \quad |||x|||_{(t)}^2 := (x, x)_{(t)} .$$

For the linear Vlasov equation the corresponding modes estimates are given in the form

$$\bar{\rho}_k^2(t) := e^{-|k|t} \rho_k^2(t) \leq e^{c\bar{g}W_k^2|k|t}$$

In case of a Coloumb potential ($W_k \approx \frac{1}{|k|}$) this leads to $\bar{\rho}_k^2(t) \leq e^{c\frac{t}{|k|}}$, which means that for fixed $t > 0$ the $H_\alpha -$ norm ($\alpha \in R$) related generalized Fourier term is damped for $|k| \rightarrow \infty$.

For the Landau equation the corresponding non-linear Landau collision operator is related to the Oseen operator with the symbol ([LeN]

$$a_{ij}(z) = \frac{1}{|z|}\left\{\delta_{ij} - \frac{z_i z_j}{|z|^2}\right\} := \frac{1}{|z|}[Id - Q](z) \quad Q(z) := (R_i R_j)_{1 \leq i,j \leq N} .$$

Its eigen-pair solutions (resp. the eigen-pair solution of the Laplacian resp. the Stokes operator ([BrK4])) enable the definition of corresponding Hilbert scales with norm $\|x\|_\alpha^2$, and an additional norm $|||x|||_{\alpha.(t)}^2$ ([BrK1] [BrK2]). An alternative (weak) Landau equation is proposed enabling a Hilbert space based analysis of the Landau damping phenomenon in a problem adequate Hilbert space framework, which is in sync with the corresponding solution framework of the 3D nonlinear, non-stationary NSE ([BrK4]). Putting $h := \underline{R}\nabla_x f$ we propose a $H_{-1/2}$ based variational representation of the Landau equation in the following form

$$(\frac{\partial}{\partial t}h, w)_{-1/2} + (v \cdot \nabla_x h, w)_{-1/2} + (F \cdot \nabla_v h, w)_{-1/2} - (\nabla_v(hL[h]), w)_{-1/2} = 0 \qquad \forall w \in H_{-1/2} .$$

whereby $\boldsymbol{L}[h]$ denotes a model collision (integral) operator of order zero (based on the Leray-Hopf operator $P(z)$ of order zero ([BrK4], [CoP] p. 115ff) given by

$$L[h] := \int_{R^N} b_{ij}(v - w)h(w)dw$$

with symbol $b_{ij}(z) = z \cdot a_{ij}(z)$ As the operator L is of order zero, it holds

$$(\nabla_v(h\mathbf{L}[h]), w)_{-1/2} \cong (\nabla_v h^2, w)_{-1/2} .$$

In ([LeN] the action of the Leray-Hopf operator on Gaussian functions is provided. In [WeP], [WeP1] self-adjoint extensions of the Laplacian operator with respect to electric and magnetic boundary conditions are provided. In combination with the proposed problem adequate norm of the related evolution (heat) equation the correspondingly defined Hilbert space provides the appropriate framework for a well-posed (variational) initial-boundary value equation representation of a corresponding Vlasov-Poisson/Lorentz-Boltzmann system.

1. A $H_{1/2}-$ (energy) Hilbert space framework for an integrated electro (kinetic) & magnetic (dynamic) (Coulomb & Lorentz potential) plasma field model

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Plasma is the fourth state of matter, where from general relativity and quantum theory it is known that all of them are fakes resp. interim specific mathematical model items. An adequate model needs to take into account the *axiom of (quantum) state* (physical states are described by vectors of a separable Hilbert space ***H***) and the *axiom of observables* (each physical observable ***A*** is represented as a linear Hermitian operator A of the state Hilbert space). The corresponding mathematical model and its solutions are governed by the Heisenberg uncertainty inequality. As the observable space needs to support statistical analysis the L_2 −Hilbert space, this Hilbert space needs to be at least a subspace of ***H***. At the same point in time, if plasma is considered as sufficiently collisional, then it can be well-described by fluid-mechanical equations. There is a hierarchy of such hydrodynamic models, where the magnetic field lines (or magneto-vortex lines) at the limit of infinite conductivity is "frozen-in" to the plasma. The "mother of all hydrodynamic models is the *continuity equation* treating observations with macroscopic character, where fluids and gases are considered as continua. The corresponding infinitesimal volume "element" is a volume, which is small compared to the considered overall (volume) space, and large compared to the distances of the molecules. The displacement of such a volume (a fluid particle) then is a not a displacement of a molecule, but the whole volume element containing multiple molecules, whereby in hydrodynamics this fluid is interpreted as a mathematical point.
Our approach below is based on the common Hilbert space framework in [BrK3] and the proposed alternative Schrödinger momentum operator ([BrK5], [BrK9]) given by

$$u(x) \rightarrow P^*[u](x) := -i\frac{d}{dx}H[u](x) := -i\frac{d}{dx}\,{}_xH[u](x) = -iH[u_x](x)$$

with domain $H_{1/2} = H_1 + H_1^{\neg}$ and corresponding quantum state Hilbert space $H_{-1/2} = H_0 + H_0^{\neg}$. The Hilbert transform is related to the Laplace equation by the concept of conjugate functions. The corresponding generalization with respect to the Yukawan potential is provided in [DuR].

In quantum mechanics, a boson is a "particle" that follows the Bose-Einstein statistics ("photon gases"). A characteristic of bosons is that their statistics do not restrict the number of them that occupy the same quantum state. All bosons can be brought into the energetically lowest quantum state, where they show the same "collective" behavior. Unlike bosons, two identical fermions cannot occupy the same quantum state. Fermions follow the Fermi statistics (e.g. [AnJ]). With respect to the above extended domain of the Schrödinger momentum operator, we propose to identify H_0 as quantum state space for the fermions (which is compactly embedded into $H_{-1/2}$), and $H_0^{\neg}$ as quantum state space for the bosons. *The "fermions quantum state" Hilbert space H_0 is dense in $H_{-1/2}$ with respect to the $H_{-1/2}$ −norm, while the (orthogonal) "bosons quantum state" Hilbert space $H_0^{\neg}$ is a closed subspace of $H_{-1/2}$, resp. the "mass/energy fermions" Hilbert space H_1 is dense in $H_{1/2}$ with respect to the $H_{1/2}$ −norm, while the "mass/energy bosons" Hilbert space is a closed subspace of $H_{1/2}$.* The concept of "vacuons" (i.e. the vacuum expectation values of scalar fields) in the context of "spontaneous" breakdown of symmetry [HiP] then corresponds to the orthogonal projection $H_{1/2} \rightarrow H_1$. For related topics see also [BrK1-5], [BrK6] Notes O58-O65, O69-O71.

In section 4 below, we provide a distributional variational Hilbert space framework for Landau type equations being enriched by an additional norm with an "exponential decay" behavior in the form ($t>0$) (in line with the statistics above) given by ([BrK5])

$$(x,y)_{\alpha.(t)}=\sum_k \sigma_k^{\alpha} e^{-\sqrt{\sigma_k}t}(x,\varphi_k)(y,\varphi_k) \quad , \quad \|x\|_{\alpha.(t)}^2 := (x,x)_{\alpha.(t)}.$$

It is based on appropriately defined eigen-pair solutions of a problem adequate linear operator A with the properties (1) A selfadjoint, positive definite, (2) A^{-1} compact.

An element $x=x_0+x_0^{\neg} \in H_{-1/2}=H_0+H_0^{\neg}$ with $\|x_0\|_0=1$ is governed by the norm of its (observation) subspace H_0 in combination with the norm $|||x|||_{(t)}^2 := (x,x)_{(t)}$ ([BrK3], [BrK5]) in the form

$$\|x\|_{-1/2}^2 \leq \theta \|x\|_0^2 + \sum_{k=1}^{\infty} e^{1-\sqrt{\sigma_k}\theta} x_k^2 \qquad \text{with} \quad \theta := \|x_0^{\neg}\|_{-1/2}^2 \ ,$$

which is a special case of the general inequality ($\alpha>0$ be fixed)

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2.$$

Plasma is an ionized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons. One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not. The continuity equation of ideal magneto-hydrodynamics is given by ([DeR] (4.1))

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0$$

with $\rho=\rho(x,t)$ denoting the mass density of the fluid and $\boldsymbol{v}$ denoting the bulk velocity of the macroscopic motion of the fluid. The corresponding microscopic kinetic description of plasma fluids leads to a continuity equation of a system of (plasma) "particles" in a phase space $(\boldsymbol{x},\boldsymbol{v})$ (where $\rho(\boldsymbol{x},t)$ is replaced by a function $f(\boldsymbol{x},\boldsymbol{v},t)$) given by ([DeR] (5.1))

$$\frac{\partial}{\partial t}f + v\cdot\nabla_x f + \frac{dv}{dt}\cdot\nabla_v f + f\frac{\partial}{\partial v}\cdot\frac{dv}{dt} = 0 \ .$$

In case of a Lorentz force the last term is zero, leading to the so-called collisions-less (kinetic) Vlasov equation ([ShF] (28.1.2)).

In fluid description of plasmas (MHD) one does not consider velocity distributions (e.g. [GuR]). It is about number density, flow velocity and pressure. This is about moment or fluid equations (as NSE and Boltzmann/Landau equations). In [EyG] it is proven that smooth solutions of non-ideal (viscous and resistive) incompressible magneto-hydrodynamic (plasma fluid) equations satisfy a stochastic (conservation) law of flux. It is shown that the magnetic flux through the fixed surface is equal to the average of the magnetic fluxes through the ensemble of surfaces at earlier times for any (unit or general) value of the magnetic Prandtl number. For divergence-free $\vec{z}=(\vec{\mathrm{u}},\vec{B}) \in C([t_0,t_f],C^{k.\alpha}), (\vec{\mathrm{u}}(0),\vec{B}(0)) \in C^{k.\alpha}$ the key inequalities are given by

- unit magnetic Prandtl number:

$$e^{-2\gamma(t_f-t_0)}\left\|\vec{z}(t_f)\right\|_2^2 + 2\int_{t_0}^{t_f} e^{-2\gamma(t-t_0)}[\varepsilon\|\vec{z}(t)\|_2^2 + \mu\|\nabla\vec{z}(t)\|_2^2]dt \leq \|\vec{z}(0)\|_2^2$$

- general magnetic Prandtl number (→ stochastic Lundquist formula):

$$e^{-2\gamma(t_f-t_0)}\left\|\vec{B}(t_f)\right\|_2^2 + 2\int_{t_0}^{t_f} e^{-2\gamma(t-t_0)}\left[\varepsilon\left\|\vec{B}(t)\right\|_2^2 + \mu\left\|\nabla\vec{\mathrm{B}}(t)\right\|_2^2\right]dt \leq \left\|\vec{B}(0)\right\|_2^2 \ .$$

The corresponding situation of the fluid flux of an incompressible viscous fluid leads to the Navier-Stokes equations. They are derived from continuum theory of non-polar fluids with three kinds of balance laws: (1) conservation of mass, (2) balance of linear momentum, (3) balance of angular momentum ([GaG]). Usually the momentum balance conditions are expressed on problem adequate "force" formula derived from the Newton formula $= m \cdot \frac{dv}{dt}$. For getting any well-posed (evolution equation) system is it necessary to define its corresponding initial-boundary value conditions.

The NSE are derived from the (Cauchy) stress tensor (resp. the shear viscosity tensor) leading to liquid pressure force $\frac{\partial p}{\partial x_i} = -\frac{\partial T_{ji}}{\partial x_j} + \mu \Delta v_i$. In electrodynamics & kinetic plasma physics the linear resp. the angular momentum laws are linked to the electrostatic (mass "particles", collision, static, quantum mechanics, displacement related; "fermions") Coulomb potential resp. to the magnetic (mass-less "particles", collision-less, dynamic, quantum dynamics, rotation related; "bosons") Lorentz potential.

In [PlJ] a mathematical mass element concept is considered, which replaces the mathematical "mass" object x (real number) by a "differential" object dx. It leads to alternative unit outer normal derivative definition enabling a Newton potential, where a density function is replaced by its differential. This goes along with Plemelj's definition of a double layer potential. From a mathematical point in view this means that a Lebesgue integral is replaced by a Stieltjes integral. This goes along with Plemelj's double layer potential definition. From a physical interpretation perspective it means that Newton's "long distance test particle" defining a potential function is influenced by the enclosed mass elements and no longer by their corresponding density function, only.

When one wants to treat the time-harmonic Maxwell equations with variational methods, one has to face the problem that the natural bilinear form is not coercive on the whole Sobolev space H_1 ([KiA]). On can, however, make it coercive by adding a certain bilinear form on the boundary of the domain (vanishing on a subspace of H_1), which causes a change in the natural boundary conditions ([CoM]).

The mathematical tool to distinguish between unperturbed cold and hot plasma is about the Debye length and Debye sphere ([DeR]). The corresponding interaction (Coulomb) potential of the non-linear Landau damping model is based on the (Poisson) potential equation with corresponding boundary conditions. A combined electro-magnetic plasma field model needs to enable "interaction" of cold and hot plasma "particles", which indicates Neumann problem boundary conditions. The corresponding double layer (hyper-singular integral) potential operator of the Neumann problem is the Prandtl operator $\overline{\mathrm{P}}$, fulfilling the following properties ([LiI] Theorems 4.2.1, 4.2.2, 4.3.2):

i) the Prandtl operator $\overline{\mathrm{P}}: \mathrm{H}_{\mathrm{r}} \rightarrow \widehat{\mathrm{H}}_{\mathrm{r}-1}$ is bounded for $0 \leq \mathrm{r} \leq 1$

ii) the Prandtl operator $\overline{\mathrm{P}}: \mathrm{H}_{\mathrm{r}} \rightarrow \widehat{\mathrm{H}}_{\mathrm{r}-1}$ is Noetherian for $0 < \mathrm{r} < 1$

iii) for $1/2 \leq \mathrm{r} < 1$, the exterior Neumann problem admits one and only one generalized solution.

Therefore, the Prandtl operator enables a combined (conservation of mass & (linear & angular) momentum balances) integral equations system, where the two momentum balances systems are modelled by corresponding momentum operator equations with corresponding domains according to $H_{1/2} = H_1 \times H_1^{\neg}$. For a correspondingly considered variational representation (e.g. for the (Neumann) potential equation or the corresponding Stokes equation) it requires a less regular Hilbert space framework than in standard theory. Basically, domain $H_1$of the standard (Dirichlet integral based) "energy" (semi) inner product $\mathrm{a}(\mathrm{u}, \mathrm{v}) = (\nabla u, \nabla v)$ is extended to $H_{1/2}$ with a corresponding alternative (semi) inner product in the form $a(u, v) = (\nabla u, \nabla v)_{-1/2} = (u, v)_{1/2}$. It enables e.g. the method of Noble ([VeW] 6.2.4), [ArA] 4.2), which is about two properly defined operator equations, to analyze (nonlinear) complementary extremal problems. The Noble method leads to a "Hamiltonian" function $W(\cdot,\cdot)$ which combines the pair of underlying operator equations (based on the "Gateaux derivative" concept)

$$Tu = \frac{\partial W(\acute{u},u)}{\partial \acute{u}} \quad , \quad T^*\acute{u} = \frac{\partial W(\acute{u},u)}{\partial u} \quad u \in E = H_{1/2} \quad , \quad \acute{u} \in \acute{E} = H_{-1/2} .$$

The proposed (Hilbert space based) model provides a truly infinitesimal geometry. Variational integrators were originally developed for geometric time integration, particularly to simulate dynamical systems in Lagrange mechanics. In [StA] the concept of (electromagnetic) "variational integrators for the Maxwell equations with sources", including free sources of charge and current in non-dissipative media, is provided in the framework of differential form. The correspondence of the inner products of the distributional Hilbert spaces $H_{-1/2}$, H_{-1} in the context of a proper ground state energy model is provided in [BrK9].

The alternative Hilbert space framework can also be applied for correspondingly modified Maxwell equations, which govern the electromagnetic field for given distributions of the electric charges and currents. The laws how those charges and currents behave are unknown. What's known, is, that electricity exists within the elementary particles (electron, positron), but the appropriate mathematical model, which is consistent with the Maxwell equations and the related Einstein field equations, is still missing. Only the energy tensor of electromagnetic fields *outside* of elementary particles is known. Modelling the elementary particles as singularities should be considered as interim "solution", only, as well as applying hydrodynamic equations (the classical mechanics approach) to describe "matter" by terms like the density of the ponderable substance (and the corresponding "Ruhemasse") and hydrodynamic pressure forces (area forces). The alternative Hilbert space framework preserves the electrostatics and magnetostatics equations ([ArA] 3.6, 3.7), while replacing Maxwell's electromagnetic "mass density / flux/flow" density concept (enabled by the Maxwell displacement current (density) concept to extend the Ampere law to the Ampere-Maxwell law) by Plemelj's "mass element / flux/flow strength" concept ([PlJ] §8). The latter one is enabled by Plemelj's alternative normal derivative concept ([PlJ] §5: space dimension n=2 & logarithmic potential), where du/dn is replaced by $\bar{u}(\sigma) := -\int_{\sigma_0}^{\sigma} \frac{du}{dn} ds$ ($\bar{u}(\sigma)$ denotes the conjugate potential to u). The latter one might be well defined, while the standard normal derivative might not be defined. We emphasis that $\bar{u}(\sigma)$ is defined with purely boundary values, i.e. it requires no information about the interior or exterior domain of a related vector field. For the generalization of the Cauchy-Riemann equations to space dimension n=3 and related translation or rotation groups we refer to [StE2]. The corresponding physical interpretations are about "source density" or "invertebrate density/rotation" with its related mathematical formulas $rot(u) = rot(\pm grad\varphi) = 0$ or $div(rot(u)) = 0$ (which are the 3-space interpretations of the Poincare lemma $d(d\omega)) = 0$). With respect to the below we also mention that the replacement of the displacement current concept avoids to "calibration (Eichung)" need to ensure well-posed PDE systems. As Plemelj's flow strength definition requires only information from the boundary/surface it can be applied to both, the Gaussian law (based on normal directions to the surface) and the Stokes law (based on tangential directions to the surface). This enables a new "double layer potential***s***" concept with two different potentials on each side of the double layer of the boundary/surface. The corresponding electrostatic and magnetostatic systems are linked by common flow strength values at each point of the surface.

In [CoM] a coercive bilinear form of the Maxwell equations in combination with appropriate boundary conditions is provided. The underlying "energy" inner product is based on the standard (energy) Hilbert space $H_1 x H_1$. We propose the extended domain $H_{1/2} x H_{1/2}$ in combination with a "boundary bilinear form" (based on the Prandtl operator T above with $r = 1/2$) in the form

$$a(u,v) := (divu, divv)_{-1/2} + \langle \overline{\mathrm{P}}u, \overline{\mathrm{P}}v \rangle_{-1/2} + (curlu, curlv)_{-1/2} \quad \forall \mathrm{u}, \mathrm{v} \in \mathring{H}_{1/2}$$

In section 2 & 4 below, we consider the Leray-Hopf (Helmholtz-Weyl) operator $P : H_\beta \to H_\beta^\sigma$ (whereby H_β^σ denotes the divergence-free (solenoidal) H_β). It is linked to the tensor product of the Riesz operators $Q \coloneqq RxR$ (which is selfadjoint and a projection operator, i.e. $Q = Q^2$) and the curl operator by the following identities ([LeN]): $P + Q = Id$, $curl = curl^*$, $curl^2 = -\Delta P$, $[P, curl]$=0, $Pcurl = curlP = curl$, and P is also a projection operator (i.e. $P = P^2$), if $divu = 0$.

The physical interpretation is then as above: The "energy" space $\mathring{H}_{1/2} = \mathrm{H}_1 \times \mathrm{H}_1^\neg$ is built on the "charged electrical particles" Hilbert space H_1 and its related distributions (which is dense in $\mathrm{H}_{1/2}$ with respect to the $\mathring{H}_{1/2} -$ norm), and the "orthogonal" Hilbert space $\mathrm{H}_1^\neg$ generating those "charged electrical particles" (which is a closed subspace of $\mathrm{H}_{1/2}$). The Hilbert space $\mathrm{H}_1^\neg$ "acts" on the "particles" in H_1 (governed by the Prandtl operator), but also "binary collisions" in H_1 acts on $\mathrm{H}_1^\neg$, giving "back" energy into the Hilbert space $\mathrm{H}_1^\neg$, w/o affecting corresponding (statistical) distribution function in H_1. Without additional energy from "outside" the "system" the probability, that two "particles" collide, is zero.

The modified Maxwell equations are proposed as non-standard model of elementary particles (NMEP). Electromagnetic waves propagation in vacuum can be described by the source-free Maxwell equations w/o specifying anything about charges or currents that might have produced them. It provides an alternative model for spontaneous symmetry breakdown with massless bosons ([HiP]). In [WeP], [WeP1], self-adjoint extensions and spectral properties of the Laplace operator with respect to electric and magnetic boundary conditions are provided.

The ("Pythagoras") split of the newly proposed energy norm $\|x\|_{1/2}^2 = \|x_0\|_{H_0}^2 + \|x_0^\neg\|_{H_0^\neg}^2$ goes along with two corresponding groups of transformations, the group of translations and the group of rotations. The corresponding theory of generalized Cauchy-Riemann equations are given in [StE1] III, 4.2, and [StE2]. There are basically two characterizations of all possible generalizations of the Cauchy-Riemann equations: (1) the existence of a harmonic function H on R_+^{n+1} so that $u_j = \frac{\partial}{\partial x_j} H$, $j = 0,1,2, \dots n$; (2) rotation group theory based, building on the (complex unitary) vector space of symmetric tensors of $2x2$ matrices. The latter one includes the "electron equation of Dirac" in the case w/o external forces, with zero mass, and independent of time ([StE2]).

The same (extended Maxwell equations, combined transformation group for kinetic and potential energy interaction) concept as above can be applied to the Einstein field equations. In this context we note that the gravity "force" is an only attractive one. At the same point in time the magnetic field of the earth is the result of the gravity "force". The electromagnetic waves propagation in vacuum of the Maxwell equations corresponds to the Weyl (curvature) tensor, while the Ricci flow equation governs the evolution of a given metric to an Einstein metric ([AnM]). The Einstein vacuum field equations allow wave solutions describing a propagating gravitation field in a geometrical Minkowski space. The wave front of this gravitation field, which is the boundary of the curved and the plane space propagates with light speed.

The distributional Hilbert space framework above is also proposed as an appropriate framework for Wheeler's geometrodynamics ([WhJ]), which is about the attempt to describe space-time and associated phenomena completely in terms of geometry. It especially would enable a consistent model for "graviton" quanta dynamics regarding gravitation waves in Einstein's vacuum field equations and corresponding "graviton" quanta properties in quantum field theory.

A Hilbert space based variational representation of the Einstein field equations goes along with a geometric structure on a 3-manifold, which is a complete, locally homogeneous Riemannian metric g.

A geometric 3-manifold (i.e. a 3-manifold admitting a geometric structure) admits eight simply connected geometries with compact quotients G/H. Those eight geometric structures are rigid in that there are no geometries which interpolate continuously between them. Among them the constant curvature geometries H^3 and S^3 are by far the most important to understand (in terms of characterizing which manifolds are geometric). The stationary points of the volume-normalized Ricci flow are exactly the class of Einstein metrics, i.e. metrics of constant Ricci curvature. Einstein metrics are of constant curvature and so give the geometries H^3, R^3 and S^3 geometries. The Ricci flow is a non-linear heat-type equation for g_{ij} ([AnM]).

The Ricci curvature is a symmetric bilinear form, as it is the metric. In [GöK] a 4-D space S is provided, where matter everywhere rotates relative to the compass of inertia with the angular velocity with the properties: (1) S is homogeneous, (2) there exists a one-parametric group of transformations of S into itself which carries each world line of matter into itself, so that any two world lines of matter are equidistant, (3) S has rotational symmetry, (4) a positive direction of time can consistently be introduced in the whole solution.

With respect to long term stability questions we note, that both, Euler and NS equations, with smooth initial data possess unique solutions which stay smooth forever in case of space dimension n=2. For 3D-NSE the question of global existence of smooth solution vs. finite time blow up is one of the Clay Institute Millennium problems. Having in mind the setting $y(t) := \|v(t)\|^2$ the ODE $\frac{\partial}{\partial t}y(t) = y^2(t)$, $y(0) = y_0$, shows the solution $y(t) = \frac{y_0}{1-t\cdot y_0}$, which, for some initial data $y_0 > 0$, becomes infinite in finite times. An $H_{-1/2}$ inner product based variational representation of the 3D-NSE enjoys a corresponding (evolution) ODE in the form $\frac{\partial}{\partial t}y(t) = c_{\|v(t)\|_1} \cdot y^{1/2}(t)$ ([BrK4]).

We note that for an initial value function $a_0 \in L_\infty(R^n)$ there is an unique local in time solution for the NSE with

$$p = \sum_{i,j=1}^n R_i R_j v_i v_j \quad , \quad R_i \text{ denotes the Riesz operator.}$$

In [GiY] it is shown that in R^2 this solution can be extended globally in time.

In [HeJ] it is shown for existing solutions of the NSE, for which the Dirichlet norm $\mathrm{a}(\mathrm{v},\mathrm{v}) = (\nabla v, \nabla v)$ with domain H_1 of the velocity is continuous as $t = 0$, this is not the case for the corresponding normalized L_2norm of the pressure. At the same point in time, the pressure can be characterized as solution of a Neumann problem by formally operating with "div" on both sides of the NSE ([GaG]). If follows that the prescription of the pressure p at the bounding walls or at the initial time independently of $\boldsymbol{v}$, could be incompatible with the initial-boundary data from the NSE system, and, therefore, could render the problem ill-posed. Both cases above supports the proposed extended Dirichlet norm of the Hilbert space $H_{1/2}$.

Nonlinear evolution equations are also analyzed in a Schauder/Lipschitz function space framework. The space of Lipschitz continuous functions is defined by the norm ([StE] V.4, proposition 6)

$$||f||_{\alpha} := \|f\|_{L_\infty} + sup_{|t|>0} \frac{\|f(x+t)-f(x)\|_{L\infty}}{|t|^{\alpha}} \quad (0 < \alpha < 1).$$

Proposition 6: Every Lipschitz continuous function may be modified on a set of measure zero so that it becomes continuous.

We note that the spaces $C^{k.\alpha}$ are compactly embedded on C^0 and there is a general principle bounding the norm in $C^{k.\alpha}$ of a linear projection operator by means of the norm of in C^0 ([NiJ], [NiJ1]). Hölder- resp. Lipschitz spaces are the adequate ones in treating nonlinear elliptic problems ([NiJ]). Hölder space based Cauchy problem solution(s) of nonlinear evolutions equations are considered in [HöK]. Integral inequalities for the Hilbert transform applied to a nonlocal transport equation are provided in [CoA].

The Gromov compactness theorem (in the context of the study of the deformation and degeneration of general Riemann metrics with merely bounded curvature in place of constant curvature) is based on $C^{1.\alpha}$ topology ([AnM]).

2. Introduction

In section 3 below, we provided a $H_{-1/2}$ – Hilbert norm based estimate for the Fourier coefficient of ρ of the linearized Vlasov equation, as an alternative proof technique of the nonlinear Landau damping phenomenon as suggested in ([MoC]). In section 4 below a $H_{-1/2}$ – Hilbert space based representation of a Landau type equation is provided following the described concept above.

The Boltzmann equation is a (non-linear) integro-differential equation which forms the basis for the kinetic theory of gases. This not only covers classical gases, but also electron /neutron /photon transport in solids & plasmas / in nuclear reactors / in super-fluids and radiative transfer in planetary and stellar atmospheres. The Boltzmann equation is derived from the Liouville equation for a gas of rigid spheres, without the assumption of "molecular chaos"; the basic properties of the Boltzmann equation are then expounded and the idea of model equations introduced. Related equations are e.g. the Boltzmann equations for polyatomic gases, mixtures, neutrons, radiative transfer as well as the Fokker-Planck (or Landau) and Vlasov equations. The treatment of corresponding boundary conditions leads to the discussion of the phenomena of gas-surface interactions and the related role played by proof of the Boltzmann H-theorem.

The Landau equation (a model describing time evolution of the distribution function of plasma consisting of charged particles with long-range interaction) is about the Boltzmann equation with a corresponding Boltzmann collision operator where almost all collisions are grazing. The mathematical tool set is about Fourier multiplier representations with Oseen kernels ([LiP1]), Laplace and Fourier analysis techniques (e.g. [LeN]) and scattering problem analysis techniques based on Garding type (energy norm) inequalities (like the Korn inequality) (e.g. [AzA]). Its solutions enjoy a rather striking compactness property, which is main result of [LiP1]. The collision operator of the Landau equation is given by

$$Q(f,f)=\frac{\partial}{\partial v_i}\left\{\int_{R^N} a_{ij}(v-w)\left[f(w)\frac{\partial f(v)}{\partial v_j}-f(v)\frac{\partial f(w)}{\partial w_j}\right]dw\right\}$$

with

$$a_{ij}(z)=\frac{a(z)}{|z|}\left\{\delta_{ij}-\frac{z_i z_j}{|z|^2}\right\}=\frac{a(z)}{|z|}P(z):=\frac{1-[1-a(z)]}{|z|}[Id-Q](z)\quad Q(z):=(R_i R_j)_{1\le i,j\le N}$$

and $a(z)$ symmetric, non-negative and even in z and R_i denote the Riesz operators, and with an unknown function f corresponding at each time t to the density of particle at the point x with velocity v.

The Landau damping (physical, observed) phenomenon is about "*wave damping w/o energy dissipation by collisions in plasma*", because electrons are faster or slower than the wave and a Maxwellian distribution has a higher number of slower than faster electrons as the wave. As a consequence, there are more particles taking energy from the wave than vice versa, while the wave is damped ([BiJ]). The (kinetic) Vlasov equation is collisions-less ([ShF] (28.1.2)) and the related "constructive" proof of the (observed) Landau damping plasma phenomenon in [MoC] is based on analytic norms. In other words, the Vlasov equations are a not appropriate model for this phenomenon.

Vlasov's mathematical argument against the Landau equation (leading to the Vlasov equation) was, that "*this model of pair collisions is formally (!) not applicable to Coulomb interaction due to the divergence of the kinetic terms*". It is being overcome by the below distributions framework.

Vlasov's formula for the plasma dielectric for the longitudinal oscillators is based on the integral ([ShF] p. 392)

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{\prime}(v)dv}{\frac{\omega}{k}-v} \quad .$$

As Landau pointed out, this model overlooks the important physical phenomenon of electrons travelling with exactly the same material speed $v_\varphi = \frac{\omega}{k}$ and the wave speed v.

In ([ShF] p. 395) the correct definition (as provided by Landau) for the Vlasov formula is given, which is basically a threefold integral definition depending from the value ω_I the imaginary part of $\omega = \omega_R + i\omega_I$:

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{\prime}(v)dv}{\frac{\omega}{k}-v} \qquad \text{for } \omega_I < 0$$

$$W\left(\frac{\omega}{k}\right) = -p.v.\int_{-\infty}^{\infty} \frac{F_0{\prime}(v)dv}{\frac{\omega}{k}-v} - \pi i\, F_0{}'\left(\frac{\omega}{k}\right) sgn(k) \qquad \text{for } \omega_I = 0$$

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{\prime}(v)dv}{\frac{\omega}{k}-v} - 2\pi i\, F_0{}'\left(\frac{\omega}{k}\right) sgn(k) \qquad \text{for } \omega_I > 0$$

If ω_I were to continue and become positive (damped disturbance), then analytical continuation yields, in addition to the integral along the real line (which also presents no difficulty of interpretation), a full residue contribution.

$Im(\omega)$ arises from the pole at $v = v_\varphi$, which is about the pole of the above integral, when the path of integration lies on the x-axis ([ChF] 7). Consequently, the effect is connected with those particles in the distribution that have a velocity nearly equal to the phase velocity – the "resonant particles". These particles travel along with the wave and do not see a rapidly fluctuating electric field: They can, therefore, exchange energy with the wave effectively. The easiest way to understand this exchange of energy is to picture a surfer trying to catch an ocean wave. If the surfboard is not moving, it merely bobs up and down as the wave goes by and does not gain any energy on the average. Similarly, a boat propeller much faster than the wave cannot exchange much energy with the wave. However, if the surfboard has almost the same velocity as the wave, it can be caught and pushed along by the wave; this is, after all, the main purpose of the exercise. In that case, the surfboard gains energy, and therefore the wave must lose energy and is damped. On the other hand, if the surfboard should be moving slightly faster that the wave, it would push on the wave as it moves as it moves uphill; then the wave could gain energy. In plasma, there are electrons both faster and slower than the wave. A Maxwellian distribution, however, has more slow electrons than fast ones. Consequently, there are more particles taking energy from the wave than vice versa, and the wave is damped. As particles with $v \approx v_\varphi$ are trapped in the wave, $f(v)$ is flattened near the phase velocity.

In the nonlinear case when the amplitude of an electron or ion wave exited, say, by a grid is followed in space, it is often found that the decay is not exponential, as predicted by linear theory, if the amplitude is large ([ChF] 8.7). Instead, one typically finds that the amplitude decays, grows again, and then oscillates before settling down to a steady state value. Although other effects may also be operative, these oscillations in amplitude are exactly what would be expected from the nonlinear effect of particle trapping. Trapping of a particle of velocity v occurs when its energy in the wave frame is smaller than the wave potential. … Small waves will trap only these particles moving at high speeds near v_φ. .... When the wave is large, its linear behavior can be expected to be greatly modified…. The quantity ω_B is called the bounce frequency of oscillation of a particle trapped at the bottom of sinusoidal potential well. The frequency ω of the equation of motion is not constant unless x is small. The condition $\omega_B \geq \omega$ turns out to define the breakdown of linear theory even when other processes besides particle trapping are responsible. Another type of nonlinear Landau damping involves the beating of two waves….

In the proof of Mouhot and Villani of the Landau damping ([MoC]) the Laplace transform technique (as introduced by Landau) is replaced by a plain Fourier (inverse formula) analysis technique. The stability of the linearized Vlasov equation is studied by *classical* Volterra (integral) equation analysis establishing the linear Landau damping under certain conditions related to the homogeneous equilibrium function and the governing modes of the function ρ. In order to prove the non-linear Landau damping the corresponding Vlasov equation is solved by the Newton schema and corresponding (analytical!) norm (stability) estimates are provided. Those analytical norms are "hybrid" and "gliding", i.e. for the latter one the norm is changing with time to take into account the transfer of regularity to small velocity scales. This results into corresponding mathematical assumptions to guarantee convergent "analytical norms". The corresponding analytical norm estimates do not provide any evidence related to the physical explanation of the Landau damping.

Mathematically, the issues about the Vlasov formula, which are about

- not well defined classical (divergent) integral formula for relevant domain values
- correct Vlasov formula is split into three different formula depending from the sign of the value ω_I, which is the imaginary part of $\omega = \omega_R + i\omega_I$,

can be addressed by the "principle-value integral" (which was also the defending argument of Vlasov) still neglecting the underlying physical interpretation issue, as long as it is about a classical PDE framework. The "principle-value integral" of the Vlasov formula is given by the Hilbert transform H defined by

$$H[\varphi](\omega) := lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|\omega-y|>\varepsilon}\frac{\varphi(y)}{\omega-y}dy \ .$$

It leads to the alternative Vlasov formula in the form

$${}_xH\left[\frac{\partial}{\partial v_x}g\right]\left(\frac{\omega}{k}\right) := lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|w-v_x|>\varepsilon}\frac{\frac{\partial}{\partial v_x}g(v_x,v_y,v_z)}{\frac{\omega}{k}-v_x}dv_x \ , \ v=(v_x,v_y,v_z)$$

with properly (i.e. problem adequate) to-be-defined domain:

Following the ideas of [BrK3-6] we propose to model the collisions of plasma particles (which are charged particles with long-range interaction, where almost all collisions are grazing) by a weak $H_{-1/2}$ –based variational representation with a corresponding $H_{1/2}$ (energy) Hilbert space. The proposed Hilbert space ensures a well-defined Hilbert transform. The observable L_2 – Hilbert space (supporting statistical analysis) is a compactly embedded subspace of $H_{-1/2}$. At the same point in time the model supports the PLemelj's alternative "mass element" concept, based on an alternative normal derivative concept ([PlJ] p.12). The normal derivative concept plays a key role to model the amount of fluid (mass) streaming in resp. out of a considered volume element defining the continuity equation. Plemelj's alternative normal derivative concept allows modelling those streams only based on boundary data w/o requiring any additional information from the interior domain.

Mathematically speaking this gains a regularity reduction in the same size as a reduction from $C^1 \to C^0$, or (in a variational form representation) from $H_\alpha \to H_{\alpha-1/2}$.

The above approach corresponds to the approach to directly deduce equations of gas dynamics by calculating moments of the Boltzmann equation for quantities that are conserved in collisions of particles composing the gas ([MiD] 3.2). This approach provides an independent derivation of the equations obtained not from macroscopic arguments, deepening the understanding of the physical meaning of the terms appearing in the equations.

Considering the corresponding approach for the 3-D NSE equations ([BrK4]) this corresponds to the a priori ("new "energy" norm) estimate of the non-linear term of the 3-D non-stationary, non-linear NSE. A $\mathrm{H}_{-1/2}$ Hilbert ("fluid") space enables the Sobolevskii-estimate of the non-linear term of the corresponding variational NSE representation, leading to the bounded, generalized energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2+\|u\|_{1/2}^2\le\left|(Bu,u)_{-1/2}\right|\le c\cdot\|u\|_{-1/2}\|u\|_1^2$$

For the correspondingly proposed NMEP we refer to [BrK4].

The corresponding Galerkin-Riesz approximation theory is provided in [BrK].

A combined L_2 – based Fourier wave and $(\mathrm{H}_{-1/2}-\mathrm{H}_0)$ – based Calderón wavelet (Non-standard MEP) analysis tool is provided in [BrK3] (see also appendix).

In sufficiently hot plasma, collisions can be neglected. If, furthermore, the force F is entirely electromagnetic, the Vlasov equation takes the following form *([ChF] 7.2).*

$$\frac{\partial}{\partial t}f+\boldsymbol{v}\cdot\nabla_x f+\frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial}{\partial v}f=0\ .$$

Based on the perturbation split $\mathrm{f(x,v,t)}=f_0(\mathrm{v})+f_1(\mathrm{x,v,t})$ the first order Vlasov equation for electrons is given by

$$\frac{\partial}{\partial t}f_1+\boldsymbol{v}\cdot\nabla_x f_1-\frac{e}{m}\boldsymbol{E}_1\frac{\partial}{\partial v}f_0=0\ .$$

If f_0 is a Maxwellian the corresponding dispersion relation (in a weak sense) is given by

$$1+\frac{\omega_p^2}{k^2}\ {}_xH\left[\frac{\partial}{\partial v_x}\tilde{f}_0\right]\left(\frac{\omega}{k}\right)=0\ .$$

There are two nonlinear equations in connection with nonlinear plasma waves, the Korteweg-Vries and the nonlinear Schrödinger equations. The latter one is given by ([ChF] 8.7):

$$i\frac{\partial}{\partial t}\varphi+p\frac{\partial^2}{\partial x^2}+q|\varphi|^2\varphi=0\ .$$

Regarding the proof of the Landau damping in[MoC] we propose an alternative interaction potential W_x (for space dimension n=1) in the form $\boldsymbol{W_x}:=H[W_x]=(H[W])_x$ resp. (for space dimension n>1) the related formula, where the derivative with respect to the variable x is replaced by the ∇ operator, while the Hilbert transform operator H is replaced by the Riesz transform operator R. The key differentiator to standard assumptions of Landau damping papers with respect to $W_k:=\widehat{W}(k)$ is about the fact, that all Hilbert (resp. Riesz) transformed functions and distributions already do have vanishing constant Fourier terms, i.e. $\widehat{\boldsymbol{W}}_{k=0}=0$.

3. A Hilbert norm estimate for the Fourier coefficient of ρ of the linearized Vlasov equation

In this section we demonstrate the $H_{-1/2}$ – Hilbert space solution concept based on the linearized Vlasov equation. For the notations of the following we refer to [MoC]. We will denote different numerical constants with the same symbol c.

We omit the analysis of the source term (the initial datum contribution) of the defining equation of the Fourier coefficients (i.e. the modes) $\rho_k(t)$ of ρ (see [MoC] (11)), as its convergence rate is not problem solution relevant.

In the linearized Vlasov equation the Fourier coefficients (i.e. the modes) of ρ are linked by the following (Volterra integral type) term ($\boldsymbol{W} := \underline{R}[W]$)

$$\rho_k(t) + 4\pi^2 \boldsymbol{W}_k |k|^2 \frac{1}{k}\int_0^t k(t-\tau)\tilde{f}^o\left(k(t-\tau)\right)\rho_k(\tau)d\tau \quad .$$

Replacing the interaction potential W by its corresponding Riesz transformation $\boldsymbol{W} := \underline{R}[W]$ results into one of the standard assumptions, which is about a vanishing constant Fourier term $\widehat{\boldsymbol{W}}_{k=0} = 0$.

Putting $g(s) := s\tilde{f}^o(s)$, $\overline{g} := \int_0^\infty g^2(v)dv$ it holds

$$\int_0^t g^2\big(k(t-\tau)\big)\,d\tau = \int_0^t g^2(ku)\,du = \frac{1}{k}\int_0^{kt} g^2(v)\,dv \le \frac{1}{k}\int_0^\infty g^2(v)\,dv = \frac{\overline{g}}{k} \,.$$

Putting $\overline{\rho}_k^2(t) := e^{-\alpha t}\rho_k^2(t)$ ($\overline{\rho}_k^2(t) := e^{-\alpha t}\rho_k^2$ ($\alpha \in [0,\infty)$)) this leads to an estimate in the form

$$\overline{\rho}_{\alpha.k}^2(t) = e^{-\alpha t}\rho_k^2(t) \le c e^{-\alpha t}\boldsymbol{W}_k^2 |k|^2 \left[\int_0^t g(k(t-\tau))\,\rho_k(\tau)d\tau\right]^2$$

and therefore

$$\overline{\rho}_{\alpha.k}^2(t) \le c\overline{g}|k| W_k^2 \int_0^t e^{-\alpha\tau}\,\rho_k^2(\tau)d\tau = c\overline{g}|k|\boldsymbol{W}_k^2 \int_0^t \overline{\rho}_k^2(\tau)\,d\tau \quad .$$

Applying the lemma of Gronwall (appendix) then results into

$$\overline{\rho}_{\alpha.k}^2(t) \le e^{c\overline{g}\boldsymbol{W}_k^2|k|t}.$$

In case of a Coulomb potential ($W_k \approx \frac{1}{|k|}$) this inequality is governed by $\overline{\rho}_{\alpha.k}^2(t) \le e^{c\frac{t}{|k|}}$. In case a $H_{-1/2}$Hilbert space is chosen this leads to the inequality in the form $\frac{1}{|k|}\overline{\rho}_{\alpha.k}^2(t) \le \frac{1}{|k|}e^{c\frac{t}{|k|}}$. The counterpart of the critical term of the linearized Vlasov equation ($(\nabla W * \rho)\cdot\nabla_v f^0$) in the Vlasov equation is given by the non-linear term $F[f]\cdot\nabla_v f$, whereby

$$F[f](t,x) := -\iint \nabla W(x-y)f(t,y,w)dwdy.$$

Because of the corresponding Vlasov-Poisson model

$$F = -\nabla W,\ -\Delta_x W = \rho, \quad W = \frac{1}{4\pi|x|} *_x \rho, \quad \rho(x,t) = \int_{R^n} f(x,v,t)dv$$

the combination of both systems is called the Vlasov-Poisson-Boltzmann (VPB) system. The extension of the VPB system, where the Vlasov force F (or self-consistent force, or mean force …) is replaced by the Lorentz force determined by the electro-magnetic field created by the particles themselves is described in [LiP2].

In first step we consider a $H_{-1/2}$ based variational representation in the following form

$$(\frac{\partial}{\partial t}f,g)_0+(v\cdot\nabla_x f,g)_0+(\boldsymbol{F}[f]\cdot\nabla_v f,g)_0=0 \quad \forall g\in H_0$$

The term $(v\cdot\nabla_x f,f)_0$ is governed by $\|f\|_{1/2}$ for L_∞ −bounded $v\in L_\infty$.

The term $(F[f]\cdot\nabla_v f,g)_0$ of the VPB system is proposed to be replaced by $(\mathbf{F}[f]\cdot\nabla_v f,g)_0$, i.e. the potential W is replaced by its Riesz transform $\mathbf{W}=R[W]$. Technically this is achieved by introducing an auxiliary function $h(x):={}_xH\left[\frac{\partial}{\partial v_x}f\right](x)$ considering the variational representation

$$(\frac{\partial}{\partial t}h,g)_{-1/2}+(v\cdot\nabla_x h,g)_{-1/2}+(F[h]\cdot\nabla_v h,g)_{-1/2}=0 \quad \forall g\in H_{-1/2}$$

In the next section this concept is extended in the following way:

- the VPB system is replaced by the VLB system, whereby ([DeR] (1.1)-(1.3))
    - a group of electrons at a given point in time in the plasma will move in response to the wave field (wave packages, Lorentz force)
    - the electromagnetic radiation through the Debye sphere is modelled via Plemelj's plasma mass element concept ([DeR] 5.4, [PlJ])
- an additional term is added to model the binary Coulomb collisions ([DeR] (1.4).

We mention that the required well-defined (variational) Lorentz system as part of the VLB model above needs proper initial and boundary value conditions in the proposed $H_{-1/2}$ framework. The corresponding electromagnetic radiation model might provide opportunities for a Hilbert space based "cycles of time" model ([PeR]).

In this context with respect to the related Maxwell resp. the gravity field equations we note that the variational space-time integrators concept for the Maxwell equations treats electromagnetic Lagrange density as a discrete differential 4-form in space-time ([StA]). The concept combines spatial and time discretization in developing geometric numerical integrators. The approach preserve, by construction, various geometric properties and invariants of the continuous physical systems that they approximate.

4. A $H_{-1/2}$ – Hilbert space based representation of a Landau type equation

In this section we propose an alternative ($H_{-1/2}$ – Hilbert space based) approach to prove the Landau damping based on the Landau equation. It provides a Fourier (modes, wave) analysis in a distributional variational Hilbert space framework being enriched by an additional norm with an "exponential decay" behavior in the form ($t > 0$) given by ([BrK5])

$$(x,y)_{\alpha.(t)} = \sum_k \sigma_k^{\alpha} e^{-\sqrt{\sigma_k}t}(x,\varphi_k)(y,\varphi_k) \quad , \quad \|x\|_{\alpha.(t)}^2 := (x,x)_{\alpha.(t)}$$

based on appropriately defined eigen-pair solutions of a problem adequate linear operator A with the property

- A selfadjoint, positive definite
- A^{-1} compact.

The standard model operator related of the potential theory is the Laplacian operator $-\Delta$. The example related to hydrodynamic theory is the Stokes operator ([TeR] (2.10)-(2.14)), whereby p denotes the density (or pressure) of the fluid. In [WeP], [WeP1] self-adjoint extensions of the Laplacian operator with respect to electric and magnetic boundary conditions are considered. Below we provide the adequate time-dependent related Hilbert scale norm for the corresponding Cauchy problem.

An element $x = x_0 + x_0^{-} \in H_{-1/2} = H_0 + H_0^{-}$ with $\|x_0\|_0 = 1$ is governed by the norm of its (observation) subspace H_0 in combination with the norm $|||x|||_{(t)}^2 := (x,x)_{(t)}$ ([BrK3], [BrK5]) in the form

$$\|x\|_{-1/2}^2 \leq \theta\|x\|_0^2 + \sum_{k=1}^{\infty} e^{1-\sqrt{\sigma_k}\theta} x_k^2 \qquad \text{with} \quad \theta := \|x_0^{-}\|_{-1/2}^2 ,$$

which is a special case of the general inequality ($\alpha > 0$ be fixed)

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2.$$

The evolution equation with respect to the Laplacian operator $-\Delta$ is given by the heat equation $Au := \dot{u} - \Delta u = h$. The Fourier analysis technique is built on the eigen-pairs of the Laplacian $-\Delta\varphi_k = \sigma_k\varphi_k$ leading to the ordinary differential equation $\dot{u}_k(t) - \sigma_k u_k(t) = h_k(t)$, which is solved by

$$u_k(t) = \int_0^t e^{-\sigma_k(t-\tau)} h_k(\tau)d\tau \quad \text{in case of } u_k(0) = 0.$$

It enables the proof of an optimal shift theorem of the heat equation in the form $|||u|||_{\alpha+2.T}^2 \leq c|||Au|||_{\alpha.T}^2$ based on the problem adequate norm $|||u|||_{\alpha.T}^2 := \int_0^T \|u(t)\|_\alpha^2 \, dt$. The proof of the shift theorem takes advantage of an "exchanging the order of integration" rule in the following sense

$$\int_0^T \int_0^t e^{-\sigma_k(t-\tau)} h_k^2(\tau)d\tau \, dt = \int_0^T \int_\tau^T e^{-\sigma_k(t-\tau)} h_k^2(\tau)dtd\tau$$

leading to the following estimates (($\beta \in [0,1)$)

$$\int_0^T t^{-\beta} u_k^2(t)dt \leq \int_0^T t^{-\beta} \left[\int_0^t e^{-\sigma_k(t-\tau)}d\tau\right] \cdot \left[\int_0^t e^{-\sigma_k(t-\tau)} h_k^2(\tau)d\tau\right]$$

$$\leq \sigma_k^{-1} \int_0^T t^{-\beta} \int_0^t e^{-\sigma_k(t-\tau)} h_k^2(\tau)d\tau \, dt \leq \sigma_k^{-2} \int_0^T \tau^{-\beta} h_k^2(\tau)d\tau$$

The corresponding shift theorem of the inhomogeneous evolution equation is given by

$$\int_0^t \tau^{-\beta}\|u(\tau)\|_{\alpha+\beta+1}^2 d\tau \leq c\int_0^t \tau^{-\beta}\|u(\tau)\|_{\alpha+\beta-1}^2 d\tau \ .$$

The term $t^{-\beta}$ is supposed to enable appropriate norm estimates for a corresponding Cauchy problem with non-regular initial value function in the following sense ([BrK2,4]):

In the 1-D case the homogeneous (initial-boundary value) heat equation is given by ([BrK7])

$$\frac{\partial}{\partial t}z - \frac{\partial^2}{\partial x^2}z = 0 \text{ in } (0,1) \times [0,T]$$

$$z(0,t) = z(1,t) = 0 \text{ for } t \in (0,T]$$

$$z(x,0) = g(x) \text{ for } \in (0,1) \ .$$

In order to ensure appropriate regularity of the solution $z(x,t)$ it requires corresponding compatibility relations for the initial value function: $g(1) = 0$, $\frac{\partial}{\partial x}g(0) = 0$, $\frac{\partial^2}{\partial x^2}g(1) = (\frac{\partial}{\partial x}g)^2(1)$ etc.. In case of a non-regular initial value function, e.g. $g \in L_2$, it holds

$$\|z(t)\|_m^2 \leq ct^{-(m-l)}\|g\|_l^2 \quad , \quad \int_0^t \tau^{-1/2}\|z(\tau)\|_{1/2}^2 d\tau \leq c\|g\|_0^2 .$$

(Proof: For the Fourier coefficients of $z(x,t) = \sum z_k(t)\varphi_k(x)$ *it holds* $z_k(t) = z_k(0)e^{-\sigma_k t}$ *with* $z_k(0) := (g,\varphi_k)$*. For the critical case* $m > l$ *the conditions*

$$(m-l)\sigma^{m-l-1}e^{-2\sigma t} + \sigma^{m-l}(-2t)e^{-2\sigma t} = 0$$

results to $\sigma_k \approx t^{-1}$*. Putting* $C_{m.l}(t) := sup_{\sigma_k \geq n > 0}\sigma_k^{m-l}e^{-2\sigma_k t}$ *this leads to*

$$\|z(t)\|_m^2 = \sum \sigma_k^m z_k^2(t) = \sum \sigma_k^m(t) g_k^2 \, e^{-2\sigma_k t} \leq C_{k.j}(t) \sum \sigma_k^l(t) g_k^2 \, e^{-2\sigma_k t}.)$$

For Landau type equation this relates to $\sqrt{|\sigma_k|} \approx |k| \approx t^{-1}$.

In [BrK8] quasi-optimal convergence of FEM Galerkin-Ritz methods for non-linear parabolic problems with non-regular initial values are considered. The approach is based on the weak one-dimension Stefan model problem with solution $u(x,t)$ and corresponding auxiliary function $v(x,t)$

- according to $u_x(x,t) = v(x,t)$ in a L_2 –Hilbert space framework, resp.
- according to to $H[u_x](x,t) = v(x,t)$ in a $H_{-1/2}$ –Hilbert space framework.

With respect to the following we note that $(u_x,w)_{-1/2} = (v,w)_{-1/2} = (u,w)_0$.

Putting $\mathrm{h} := \underline{\mathrm{R}}\nabla_x f$ we propose a $\mathrm{H}_{-1/2}$ based variational representation in the following form

$$(\frac{\partial}{\partial t}\mathrm{h},w)_{-1/2} + (v \cdot \nabla_x \mathrm{h},w)_{-1/2} + (\mathbf{F} \cdot \nabla_v \mathrm{h},w)_{-1/2} + (\nabla_v(\mathrm{h}\boldsymbol{L}[\mathrm{h}]),w)_{-1/2} = 0 \qquad \forall w \in H_{-1/2} \ .$$

whereby $\mathbf{F}$ denotes the Lorentz force.

With respect to the chosen (Hilbert space) domain to be in sync with the estimates of the previous section we note that ([LiP2])

$$\frac{\partial^2}{\partial x_i \partial x_j}(\frac{1}{|x|} * \rho) = cR_iR_j\rho \ ,$$

where R_i denotes the Riesz transform $\frac{\partial}{\partial x_i}(-\Delta)^{-1/2})$.

With respect to the proposed replacement in the previous section of the potential function W by its Riesz transform $\mathbf{W}=\mathrm{R}[\mathrm{W}]$ we recall the following identities ([StE] V.2, V.3)

$$\left[R_i\left(\frac{\partial f}{\partial x_i}\right)\right]^{\frown}(x)=2\pi\frac{x_i^2}{|x|}\hat{f}(x)\ ,\ \left[\left(\frac{\partial f}{\partial x_j}\right)\right]^{\frown}(x)=-2\pi i x_j(x).$$

The Riesz transforms enjoy nice properties as

- $R_i^*=-R_i$, $\sum_{i=1}^n R_i^2=-I$, R_i commutes with translations and homotheties
- It transforms in the same manner as the components of a vector with respect to rotation ([StE] III.1), i.e.

$$\theta R_i\theta^{-1}g=\sum_k\theta_{ik}R_k g\qquad(\theta=\theta_{ik}\text{ rotation matrix}).$$

The identity $u=(-\Delta^{-\frac{1}{2}})(\sum_i R_i\frac{\partial u}{\partial x_i})$ in combination with the definition $(u,v)_{-1/2}=(\left(-\Delta^{-\frac{1}{2}}\right)u,v)$ leads to

$$(u,v)=\left(\left(-\Delta^{-\frac{1}{2}}\right)\left(\sum_i R_i\frac{\partial u}{\partial x_i}\right),v\right)=\left(\sum_i R_i\frac{\partial u}{\partial x_i}\right),v)_{-1/2}=:(\underline{R}\nabla u,v)_{-1/2}.$$

This corresponds to the 1D model identity ([BrK5])

$$\left(-H\left[\frac{d}{dx}(u)\right],v\right)_{-1/2}=\left(-HA\left[\frac{d}{dx}(u)\right],v\right)_0=\left(-AH\left[\frac{d}{dx}(u)\right],v\right)_0=\left(-HA\left[\frac{d}{dx}(u)\right],v\right)_0=(-H^2[u],v)=(u,v).$$

Regarding the fourth term $(\nabla_v(f\boldsymbol{L}[h]),g)_{-1/2}$ (modelling cold plasma with particle collisions) in the above equation the linear operator $\boldsymbol{L}[h]$ is defined by

$$\boldsymbol{L}[h]:=-\int_{R^N}b_{ij}(v-w)h(w)dw:=-\int_{R^N}(v-w)a_{ij}(v-w)h(w)dw$$

with symbol $b_{ij}(z)=z\cdot a_{ij}(z)$, based on the Oseen kernel ([LeN])

$$(*)\quad a_{ij}(z)=\frac{1}{|z|}\left\{\delta_{ij}-\frac{z_iz_j}{|z|^2}\right\}\coloneqq\frac{1}{|z|}P(z)\coloneqq\frac{1}{|z|}\left[Id-\overline{Q}\right](z)\quad\overline{Q}\coloneqq\left(R_iR_j\right)_{1\le i,j\le N}=\overline{Q}^2.$$

The operator $\boldsymbol{L}[h]$ is of order zero. It therefore holds

$$(\nabla_v(h\boldsymbol{L}[h]),w)_{-1/2}\cong(\nabla_v h^2,w)_{-1/2}.$$

The operator $\boldsymbol{L}[h]$ is proposed to define a model collision operator. It is built from the collision operator

- by splitting the symbol in the form

  $$\left(\frac{1}{|z|}-\frac{[1-a(z)]}{|z|}\right)[Id-Q](z)$$

  whereby the second term can be interpreted as compact perturbation
- by splitting

  $$h_v(v)h(w)=(v-w)h(v)h(w)+l(v,w)$$

  with $(v,w)\coloneqq h(w)[h_v(v)-vh(v)]-h(v)[h_w(w)-wh(w)]$.

The corresponding model collision operator with corresponding appropriately defined domain is then given by

$$\tilde{Q}(h,h)=\frac{\partial}{\partial v_i}h(v)\left\{\iint_{R^N}b_{ij}(v-w)h(w)dw\right\}.$$

It enables the governance of the collision operator by Garding type inequalities (as e.g. Korn's second inequality for elasticity mathematical models) in combination with (Gateaux derivative based) nonlinear functional analysis techniques and variational methods (e.g. [AzA], [BrK8], [LeN], [LiP], [LiP2], [VeW]). A corresponding coercive bilinear form for the Maxwell equations is provided in [CoM]).

The Leray-Hopf operator $P(z)$ is of order zero ([CoP] p. 115 ff.). It is an orthogonal projection operator from $(L^3(R^3))^3$ onto the closed subspace of divergence free vector fields. It can be computed through the following identity

$$P = (-\Delta)^{-1} curlcurl.$$

In ([LeN] the action of on Gaussian functions is provided.

"*The uniqueness of Leray's solutions of the NSE is one of the most fascination problem concerning Navier-Stokes equations*". A Leray solution to the NSE satisfies the following properties

$$v \in L^2([0,T], H^1(R^3)) \quad , \quad v \in C^0([0,T], L^2_\omega(R^3) \ .$$

The lack of stability of Leray's solutions has several meanings. In ([CoP] p. 115 ff.) a proof is given that the L_2 –theory is somehow instable. We take this as another indicator that the proposed quantum state Hilbert space $H_{-1/2}$ is the appropriate one to overcome mathematical constraints for stability proof challenges.

Plasma physics is about quantum physics. The key concept of this paper is the quantum state Hilbert space $\mathrm{H}_{-1/2} = \mathrm{H}_0 + \mathrm{H}_0^{\neg}$. With respect to the above we propose to identify the Hilbert space H_0 as the "cold plasma particle" space ("matter" plasma particles with relevant collision energy), while the closed subspace $\mathrm{H}_0^{\neg}$ of $\mathrm{H}_{-1/2}$ is being identified with the "hot plasma particle" space ("mass-less" plasma particles with relevant "potential" energy). In the appendix we provide a corresponding adequate mathematical analysis tool.
We further mention the relationship of the term $f\boldsymbol{L}[f]$ to the Constantin-Lax-Majda equation (e.g. [MaA]) [SaT]), which is a one-dimensional model for the 3-D Euler vorticity equation given by

$$\frac{\partial f}{\partial t} = fH[f] \ , \ \ x \in R, t > 0.$$

The non-linear character of the above equation indicates a complementary variational technique approach, where the original variational problem is split into two dual or complementary problems (with two corresponding operators) leading to a (to be minimized) energy functional and a (to be maximized) complementary functional ([ArA], [VeW]). For electrostatic field problems this is about the principle of minimizing the potential energy and its complementary principle, which is the principle of Thomson ([VeW] 4).

Appendix

A combined L_2 – based Fourier wave and $(H_{-1/2} - H_0)$ – based Calderón wavelet analysis tool

The trilinear form of the non-linear NSE term is antisymmetric. Therefore the energy inequality of the NSE with respect to the physical H_0 – space does not take into account any contribution from the non-linear term. At the same time the regularity of the non-linear term cannot be smoother than the linear term. An alternative physical space $H_{-1/2}$ is the baseline of the unique 3D-NSE solution of this page.

Kolmogorov's turbulence theory is a purely statistical model, based on Brownian motion, which describes the qualitative behavior of turbulent flows ([FrU]). There is no linkage to the quantitative model of fluid behavior, as it is described by the Euler or Navier-Stokes equations.

Kolmogorov's famous 4/5 law is based on an analysis of low- and high-pass filtering Fourier coefficients. The physical counterpart to this is about a "local Fourier spectrum" which is (according to ([FaM]) nonsensical because, as, either it is non-Fourier, or it is nonlocal.

In Kolmogorov's spectral theory the two central concepts of a turbulent flow are homogeneous and isotropic flows (unfortunately they never encounter in nature). A flow is homogeneous if there is no "space" gradient in any averaged quantity, i.e. the statistics of turbulent flow is not a function of space. A flow is isotropic, if rotation and buoyancy are not relevant (they can be neglected) and there is no mean flow.

[FaM1] "The definition of the appropriate "object" that composes a turbulent field is still missing. It would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution. If the appropriate "object" has been defined that composes a turbulent field it would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution.

Turbulent flows have non-zero vorticity and are characterized by a strong three-dimensional vortex generation mechanism (vortex stretching). Brownian motion describes the random motion of particles suspended in a fluid resulting from their collision with quick atoms or molecules in a gas or a liquid. In mathematics it is described by the Wiener process. It is related to the normal density function. A Brownian (=red) noise is produced by a Brownian motion (i.e. a random walk noise). It is obtained as the integral of a white noise signal.

[FaM1] "The notion of "local spectrum" is antinomic and paradoxical when we consider the spectrum as decomposition in terms of wave numbers for as they cannot be defined locally. Therefore a "local Fourier spectrum" is nonsensical because, either it is non-Fourier, or it is nonlocal. There is no paradox if instead we think in terms of scales rather than wave numbers. Using wavelet transform then there can be a space-scale energy be defined with a correspondingly defined scale decomposition in the vicinity of location x and a correspondingly defined local wavelet energy spectrum. By integration this defines a local energy density and a global wavelet energy spectrum. The global wavelet spectrum can be expressed in terms of Fourier energy spectrum. It shows that the global wavelet energy spectrum corresponds to the Fourier spectrum smoothed by the wavelet spectrum at each scale. … … The concept enables the definition of a space-scale Reynolds number, where the average velocity is being replaced by the characteristics root mean square velocity $Re(l, x)$ at scale l and location x.

At large scale (i.e. $l \sim L$) $Re(L)$ coincides with the usual large-scale Reynolds number, where $Re(L)$ is defined as

$$\operatorname{Re}(L) = \iint_{R^n} \operatorname{Re}(L,x)dx$$

A wavelet series of a function $g(x)$ converges locally to $g(x)$, even if $g(x)$ is a distribution as long as the order of the distribution does not exceed the regularity of the analyzing wavelet. The admissibility condition ensures the validity of the inverse wavelet transform which then is valid for all Hilbert scale values.

A L_2 − based Fourier wave analysis is the baseline for statistical analysis, as well as for PDE and PDO theory. There are at least two approaches to wavelet analysis, both are addressing the somehow contradiction by itself, that a function over the one-dimensional space R can be unfolded into a function over the two-dimensional half-plane. The Fourier transform of a wavelet transformed function f is given by ([LoA], [MeY]):

$$\widehat{\mathrm{W}_\vartheta[\mathrm{f}]}(\mathrm{a},\omega) := (2\pi|\mathrm{a}|)^{\frac{1}{2}} \mathrm{c}_\vartheta^{-\frac{1}{2}} \hat{\vartheta}(-\mathrm{a}\omega)\hat{\mathrm{f}}(\omega) \quad .$$

For $\varphi, \vartheta \in \mathrm{L}_2(\mathrm{R})$, $\mathrm{f}_1, \mathrm{f}_2 \in \mathrm{L}_2(\mathrm{R})$,

$$0 < \left|\mathrm{c}_{\vartheta\varphi}\right| := 2\pi \left| \int_{\mathrm{R}} \frac{\hat{\vartheta}(\omega)\bar{\hat{\varphi}}(\omega)}{|\omega|} \mathrm{d}\omega \right| < \infty$$

and $\left|\mathrm{c}_{\vartheta\varphi}\right| \leq \mathrm{c}_\vartheta \mathrm{c}_\varphi$ one gets the duality relationship ([LoA])

$$(\mathrm{W}_\vartheta \ f_1, \mathrm{W}_\varphi^* f_2)_{L_2(R^2, \frac{dadb}{a^2})} = \mathrm{c}_{\vartheta\varphi}(f_1, f_2)_{L_2}$$

i.e.

$$\mathrm{W}_\varphi^* \mathrm{W}_\vartheta \ [\mathrm{f}] = \mathrm{c}_{\vartheta\varphi}\mathrm{f} \quad \text{in a } \mathrm{L}_2 \text{−sense.}$$

For $\varphi, \vartheta \in \mathrm{L}_2(\mathrm{R})$, $\mathrm{f}_1, \mathrm{f}_2 \in \mathrm{L}_2(\mathrm{R})$,

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i.e.

$$\mathrm{W}_\varphi^* \mathrm{W}_\vartheta \ [\mathrm{f}] = \mathrm{c}_{\vartheta\varphi}\mathrm{f} \quad \text{in a } \mathrm{L}_2 \text{−sense.}$$

This identity provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "*microscope observations*" of _two_ wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula. The prize to be paid is about additional efforts, when re-building the reconstruction wavelet. We further note that for a convenient choice of the two wavelet functions the Gibbs phenomenon disappears ([HoM] 2.7).

We note the Gaussian function related “Mexican hat” (wavelet) function

$$g(x) := -\frac{d^2}{dx^2}(e^{-\frac{x^2}{2}}) = (1-x^2)e^{-\frac{x^2}{2}}.$$

being successfully applied e.g. in wavelet theory (see also below section D), as well as the Poisson wavelet ([HoM], example 7.0.2).

We further mention that the Hilbert transform of a wavelet is again a wavelet.

In [BrK3] an alternative quantum state Hilbert space $\mathrm{H}_{-1/2}$ is provided, which includes an alternative concept to the "Dirac function" calculus. This overcomes current handicaps concerning the regularity of the Dirac function, which depends from the space dimension, i.e. $\delta \in \mathrm{H}_{-\mathrm{s}}(R^n)$ for $\mathrm{s} > \mathrm{n}/2$.

The alternatively proposed Hilbert space $\mathrm{H}_{-1/2}$ provides a truly "microscopic" mathematical frame (independently from the space dimension), while still supporting the existing physical observation (statistical analysis) subspace. It is also proposed to replace the (continuous & differentiable) manifold concept (and exterior products of differential forms) in Einstein's field theory.

The extended admissibility condition above indicates that wavelet "pairs" in the form $(\varphi, \vartheta) \in L_2 x H_{-1} \cong H_{-1/2} x H_{-1/2}$ would be an appropriate good baseline to start from, when analyzing in the Hilbert space frame $H_{-1/2} = L_2 x L_2^{\neg}$, , resp. $L_2^{\neg}$, where $L_2^{\neg}$ denote the complementary space of L_2 with respect to the $H_{-1/2}$ −norm, while still analyzing the "observation measurement" Hilbert space L_2 by Fourier waves.

In line with the proposed distributional $\mathrm{H}_{-1/2}$ −Hilbert space concept of this paper, we suggest to define "continuous entropy" in a weak $\mathrm{H}_{-1/2}$ − frame in the form

$$\mathrm{h}(\mathrm{X}) := (\mathrm{f}, \log\frac{1}{\mathrm{f}})_{-1/2} \ ,$$

where X denotes a continuous random variable with density $\mathrm{f}(\mathrm{x})$. In this case it can be derived from a Shannon (discrete) entropy in the limit of n, the number of symbols in distribution $\mathrm{P}(\mathrm{x})$ of a discrete random variable X ([MaC]):

$$\mathrm{H}(\mathrm{X}) := \sum_{\mathrm{i}} \mathrm{P}(\mathrm{x_i}) \log(\frac{1}{\mathrm{P}(\mathrm{x_i})}) \ .$$

This distribution $\mathrm{P}(\mathrm{x})$ can be derived from a set of axioms. This is not the case, in case of the standard entropy (which cannot be derived from dynamic laws (!), anyway, [PeR]) in the form

$$\mathrm{h}(\mathrm{X}) := (\mathrm{f}, \log\frac{1}{\mathrm{f}})_0 .$$

Lemmata of Gronwall

The lemma of Gronwall is a well-established tool for instance to derive evolution equation based classical or variational inequalities. However, applying this tool to Hilbert norm base estimates jeopardizes the balance of any problem adequate norm, e.g. energy / conservation law equations or related inequality estimate (e.g. Garding type inequalities). In this sense, every Gronwall lemma based proof is valid, but related to the underlying physical model the proof is not problem adequate, means that there is still room of mathematical improvements to fit to purely physical (" reality") modelling requirements.

Generalized Lemma of Gronwall (version 1): Let $\psi(t) \in C^0[0,a]$ be a real valued function and $h(t) \in L_1(0,a)$ be non-negative function with

$$\psi(t) \le \alpha + \int_0^t h(\tau)\psi(\tau)d\tau \quad , \quad \alpha \in R \ .$$

Then

$$\psi(t) \le \alpha * e^{\int_0^t h(\tau)d\tau} \ .$$

Generalized Lemma of Gronwall (version 2): Let $\psi(t) \in C^0[0,a]$ be a real valued function and $h(t) \in L_1(0,a)$ be non-negative function with

$$\psi(t) \le \alpha(t) + \int_0^t h(\tau)\psi(\tau)d\tau \quad , \quad \alpha \in R \ .$$

Then

$$\psi(t) \le \alpha(t) + \int_0^t \alpha(\tau)h(\tau)e^{H(t)-H(\tau)}d\tau$$

with

$$H(\tau) := \int_0^\tau h(s)ds \ .$$

Generalized Lemma of Gronwall (version 3: log type): ([YGi1]) Let a, β be non-negative constants. Assume that a non-negative function $a(t,s)$ satisfies $a(*,*) \in C(0 \le s < t \le T)$, $a(t,*) \in L_1(0,t)$ for all $t \in ((0,T)]$. Furthermore, we assume that there exists a positive constant ε_0 such that

$$\sup_{0 \le t \le T} \int_{t-\varepsilon_0}^{t} a(t,s)ds \le 1/2 \qquad .$$

If a non-negative function $f \in C([0,T])$ satisfies

$$f(t) \le \alpha + \int_0^t a(t,s) f(s) ds + \beta \int_0^t \{1 + \log(1 + f(s))\} f(s) ds$$

for all $t \in [0,T]$. Then we have

$$f(t) \le e^{\left\{1 + \frac{\gamma}{\beta} + \log(1 + 2\alpha)\right\} e^{2\beta t}}$$

for all $t \in [0,T]$. Here we put

$$\gamma := \sup_{0 \le t \le T} \left\{ \sup_{0 \le s \le t-\varepsilon_0} a(t,s) \right\} \ .$$

Lemma of Gronwall (version 4): Let $a(t)$ and $b(t)$ nonnegative functions in $[0,A)$ and $0 < \delta < 1$. Suppose a nonnegative function $y(t)$ satisfies the differential inequality

$$y'(t) + b(t) \le \alpha(t) y^{\delta}(t) \qquad \text{on} \qquad [0,A)$$
$$y(0) = y_0 \, .$$

Then for $0 \le t < A$

$$y(t) + \int_0^t b(\tau) d\tau \le (2^{\delta/(1-\delta)} + 1) y_0 + 2^{\delta/(1-\delta)} \left[\int_0^t \alpha(\tau) d\tau \right]^{\delta/(1-\delta)}$$

Proof: solving

$$y'(t) \le \alpha(t) y^{\delta}(t)$$

leads to

$$y(t) \le y_0 + \left[\int_0^t \alpha(\tau) d\tau \right]^{\delta/(1-\delta)}$$

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