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The Distribution of Zeros of Certain Entire Functions*

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1. INTRODUCTION

Linear difference and difference-differential equations with constant coefficients have long been of interest (cf. [1], [2], [3], [4]). More recently, generalizations to difference-integral and difference-differential-integral equations with the integrals of convolution type have been studied (cf. [5], [6], and (1.1) below). Much of the analysis concerns the homogeneous equations, which admit so-called *fundamental solutions* of the form $\varphi(t) = t^m e^{zt}$, where z is complex. Direct substitution shows that $\varphi(t) = t^m e^{zt}$ satisfies the equation iff z is a zero of multiplicity $\geqslant m$ of a certain *characteristic* (entire) *function* $\psi(z)$. The zeros of $\psi(z)$ are also poles of the Laplace transform of solutions defined for $t \geqslant 0$. Residue evaluations of the inverse transform integrals yield expansions of more general solutions in terms of fundamental solutions. Thus, the distribution of the zeros of such entire functions $\psi(z)$ is of some importance.

If the differences in the equation are commensurable, then a simple change of independent variable makes them integers. In the latter case, the difference terms and the difference-derivative terms contribute polynomials in e^z and in z and e^z, respectively, to $\psi(z)$. If the differences are incommensurable, then the characteristic function is more complicated and the problem of finding its zeros is accordingly more difficult.

We first consider the distribution of the zeros of the characteristic function of a difference-integral equation with integer differences.

Thus, consider the difference-integral equation

$$\sum_{\nu=0}^{N} a_\nu \varphi(t-\nu) = \int_0^N K(s)\, \varphi(t-s)\, ds, \qquad t \text{ real} \tag{1.1}$$

where N is a fixed positive integer; K is an arbitrary function in complex $L_1(0, N)$; the a_ν are complex constants with $a_0 \neq 0$ and either $a_N \neq 0$ or

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$K(s) \neq 0$ on a subset of $(N-1, N)$ of positive measure. It should be noted that every difference-integral equation of the form (1.1) in which at least one $a_\nu \neq 0$ and with kernel which does not vanish a.e. can be made to conform to our conditions by appropriate changes of variables.

As indicated above, (1.1) has a *fundamental solution* $\varphi(t) = t^m e^{zt}$ iff z is a zero of multiplicity $\geqslant m$ of the *characteristic function*

$$\psi(z) = \sum_{\nu=0}^{N} a_\nu e^{-\nu z} - \int_0^N K(s)\, e^{-zs} ds. \tag{1.2}$$

Anselone and Greenspan [6] have used functional analysis methods to relate the zeros of $\psi(z)$ and the corresponding fundamental solutions to the asymptotic behavior, as $t \to \infty$, of general solutions of (1.1). However, they did not obtain a complete "description" of the distribution of the zeros of $\psi(z)$. They showed that $\psi(z)$ has one and only one zero (counting multiplicities) near each zero of

$$\sum_{\nu=0}^{N} a_\nu e^{-\nu z}$$

for $|z|$ large and only a finite number of other zeros in any right half-plane. Cartwright [7] considered the special case

$$\Phi(z) = 1 - e^z - \int_{-1}^{1} e^{-zs} K(s)\, ds$$

and showed that $\Phi(z)$ has an infinite sequence of zeros with real parts converging to $-\infty$ iff $K(s) \neq 0$ on a subset of $(0, 1)$ of positive measure. Anselone and Boas [8] have treated the distribution of the zeros of $\Phi(z)$ with more restrictive conditions on $K(s)$.

In Section 2 of the present work we extend Cartwright's methods and results to the characteristic function (1.2) and in Section 3 we show that the methods of this paper are applicable to an even more general entire function. It seems that our methods can be used in still more general cases but this is deferred to a later occasion.

2. The Zeros of the Characteristic Function

Let (1.2) be written as

$$\psi(z) = e^{-Nz} P(e^z) - \int_0^N e^{-zs} K(s)\, ds \tag{2.1}$$

where

$$P(\lambda) = \sum_{\nu=0}^{N} a_\nu \lambda^{N-\nu}. \tag{2.2}$$

Suppose that λ is a nonzero zero of $P(\lambda)$ of multiplicity μ. Define ζ so that

$$e^\zeta = \lambda, \qquad \zeta = \xi + i\eta, \qquad 0 \leqslant \eta < 2\pi. \tag{2.3}$$

Then $e^{-Nz} P(e^z)$ and $P(e^z)$ have common zeros of multiplicity μ given by

$$\zeta + 2k\pi i \qquad (k = 0, \pm 1, \cdots).$$

Anselone and Greenspan [5, p. 73] have shown that there is one and only one zero of $\psi(z)$ near each zero of $e^{-Nz} P(e^z)$; however, there is a slight solecism in their statement, which we rectify. There is a positive minimum distance Δ between any two distinct zeros of $e^{-Nz} P(e^z)$. For each $\delta > 0$ such that $\delta < \frac{1}{2}\Delta$ and each $k = 0, \pm 1, \cdots$ let

$$\Gamma_k(\delta) = \{z : | z - (\zeta + 2k\pi i) | \leqslant \delta\}.$$

Then if λ is a nonzero zero of $P(\lambda)$ of multiplicity μ, an application of Rouché's theorem yields that there are μ zeros (counting multiplicities) of $\psi(z)$ in each $\Gamma_k(\delta)$ for $| k |$ sufficiently large.

Let Ω denote the set of zeros of $\psi(z)$ other than those near the zeros of $e^{-Nz} P(e^z)$. Anselone and Greenspan have also shown that if $a_N \neq 0$, then Ω is a finite set, and i f Ω is infinite, it consists of a sequence of complex numbers whose real parts converge to $-\infty$.

The following theorem completes the description of the distribution of the zeros of $\psi(z)$:

THEOREM 2.1. *If* $a_N = 0$, $a_{N-1} \neq 0$, *then* Ω *is an infinite set if and only if* $K(s) \neq 0$ *on a subset of* $(N - 1, N)$ *of positive measure.*

PROOF. That Ω is finite if $K(s) = 0$ a.e. on $(N - 1, N)$ follows from the work of Anselone and Greenspan, as indicated above.

For the remainder of the proof we will employ the following lemma of Cartwright [9, Theorem 55, p. 87]:

LEMMA 2.2. *If* $\psi(z)$ *is an entire function of order one and mean type such that*

$$\int_1^\infty \{\log^+ | \psi(iy) | + \log^+ | \psi(-iy) |\} \frac{dy}{y^2} < \infty$$

where $\log^+ a = \max(\log a, 0)$, *then*

(i) *there exist* A_1 *and* A_2 *with* $-\infty < A_1 \leqslant A_2 < \infty$ *such that*

$$h(\theta) = \max(A_1 \cos\theta, A_2 \cos\theta),$$

where $h(\theta)$ *is the Phragmén-Lindelöf indicator function defined by*

$$h(\theta) = \overline{\lim_{r\to\infty}}\, r^{-1} \log |\psi(re^{i\theta})|.$$

(ii) $n(r) \sim (A_2 - A_1)\, r/\pi$ *as* $r \to \infty$,

where $n(r)$ *is the number of zeros of* $\psi(z)$ *in the disk* $|z| \leqslant r$.

Clearly $\psi(z)$ is of order one and mean type. By the Riemann-Lebesgue lemma [10, p. 11]

$$\int_0^N K(s)\, e^{-(x_0+iy)s} ds \to 0 \qquad \text{as} \qquad |y| \to \infty$$

for each fixed x_0. Also

$$\left| \int_0^N K(s)\, e^{-zs} ds \right| \leqslant \int_0^N |K(s)|\, ds \qquad \text{for} \qquad \operatorname{Re} z \geqslant x_0$$

and so, by the Phragmén-Lindelöf theorem [11, p. 47]

$$\int_0^N K(s)\, e^{-zs} ds \to 0 \qquad \text{as} \qquad |y| \to \infty$$

uniformly on each finite x-interval. Therefore, on each finite x-interval, the asymptotic behavior of $\psi(z)$ as $|y| \to \infty$ is that of $e^{-Nz} P(e^z)$, i.e.,

$$\psi(x + iy) = O(1) \qquad \text{as} \qquad |y| \to \infty,$$

uniformly on each finite x-interval. Hence

$$\int_1^\infty \{\log^+ |\psi(iy)| + \log^+ |\psi(-iy)|\} \frac{dy}{y^2} < \infty.$$

Since $a_N = 0$, we write

$$\sum_{\nu=0}^{N-1} a_\nu e^{-\nu z} \qquad \text{for} \qquad \sum_{\nu=0}^{N} a_\nu e^{-\nu z},$$

and consider

$$\psi(re^{i\pi}) = \sum_{\nu=0}^{N-1} a_\nu e^{\nu r} - \int_0^N K(s)\, e^{rs} ds.$$

Since $K(s) \neq 0$ on a subset of $(N-1, N)$ of positive measure,

$$\psi(re^{i\pi}) \sim \int_{N-1}^{N} K(s)\, e^{rs} ds.$$

By a lemma of Titchmarsh [12, Lemma 2.3], the assumptions on K insure that there exists a sequence $\{R_n\}$ of real numbers with $R_n \to \infty$ such that

$$\left| \int_{N-1}^{N} K(s)\, e^{R_n s} ds \right| > A e^{(N-1+\epsilon)R_n}$$

for some $A > 0$ and some $\epsilon > 0$. Consequently

$$h(\pi) = \overline{\lim_{r\to\infty}}\, r^{-1} \log | \psi(re^{i\pi}) | \geqslant N - 1 + \epsilon.$$

If $\cos\theta > 0$, $| \psi(re^{i\theta}) | \sim | a_0 | \neq 0$ and hence

$$h(\theta) = \overline{\lim_{r\to\infty}}\, r^{-1} \log | a_0 | = 0.$$

Now $h(\theta)$ is the support function of a convex set (the indicator diagram of ψ) which we have just shown contains the points 0 and $-(N-1+\epsilon)$, cf. Levin [11, p. 77]. Hence

$$h(\theta) \geqslant -(N-1+\epsilon)\cos\theta \qquad \text{for} \qquad \frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

By (i) of Lemma 2.2,

$$h(\theta) = \max(A_1 \cos\theta, A_2 \cos\theta),$$

where $A_1 \leqslant -(N-1+\epsilon)$ and $A_2 = 0$. Consequently there exists $p(r)$ with $n(r) \sim p(r)$ and

$$p(r) \geqslant (N-1+\epsilon)\frac{r}{\pi}$$

for r sufficiently large. Let $n_1(r)$ be the number of zeros of

$$\sum_{\nu=0}^{N-1} a_\nu e^{-\nu z}.$$

Since

$$\sum_{\nu=0}^{N-1} a_\nu \lambda^\nu$$

has at most $N-1$ distinct zeros, there exists $q(r)$ such that $n_1(r) \sim q(r)$ and

$$q(r) \leqslant (N-1)\frac{r}{\pi}$$

for r sufficiently large. Thus the number of points in Ω is

$$n(r) - n_1(r) \sim p(r) - q(r) \geqslant \frac{\epsilon r}{\pi}$$

for r sufficiently large, i.e., Ω is an infinite set.

3. Zeros of a More General Entire Function

The methods of this paper are also applicable to the more general function

$$\psi(z) = \sum_{\nu=0}^{m} a_\nu[1 + \epsilon_\nu(z)]\, e^{-\beta_\nu z} - \int_0^\beta K(s)\, e^{-zs} ds \tag{3.1}$$

where the β_ν are commensurable real numbers such that

$$0 = \beta_0 < \beta_1 < \cdots < \beta_m \leqslant \beta; \qquad a_\nu \neq 0, \qquad \nu = 0, 1, \cdots, m,$$

and

$$\lim_{|z| \to \infty} \epsilon_\nu(Z) = 0, \qquad \nu = 0, 1, \cdots, m.$$

We again assume that $K \in L_1(0, \beta)$.

By Rouché's theorem $\psi(z)$ has one and only one zero near each zero of

$$g(z) = \sum_{\nu=0}^{m} a_\nu[1 + \epsilon_\nu(z)]\, e^{-\beta_\nu z}. \tag{3.2}$$

Let Ω be the set of zeros of $\psi(z)$ other than those near the zeros of $g(z)$. If $\beta_m = \beta$, we again have that Ω is finite.

It has been shown by Bellman and Cooke [1, p. 405] that the number $n_1(r)$ of zeros of $g(z)$ in the disk $|z| \leqslant r$ is asymptotic to $q(r)$ where

$$q(r) \leqslant \frac{r\beta_m}{\pi}$$

Applying the methods of Section 2 we obtain

Theorem 3.1. *If $\beta_m < \beta$, Ω is infinite if and only if $K(s) \neq 0$ on a subset of (β_m, β) of positive measure.*

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