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# On Exponential Bases for the Sobolev Spaces over an Interval*

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We suppose that $K$ is a countable index set and that $\Lambda = \{\lambda_k \mid k \in K\}$ is a sequence of distinct complex numbers such that $E(\Lambda) = \{e^{\lambda_k t} \mid \lambda_k \in \Lambda\}$ forms a Riesz (strong) basis for $L^2[a, b]$, $a < b$. Let $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_m\}$ consist of $m$ complex numbers not in $\Lambda$. Then, with $p(\lambda) = \prod_{k=1}^m (\lambda - \sigma_k)$, $E(\Sigma \cup \Lambda) = \{e^{\sigma_1 t}, ..., e^{\sigma_m t}\} \cup \{e^{\lambda_k t}/p(\lambda_k) \mid k \in K\}$ forms a Riesz (strong) basis for the Sobolev space $H^m[a, b]$. If we take $\sigma_1, \sigma_2, ..., \sigma_m$ to be complex numbers already in $\Lambda$, then, defining $p(\lambda)$ as before, $E(\Lambda - \Sigma) = \{p(\lambda_k) e^{\lambda_k t} \mid k \in K, \lambda_k \neq \sigma_j, j = 1, ..., m\}$ forms a Riesz (strong) basis for the space $H^{-m}[a, b]$. We also discuss the extension of these results to "generalized exponentials" $t^n e^{\lambda_k t}$.

## 1. INTRODUCTION

One of the best known theorems in analysis states that the functions

$$p_k(t) = \frac{1}{\sqrt{b-a}} e^{ik(2\pi/(b-a))t}, \qquad -\infty < k < \infty,$$

form an orthonormal basis for the space $L^2[a, b]$. Each $f \in L^2[a, b]$ has the unique convergent expansion

$$f(t) = \frac{1}{\sqrt{b-a}} \sum_{k=-\infty}^{\infty} f_k e^{ik(2\pi/(b-a))t},$$

$$f_k = \frac{1}{\sqrt{b-a}} \int_a^b f(t)\, e^{-ik(2\pi/(b-a))t}\, dt,$$

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and we have Parseval's identity

$$\|f\|_{L^2[a,b]}^2 = \sum_{k=-\infty}^{\infty} |f_k|^2.$$

A generalization of the concept of an orthonormal basis is that of a *Riesz* basis. A set of elements $p_k$, $k \in K$, in a Hilbert space $H$ is a Riesz basis for $H$ if every $f \in H$ has a unique expansion, norm convergent in $L^2[a, b]$,

$$f = \sum_{k \in K} f_k p_k \tag{1.1}$$

with

$$d^{-2} \|f\|_H^2 \leqslant \sum_{k \in K} |f_k|^2 \leqslant D^2 \|f\|_H^2 \tag{1.2}$$

for positive numbers, $d$, $D$, independent of $f$. It is easy to show that if $e_k$, $k \in K$, is an orthonormal basis for $H$, then the map

$$T: e_k \to p_k, \qquad k \in K, \tag{1.3}$$

extends to a bounded and boundedly invertible operator on $H$. The elements $q_k \in H$ defined by

$$q_k = (T^*)^{-1} e_k \tag{1.4}$$

form a dual, biorthogonal, Riesz basis for $H$, where "biorthogonal" refers to the relationship

$$\begin{aligned} (p_k, q_l)_H &= 1, \qquad k = l, \\ &= 0, \qquad k \neq l. \end{aligned} \tag{1.5}$$

A more general notion is that of a *strong* basis for $H$: each $f \in H$ has a convergent representation (1.1), possibly after suitable grouping of the terms, but (1.2) is replaced by the weaker statement that no $p_k$ lies in the closed span of $\{p_l \mid l \in K, l \neq k\}$. With this assumption the biorthogonal elements $q_k$ characterized by (1.5) still exist, are unique, and again form a strong basis for $H$. The identity

$$f_k = (f, q_k)_H \tag{1.6}$$

holds in the strong as well as the Riesz case and shows expansion (1.1) to be unique in both instances.

The phrase "suitable grouping of the terms" in the preceding paragraph refers to the fact that, in the case of a strong basis, we do not insist that the entire sequence of partial sums derived from series (1.1) should converge. To explain precisely what we mean we require a certain type of ordering in $K$. We shall suppose that $\{K_m \mid m = 1, 2, 3, \ldots\}$ is a sequence of finite subsets of $K$ whose union is $K$ and such that $K_{m+1} \supset K_m$, $m = 1, 2, 3, \ldots$. We define

$$S_m = \sum_{k \in K_m} f_k p_k$$

to be the partial sum of (1.1) over $K$. For a strong basis it is required that $\{K_m\}$, as here described, should exist such that $S_m$ $(= S_m(f))$ is norm convergent to $f$ as $m \to \infty$ for every $f$ in the space $L^2[a, b]$. Essentially the same ideas are presented by Schwartz in Chapter III of [19]. The inequalities in (1.2) show that in the case of a Riesz basis the $S_m$ converge to $f$ for *every* $\{K_m\}$ with the above properties.

The study of sequences of exponentials

$$E(\Lambda) = \{e^{\lambda_k t} \mid \lambda_k \in \Lambda\}, \qquad \Lambda = \{\lambda_k \mid k \in K\}, \tag{1.7}$$

which form Riesz, or strong, bases for $L^2[a, b]$ is now almost classical, going back at least to the 1927 paper of Wiener [22], who later referred to the corresponding series as nonharmonic Fourier series. The subject is further developed in his book (with Paley) [11] and by many subsequent authors, including Ingham [5], Levinson [9], Schwartz [19, 20], Boas [2], Duffin and Eachus [3], Duffin and Schaeffer [4], Riesz and Sz.-Nagy [14] Redheffer [12, 13], Kadec [6], Ullrich [21], and many others too numerous to cite completely. The "standard" result, developed over a number of years, is that if

$$\lambda_k = ik(2\pi/(b - a)) + \alpha + \varepsilon_k, \tag{1.8}$$

where $\alpha$ is a fixed complex number and the $\varepsilon_k$ are arbitrary complex numbers such that

$$\sup_{-\infty < k < \infty} |\varepsilon_k| = M < \infty, \tag{1.9}$$

$$\sup_{-\infty < k < \infty} |\mathrm{Re}(\varepsilon_k)| = \gamma < \tfrac{1}{4}(2\pi/(b - a)) \tag{1.10}$$

then $\{e^{\lambda_k t} | -\infty < k < \infty\}$ forms a uniform basis for $L^2[a, b]$. There are other sufficient conditions. For example, if $\rho_1, \rho_2, \ldots, \rho_N$ are distinct complex numbers, no two differing by an integer multiple of $(2\pi/(b - a))\, N$, the result remains true for distinct exponents

$$\begin{gathered} \lambda_{k,n} = \rho_n + ik(2\pi/(b - a))\, N + \varepsilon_{k,n}, \\ -\infty < k < \infty, \qquad n = 1, 2, \ldots, N, \end{gathered} \tag{1.11}$$

provided conditions of the form (1.9), (1.10), with modified restrictions on $\gamma$, are satisfied by the $\varepsilon_{k,n}$. (It is known that the $\frac{1}{4}$ in (1.10) is best possible for (1.8); see [9, p. 48, 6].) The result for (1.11) is stated in [4]. If the groups of exponents $\Lambda_k = \{\lambda_{k,n} \mid n = 1, 2, ..., N\}$ in (1.11) are allowed to cluster together as $|k| \to \infty$, but the $\lambda_{k,n}$ remain distinct, a strong basis results, as established by Ullrich in [21]. The theory is very strongly related to the theory of scalar neutral functional equations [15] and has applications to certain problems of controllability and/or observability for certain infinite dimensional systems (see, e.g., [16–18]).

Both Levinson [9] and Schwartz [20] have observed that as one passes from $L^1[,b]$ to $L^\infty[a,b]$, or $C[a,b]$, the "number" of exponentials needed to span the space increases by at most one. In this paper we present results which are related to that observation, but which are set in the framework of the Sobolev spaces on the interval $[a,b]$. Let us recall that if $m$ is a positive integer, the Sobolev space $H^m[a,b]$ consists of complex valued functions $f$, continuous on $[a,b]$, such that $f^{(m-1)}$ is absolutely continuous there and its almost-everywhere-defined derivative $f^{(m)}$ is square integrable on $[a,b]$. The usual inner product and norm are

$$(f, g)_{H^m[a,b]} = \int_a^b \left( \sum_{k=0}^{m} f^{(k)}(t)\, \overline{g^{(k)}(t)} \right) dt, \tag{1.12}$$

$$\|f\|_{H^m[a,b]} = [(f, f)]^{1/2}. \tag{1.13}$$

We will discuss a different, but equivalent, inner product and norm presently. Our main results consist of the following two theorems, which are proved in Sections 2 and 3, respectively. The statements here are restricted to distinct exponents $\lambda_k$ but the modifications necessary for discussing "generalized exponentials" $t^n e^{\lambda_k t}$ are discussed in Section 4.

THEOREM 1. *Let $K$ be a countable index set and let $\Lambda = \{\lambda_k \mid k \in K\}$ be a sequence of distinct complex numbers such that*

$$E(\Lambda) = \{e^{\lambda_k t} \mid \lambda_k \in \Lambda\} \tag{1.14}$$

*forms a Riesz (strong) basis for $L^2[a,b]$. Let*

$$\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_m\}$$

*consist of $m$ distinct complex numbers not included in $\Lambda$ and let*

$$p(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k). \tag{1.15}$$

*Then*

$$E(\Sigma \cup \Lambda) = \{e^{\sigma_1 t},..., e^{\sigma_m t}\} \cup \{e^{\lambda_k t}/p(\lambda_k) \mid \lambda_k \in \Lambda\} \tag{1.16}$$

*forms a Riesz (strong) basis for $H^m[a, b]$.*

THEOREM 2. *As in Theorem 1, let $E(\Lambda)$ form a Riesz (strong) basis for $L^2[a, b]$. Let*

$$\Sigma = \{\sigma_1, \sigma_2,..., \sigma_m\}$$

*be a set of m complex numbers in $\Lambda$, i.e., $\Sigma \subset \Lambda$. Let $p(\lambda)$ again be defined by (1.15). Then*

$$E(\Lambda - \Sigma) = \{p(\lambda_k)\, e^{\lambda_k t} \mid \lambda_k \in \Lambda - \Sigma\} \tag{1.17}$$

*is a Riesz (strong) basis for $H^{-m}[a, b]$.*

The definition of $H^{-m}[a, b]$ will be given in Section 3. It is evident that the coefficients $1/p(\lambda_k)$, $p(\lambda_k)$ in (1.16), (1.17) can be omitted when strong bases are being discussed and may be replaced by equivalent expressions, such as $1/(1 + |\lambda_k|)^m$, $(1 + |\lambda_k|)^m$ in the Riesz case.

The proofs of Theorems 1 and 2 are carried out in the framework of the theory of ordinary differential equations and require the following elementary lemma, which also provides an alternative inner product and norm for $H^m[a, b]$ equivalent to (1.12), (1.13).

LEMMA 1. *Let*

$$P(\lambda) = \prod_{k=1}^{m} (\lambda - \mu_k) = \sum_{k=0}^{m} a_k \lambda^{m-k}, \qquad a_0 = 1,$$

*be a monic polynomial of degree m. Let*

$$P(D) = \sum_{k=0}^{m} a_k D^{m-k}.$$

*be the corresponding differential operator ($D = d/dt$). Let $\mathscr{P}$: $E^m \oplus L^2[a, b] \to H^m[a, b]$ be defined by*

$$\mathscr{P}(x_1, x_2,..., x_m, x) = y, \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in E_m, \qquad x \in L^2[a, b], \tag{1.18}$$

*where $y$ is the unique solution of*

$$P(D)\,y = x, \qquad y^{(k-1)}(c) = x_k, \qquad k = 1,\ldots, m, \tag{1.19}$$

*$c$ being a point in $[a,b]$ independent of $y$. Defining the inner product and norm in $E^m \oplus L^2[a,b]$ by*

$$((x_1,\ldots, x_m, x), (\xi_1,\ldots, \xi_m, \xi)) = \sum_{k=1}^{m} x_k \overline{\xi_k} + \int_a^b x(t)\,\overline{\xi(t)}\,dt, \tag{1.20}$$

$$\|(x_1,\ldots, x_m, x)\| = [((x_1,\ldots, x_m, x), (x_1,\ldots, x_m, x))]^{1/2}, \tag{1.21}$$

*$\mathscr{P}$ is one to one, onto, bounded, and boundedly invertible.*

COROLLARY 1. *For each $x, y \in H^m[a,b]$ let*

$$(x, y)_m = \sum_{k=0}^{m-1} x^{(k)}(c)\,\overline{y^{(k)}(c)} + \int_a^b (P(D)\,x)(t)\overline{(P(D)\,y)(t)}\,dt, \tag{1.22}$$

$$\|x\|_m = [(x, x)_m]^{1/2}.. \tag{1.23}$$

*Then $(\ ,\ )_m$, $\|\ \|_m$ constitute an inner product and norm for $H^m[a,b]$ equivalent to* (1.12), (1.13).

The proof of Corollary 1, given Lemma 1, is clear.

*Proof of Lemma* 1. The inverse map

$$\mathscr{P}^{-1}: y \to (y(c),\ldots, y^{(m-1)}(c), P(D)\,y)$$

is clearly bounded if we can show that $y(c),\ldots, y^{(m-1)}(c)$ are each bounded by a multiple of the norm of $y$ in $H^m[a,b]$. For any $s \in [a,b]$ and $k = 1, 2,\ldots, m$

$$y^{(k-1)}(c) = y^{(k-1)}(s) + \int_s^c y^{(k)}(t)\,dt.$$

Integrating both sides with respect to $s$ over $[a,b]$ and dividing by $b-a$, we obtain

$$\begin{aligned} y^{(k-1)}(c) &= \frac{1}{b-a}\left[\int_a^b y^{(k-1)}(s)\,ds + \int_a^b \int_s^c y^{(k)}(t)\,dt\,ds\right] \\ &= \frac{1}{b-a}\left[\int_a^b y^{(k-1)}(s)\,ds + \int_a^c (t-a)\,y^{(k)}(t)\,dt \right. \\ &\qquad \left. + \int_c^b (t-b)\,y^{(k)}(t)\,dt\right] \end{aligned}$$

and from this the desired bound follows immediately.

To bound $\mathscr{P}$ itself we convert the differential equation $P(D)\,y = x$ to first order form

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(m-2)}(t) \\ y^{(m-1)}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_m & a_{m-1} & \cdots & a_2 & a_1 \end{pmatrix} \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(m-2)}(t) \\ y^{m-1)}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x(t) \end{pmatrix}$$

or, with $\eta = (y, y',..., y^{(m-1)})$, $\beta = (0, 0,..., 0, 1)$,

$$d\eta/dt = A\eta + \beta x. \tag{1.24}$$

We have $\eta(c) = (x_1, x_2,..., x_m)$ and the variation of parameters formula for (1.24) gives

$$\eta(t) = e^{A(t-c)}\eta(c) + \int_c^t e^{A(t-s)}bx(s)\,ds.$$

From this $y(t)$, $y'(t)$,..., $y^{(m-1)}(t)$ are all continuous and bounded on $[a, b]$ with bounds easily given in trms of $\|x\|_{L^2[a,\,b]}$ and $\|(x_1, x_2,..., x_m)\|_{E^m}$. Then $P(D)\,y = x$ gives

$$y^{(m)}(t) = x(t) - \sum_{k=1}^{m} a_k y^{(m-k)}(t)$$

so that $y^{(m)} \in L^2[a, b]$ and we can bound $\|y^{(m)}\|_{L^2[a,b]}$ in terms of $\|x\|_{L^2[a,b]}$, $\|(x_1, x_2,..., x_m)\|_{E^m}$, using the result obtained earlier for $y(t)$, $y'(t)$,..., $y^{(m-1)}(t)_n$. It follows that there is a constant $K > 0$ such that

$$\sum_{k=0}^{m} \|y^{(k)}\|^2_{L^2[a,b]} \leqslant K(\|(_1, x_2,..., x_m)\|^2_{E^m} + \|x\|^2_{L^2[a,b]}),$$

and since the left-hand side is $\|y\|^2_{H^m[a,b]}$, we have the desired bound. The other properties of $\mathscr{P}$ follow easily from the usual existence and uniqueness theorems for ordinary differential equations.

## 2. Proof of Theorem 1

The proof of Theorem 1 is quite straightforward, making use, again, of elementary properties of ordinary differential equations. We do need one property of the $\lambda_k$ which is not completely elementary. It may be derived from the assumption that the $e^{\lambda_k t}$ form a basis for $L^2[a, b]$. Using the fact that the $\lambda_k$ must be zeros of an entire function $\varphi(\lambda)$ of order 1 (see [8, Theorem 5, p. 419], e.g.) and the fact that the exponent of convergence of the zeros of such a function does not exceed its order, we have

$$\sum_{k\in K}\left(\frac{1}{1+|\lambda_k|}\right)^2<\infty. \tag{2.1}$$

Given this, we proceed, first under the assumption that $E(\Lambda)$ is a Riesz basis. Let $p(\lambda)$ be given by (1.15) and let

$$p(D)=\prod_{k=1}^{m}(D-\sigma_k I)$$

be the corresponding differential operator. If $y\in H^m[a,b]$, then

$$p(D)\,y=x\in L^2[a,b] \tag{2.2}$$

and $x$ has a unique representation

$$x(t)=\sum_{k\in K}x_k e^{\lambda_k t}, \tag{2.3}$$

convergent in $L^2[a,b]$, with

$$d^{-2}\,\|x\|^2_{L^2[a,b]}\leqslant\sum_{k\in K}|x_k|^2\leqslant D^2\,\|x\|^2_{L^2[a,b]} \tag{2.4}$$

for some positive constants $d$, $D$, independent of $x$. Now the differential equation (2.2) has a particular solution $\hat{y}$, not equal to $y$ in general,

$$\hat{y}(t)=\sum_{k\in K}\frac{x_k}{p(\lambda_k)}e^{\lambda_k t}. \tag{2.5}$$

Using (2.1) and (2.4) together with the fact that

$$\frac{1}{p(\lambda_k)}=\mathcal{O}\left(\frac{1}{|\lambda_k|^m}\right),\qquad |\lambda_k|\to\infty,$$

it may be verified that

$$\hat{y}^{(j)}(t)=\sum_{k\in K}\frac{\lambda_k^j x_k}{p(\lambda_k)}e^{\lambda_k t} \tag{2.6}$$

is uniformly convergent in $L^2[a,b]$ for $j=0,1,2,\ldots,m-1$. Along with $p(D)\,\hat{y}=x\in L^2[a,b]$, this shows (2.5) to be convergent in $H^m[a,b]$. Now

$$y=\hat{y}+\tilde{y}, \tag{2.7}$$

where $\tilde{y}$ is the unique solution on $[a,b]$ of the homogeneous equation

$$p(D)\,\tilde{y}=0 \tag{2.8}$$

satisfying, at a particular point $c \in [a,b]$,

$$\begin{aligned}\tilde{y}^{(j)}(c) &= y^{(j)}(c)-\hat{y}^{(j)}(c)\\ &= y^{(j)}(c)-\sum_{k\in K}\frac{\lambda_k^j x_k}{p(\lambda_k)}e^{\lambda_k c}.\end{aligned} \tag{2.9}$$

Since $\tilde{y}$ satisfies (2.8) there are unique coefficients $c_1, c_2, \ldots, c_m$ such that

$$\tilde{y}(t)=\sum_{k=1}^{m} c_k e^{\sigma_k t}. \tag{2.10}$$

Differentiating (2.10) and using (2.9) it may be seen that the $c_k$ are uniquely determined by the system of equations

$$\begin{pmatrix} 1 & 1 & \cdots & 1\\ \sigma_1 & \sigma_2 & \cdots & \sigma_m\\ \vdots & \vdots & & \vdots\\ (\sigma_1)^{m-1} & (\sigma_2)^{m-1} & \cdots & (\sigma_m)^{m-1}\end{pmatrix}\begin{pmatrix} e^{\sigma_1 c}c_1\\ e^{\sigma_2 c}c_2\\ \vdots\\ e^{\sigma_m c}c_m\end{pmatrix}=\begin{pmatrix}\tilde{y}(c)\\ \tilde{y}'(c)\\ \vdots\\ \tilde{y}^{(m-1)}(c)\end{pmatrix} \tag{2.11}$$

involving the familiar nonsingular Vandermonde matrix. Combining (2.5) and (2.10) we see that $y(t)$ has the representation

$$y(t)=\sum_{k=1}^{m} c_k e^{\sigma_k t}+\sum_{k\in K}\frac{x_k}{p(\lambda_k)}e^{\lambda_k t}, \tag{2.12}$$

convergent in $H^m[a,b]$. The uniqueness is easily established. If

$$y(t)=\sum_{k=1}^{m}\gamma_k e^{\sigma_k t}+\sum_{k\in K}\xi_k e^{\lambda_k t}, \tag{2.13}$$

convergent in $H^m[a,b]$, then

$$\begin{aligned}0 &= p(D)\left[\sum_{k=1}^{m}(c_k-\gamma_k)\,e^{\sigma_k t}+\sum_{k\in K}\left(\frac{x_k}{p(\lambda_k)}-\xi_k\right)e^{\lambda_k t}\right]\\ &= \sum_{k\in K}(x_k-p(\lambda_k)\,\xi_k)\,e^{\lambda_k t},\end{aligned} \tag{2.14}$$

convergent in $L^2[a,b]$, from which the strong independence of the $e^{\lambda_k t}$ in $L^2[a,b]$ gives

$$\xi_k = \frac{x_k}{p(\lambda_k)}, \qquad k \in K.$$

Then (2.12), (2.13) give $0 \equiv \sum_{k=1}^{m} (c_k - \gamma_k)\, e^{\sigma_k t}$, from which it follows that $\gamma_k = c_k$, $k = 1, 2, \ldots, m$.

It remains to show that there are positive numbers $d_1$, $D_1$ such that

$$d_1^{-2} \| y \|^2_{H^m[a,b]} \leqslant \sum_{k=1}^{m} |c_k|^2 + \sum_{k \in K} |x_k|^2 \leqslant D_1^2 \| y \|^2_{H^m[a,b]}. \tag{2.15}$$

From Lemma 1, $\| y \|^2_{H^m[a,b]}$ may be replaced by

$$\| y \|_m^2 = \sum_{k=1}^{m} | y^k(c) |^2 + \| p(D)\, y \|^2_{L^2[a,b]}.$$

Since $p(D)\, y = x$ with $x$ given by (2.3), (2.4) gives

$$\| p(D)\, y \|^2_{L^2[a,b]} \leqslant \gamma^2 \sum_{k \in K} |x_k|^2. \tag{2.16}$$

Clearly (2.11) gives

$$\sum_{j=1}^{m} | \tilde{y}^{(j)}(c) |^2 \leqslant \tilde{G}^2 \sum_{j=1}^{m} |c_j|^2 \tag{2.17}$$

for some $\tilde{G} > 0$. Since (2.7), (2.9) give

$$\sum_{j=1}^{m} | y^{(j)}(c) |^2 \leqslant \frac{1}{2} \left( \sum_{j=1}^{m} | \tilde{y}^{(j)}(c) |^2 + \sum_{j=1}^{m} | \hat{y}^{(j)}(c) |^2 \right), \tag{2.18}$$

it remains to bound the numbers $\hat{y}^{(j)}(c)$. Using (2.4) with all but one coefficient equal to zero and that one coefficient equal to 1, for all $k \in K$,

$$d^{-2} \| e^{\lambda_k t} \|^2_{L^2[a,b]} \leqslant 1 \leqslant D^2 \| e^{\lambda_k t} \|^2_{L^2[a,b]},$$

from which it is readily seen that $\mathrm{Re}(\lambda_k)$ is uniformly bounded. Since $\{\lambda_k^j / p(\lambda_k)\}$ is square summable for $j = 0, 1, \ldots, m-1$, the Schwarz inequality, with (2.6), gives

$$\begin{aligned} | \hat{y}^{(j)}(c) |^2 &= \left| \sum_{k \in K} \frac{\lambda_k^j x_k}{p(\lambda_k)} e^{\lambda_k c} \right|^2 \\ &\leqslant e^{2|c| \max |\mathrm{Re}(\lambda_k)|} \sum_{k \in K} \left| \frac{\lambda_k^j}{p(\lambda_k)} \right|^2 \cdot \sum_{k \in K} |x_k|^2. \end{aligned} \tag{2.19}$$

Combining (2.16), (2.17), (2.18) and (2.19) there must be a constant $d_0 > 0$ such that

$$\sum_{k=1}^{m} |y^{(k)}(c)|^2 + \|p(D)y\|^2_{L^2[a,b]} \leqslant d_0^2 \left( \sum_{k=1}^{m} |c_k|^2 + \sum_{k\in K} |x_k|^2 \right). \quad (2.20)$$

In the other direction, $p(D)y = x$ together with (2.3), (2.4) gives, for some $\Delta > 0$,

$$\|P(D)_y\|^2_{L^2[a,b]} \geqslant \Delta^2 \sum_{k\in K} |x_k|^2. \quad (2.21)$$

From the nonsingularity of the Vandermonde matrix in (2.11) and (2.7) it follows that

$$\sum_{k=1}^{m} |c_k|^2 \leqslant B^2 \left[ \sum_{k=0}^{m-1} (|y^{(k)}(c)|^2 + |\hat{y}^{(k)}(c)|^2) \right].$$

Using (2.19) again, and then (2.21) twice, we see that

$$\sum_{k=1}^{m} |c_k|^2 + \sum_{k\in K} |x_k|^2$$
$$\leqslant D_0^2 \left( \sum_{k=0}^{m-1} |y^{(k)}(c)|^2 + \|P(D)y\|^2_{L^2[a,b]} \right) \quad (2.22)$$

for some $D_0 > 0$. Combining (2.20), (2.22)

$$d_0^{-2} \left( \sum_{k=0}^{m-1} |y^k(c)|^2 + \|P(D)y\|^2_{L^2[a,b]} \right)$$
$$\leqslant \sum_{k=1}^{m} |c_k|^2 + \sum_{k\in K} |x_k|^2 \leqslant D_0^2 \left( \sum_{k=0}^{m-1} |y^k(c)|^2 + \|P(D)y\|^2_{L^2[a,b]} \right),$$

Using Corollary 1 there must be $d_1, D_1$ such that (2.15) is valid and we have shown that $E(\Sigma \cup \Lambda)$ is a uniform basis for $H^m[a,b]$.

If we insist only on the strong basis property for $E(\Lambda)$ in $L^2[a,b]$, the strong basis property of $E(\Sigma \cup \Lambda)$ in $H^m[a,b]$ is obtained with a slight modification of the above argument. Let us suppose that $K = \bigcup_{l=1}^{\infty} \hat{K}_l$, the $\hat{K}_l$ being finite subsets of $K$ (corresponding to $K_m = \sum_{l=1}^{m} \hat{K}_l$ in Section 1), and that each $x \in L^2[a,b]$ can be written uniquely as

$$x = \sum_{l=1}^{\infty} \left( \sum_{k\in K_l} x_k e^{\lambda_k t} \right) \equiv \sum_{l=1}^{\infty} \xi_l, \quad (2.23)$$

convergent in $L^2[a,b]$. For $L = 1, 2, 3, \ldots$, let

$$x_L = \sum_{l=1}^{L} \xi_l. \tag{2.24}$$

Given $y \in H^m[a,b]$, we define $x$ by $p(D)y = x$ and expand $x$ as in (2.23), (2.24). Define

$$\hat{y}_L(t) = \sum_{l=1}^{L} \left( \sum_{k \in K_l} \frac{x_k}{p(\lambda_k)} e^{\lambda_k t} \right),$$

which, being a finite sum, lies in $H^m[a,b]$. Choose

$$\tilde{y}_L(t) = \sum_{k=1}^{m} c_{L,k} e^{\sigma_k t}$$

so that

$$y_L(t) = \hat{y}_L(t) + \tilde{y}_L(t)$$

satisfies, for all $L$, and $j = 0, 1, \ldots, m-1$,

$$y_L^{(j)}(t) = y^{(j)}(t). \tag{2.25}$$

Since (2.25) is true for all $L$ and

$$P(D)\, y_L = \sum_{l=1}^{L} \left( \sum_{k \in K_l} x_k e^{\lambda_k t} \right) = \sum_{l=1}^{l} \xi_l$$

converges to $x = P(D)y$ in $L^2[a,b]$, Lemma 1 implies that

$$y_L(t) = \sum_{k=1}^{m} c_{L,k} e^{\sigma_k t} + \sum_{l=1}^{L} \left( \sum_{k \in K_l} \frac{x_k}{p(\lambda_k)} e^{\lambda_k t} \right)$$

converges to $y$ in $H^m[a,b]$. Of course, this does not give us fixed coefficients $c_k$ for the terms involving $e^{\sigma_k t}$, $k = 1, 2, \ldots, m$. For this we need the strong independence of $e^{\sigma_1 t}, \ldots, e^{\sigma_m t}, e^{\lambda_k t}$, $k \in K$, in $H^m[a,b]$, which we present at the end to maintain the train of thought. Assuming that result for the moment, none of these exponentials lies in the closed subspace spanned by the others and we can find, in particular, functions $q_1, q_2, \ldots, q_m$ in $H^m[a,b]$ such that

$$(e^{\sigma_k t}, q_j)_{H^m[a,b]} = \delta_{jk}, \qquad j, k = 1, 2, \ldots, m,$$
$$(e^{\lambda_k t}, q_j)_{H^m[a,b]} = 0, \qquad k \in K, \quad j = 1, 2, \ldots, m.$$

Forming the inner product of $y_L$ with $q_k$, we see that for $k = 1, 2, \ldots, m$, as $L \to \infty$,

$$c_{L,k} = (y_L, q_k)_{H^m[a,b]} \to (y, q_k)_{H^m[a,b]} \equiv c_k.$$

Then

$$\lim_{L \to \infty} \left\| \sum_{k=1}^{m} c_{L,k} e^{\sigma_k t} - \sum_{k=1}^{m} c_k e^{\sigma_k t} \right\|_{H^m[a,b]} = 0$$

and we have

$$y(t) = \sum_{k=1}^{m} c_k e^{\sigma_k t} + \sum_{l=1}^{\infty} \left( \sum_{k \in K_l} \frac{x_k}{p(\lambda_k)} e^{\lambda_k t} \right),$$

convergent in $H^m[a, b]$.

Finally, the strong independence result. Assuming that the $e^{\lambda_k t}$, $k \in K$, are strongly independent in $L^2[a, b]$, we select one of these, which we call $e^{\lambda_1 t}$, and observe that there is a unique element $r \in L^2[a, b]$ such that

$$\begin{aligned} (e^{\lambda_1 t}, r)_{L^2[a,b]} &= 1, \qquad (e^{\lambda_k t}, r)_{L^2[a,b]} = 0, \\ \lambda_k &\in \Lambda, \qquad \lambda_k \neq \lambda_1. \end{aligned} \tag{2.26}$$

Let $\hat{r}(\lambda)$ be the Fourier transform of $\bar{r}$:

$$\hat{r}(\lambda) = \int_a^b e^{\lambda t} \bar{r}(t)\, dt.$$

Then it is well known [9, 12] that $\hat{r}(\lambda)$ is an entire function of order 1 such that

$$\begin{aligned} \hat{r}(\lambda) &= \mathcal{O}(e^{b \operatorname{Re}(\lambda)}), \qquad \operatorname{Re}(\lambda) \to +\infty, \\ \hat{r}(\lambda) &= \mathcal{O}(e^{a \operatorname{Re}(\lambda)}), \qquad \operatorname{Re}(\lambda) \to -\infty, \end{aligned}$$

and $\hat{r}(\lambda)$ is square integrable on any line parallel to the imaginary axis. From (2.26) we see that $\hat{r}(\lambda_k) = 0$, $\lambda_k \neq \lambda_1$. There are no other zeros of $\hat{r}(\lambda)$. (If $\sigma$ were such a zero, $\hat{r}_\sigma(\lambda) = (\lambda - \lambda_1)\, \hat{r}(\lambda)/(\lambda - \sigma)$ would be the Fourier transform of the function

$$r_\sigma(t) = \lim_{A \to \infty} \frac{1}{2\pi i} \int_{-iA}^{iA} e^{-\lambda t} \hat{r}_\sigma(\lambda)\, d\lambda$$

in $L^2[a, b]$ orthogonal to all of the $e^{\lambda_k t}$, $k \in K$, contrary to the assumed completeness of $E(\Lambda)$. That the support of $r_\sigma(t)$ is confined to the interval $[a, b]$ follows [9, Chap. IV, p. 58] from the fact that $\hat{r}_\sigma(\lambda)$ has the same growth bounds as $\hat{r}(\lambda)$ as $\operatorname{Re}(\lambda) \to \pm\infty$.)

Thus $\hat{r}(\lambda)$ may be seen to be the Fourier transform of a function $r$ in $L^2[a,b]$ such that

$$(e^{\sigma_k t}, r)_{L^2[a,b]} \neq 0, \qquad k = 1, 2, \ldots, m,$$
$$(e^{\lambda_k t}, r)_{L^2[a,b]} = 0, \qquad \lambda_k \in \Lambda, \quad \lambda_k \neq \lambda_1.$$

For $k = 1, 2, \ldots, m$, define

$$p_k(\lambda) = (\lambda - \sigma_1) \cdots (\lambda - \sigma_{k-1})(\lambda - \lambda_1)(\lambda - \sigma_{k+1}) \cdots (\lambda - \sigma_m)$$

and let $p_k(D)$ be the corresponding differential operator. The bounded linear functions $l_k(y)$ defined on $H^m[a,b]$ by

$$l_k(y) = (p_k(D)y, r)_{L^2[a,b]}, \qquad k = 1, 2, \ldots, m,$$

may then be seen to have the property

$$\begin{aligned} l_k(e^{\sigma_j t}) &= p_k(\sigma_j)(e^{\sigma_j t}, r) \neq 0, \quad j = k, \\ &= 0, \quad j \neq k, \end{aligned} \qquad j, k = 1, 2, \ldots, m,$$
$$l_k(e^{\lambda_l t}) = 0, \qquad \lambda_l \in \Lambda \quad (\text{including } \lambda = \lambda_1).$$

With $p(D)$ as defined earlier and

$$l(y) = (p(D)y, r)_{L^2[a,b]},$$

$l$ is a continuous linear functional on $H$ such that

$$l(e^{\sigma_k t}) = 0, \qquad k = 1, 2, \ldots, m,$$
$$l(e^{\lambda_l t}) = 0, \qquad \lambda_l \in \Lambda, \quad \lambda_l \neq \lambda_1,$$
$$l(e^{\lambda_1 t}) \neq 0.$$

Since $\lambda_1$ could be any element of $\Lambda$, the strong independence of $E(\Sigma \cup \Lambda)$ in $H^m[a,b]$ follows and the proof of Theorem 1 is complete.

## 3. Proof of Theorem 2

We will begin by defining $H^{-m}[a,b]$ for a positive integer, $m$. We can construct a closed subspace of $H^m[a,b]$, which we will call $H_0^m[a,b]$, by setting

$$H_0^m[a,b] = \{ y \in H^m[a,b] \mid y^{(k)}(a) = y^{(k)}(b) = 0, \quad k = 0, 1, \ldots, m-1 \}. \tag{3.1}$$

Since it is a closed subspace, $H_0^m[a,b]$ is a Hilbert space with the norm and inner product of $H^m[a,b]$. We follow a procedure developed in [1, 7, 10], e.g., to define the space $H^{-m}[a,b]$ as the dual of $H_0^m[a,b]$ with respect to $L^2[a,b]$. The space $H_0^m[a,b]$ may be imbedded in $L^2[a,b]$ as a dense subspace relative to $\| \ \|_{L^2[a,b]}$—this is a familiar and easily established result. For each $z \in L^2[a,b]$, the linear functional

$$l_z(y) = \langle y, z \rangle_{L^2[a,b]} \equiv \int_a^b y(t)\, z(t)\, dt \tag{3.2}$$

is continuous with respect $\|y\|_{L^2[a,b]}$ as $y$ ranges over $H_0^m[a,b]$. But since $\| \ \|_{H^m[a,b]}$ gives a stronger topology than $\| \ \|_{L^2[a,b]}$, $l_z(y)$ is also continuous with respect to $\|y\|_{H^m[a,b]}$. Since $H_0^m[a,b]$ is a Hilbert space relative to $( \ , \ )_{H^m[a,b]}$, there is a unique element $\zeta \in H_0^m[a,b]$ such that for $y \in H_0^m[a,b]$,

$$l_z(y) = \langle y, z \rangle_{L^2[a,b]} = \langle y, \zeta \rangle_{H^m[a,b]} \equiv \sum_{j=0}^{m} \int_a^b y^{(j)}(t)\, \zeta^{(j)}(t)\, dt. \tag{3.3}$$

Since $H_0^m[a,b]$ is dense in $L^2[a,b]$, the linear map

$$\zeta = K(z), \qquad K: L^2[a,b] \to H_0^m[a,b],$$

is well defined and one to one. Defining

$$(z_1, z_2)_{H^{-m}[a,b]} = (K(z_1), K(z_2))_{H^m[a,b]}, \tag{3.4}$$

$$\|z\|_{H^{-m}[a,b]} = [(z,z)_{H^{-m}[a,b]}]^{1/2}, \tag{3.5}$$

we have an inner product and norm on $L^2[a,b]$—which now becomes an inner product space with respect to this new norm. We define $H^{-m}[a,b]$ to be the completion of $L^2[a,b]$ with respect to $\| \ \|_{H^{-m}[a,b]}$. Then $H^{-m}[a,b]$ is a Hilbert space having $L^2[a,b]$ as a proper, dense subspace, again relative to $\| \ \|_{H^{-m}[a,b]}$. That $L^2[a,b]$ is a proper subspace follows from the fact that $\| \ \|_{H^{-m}[a,b]}$ is weaker than $\| \ \|_{L^2[a,b]}$ on $L^2[a,b]$. Indeed, given $z \in L^2[a,b]$

$$\|z\|_{L^2[a,b]} = \sup_{y \in H_0^m[a,b]} \frac{|l_z(y)|}{\|y\|_{L^2[a,b]}}$$
$$\geqslant \frac{1}{M} \sup_{y \in H_0^m[a,b]} \frac{|l_z(y)|}{\|y\|_{H^m[a,b]}},$$

where $M$ (which can be shown $< 1$) is such that

$$\|y\|_{L^2[a,b]} \leqslant M \|y\|_{H^m[a,b]}, \qquad y \in H_0^m[a,b].$$

But $l_z(y) = \langle y, K(z)\rangle_{H^m[a,b]}$, so

$$\sup_{y\in H_0^m[a,b]} \frac{|l_z(y)|}{\|y\|_{H^m[a,b]}} = \sup_{y\in H_0^m[a,b]} \frac{|\langle y, K(z)\rangle_{H^m[a,b]}|}{\|y\|_{H^m[a,b]}}$$

$$= \|K(z)\|_{H^m[a,b]} = \|z\|_{H^{-m}[a,b]}$$

and we have

$$\|z\|_{L^2[a,b]} \geqslant \frac{1}{M} \|z\|_{H^{-m}[a,b]}.$$

We remark that the norm $\| \ \|_m$ introduced in Corollary 4 could be used in $H^m[a,b]$ and $H_0^m[a,b]$ instead of the usual Sobolev norm. The resulting norm $\| \ \|_{-m}$ on $H^{-m}[a,b]$ is then equivalent to $\| \ \|_{H^{-m}[a,b]}$.

Proceeding with the proof of Theorem 2, we note that if $y \in H_0^m[a,b]$ and $z \in L^2[a,b]$, the linear functional (3.2) has been seen to be continuous with respect to $\|y\|_{H^m[a,b]}$ and $H^{-m}[a,b]$ has been seen to consist of limits of sequences $\{z_k\}$ in $L^2[a,b]$ relative to norm (3.5) thus defined for $z \in L^2[a,b]$. The form (3.2) may then be extended to $y \in H_0^m[a,b]$, $z \in H^{-m}[a,b]$ by continuity. For $y \in H_0^m[a,b]$, $\|y\|_{H^m[a,b]}$ is equivalent to $\|p(-D)\,y\|_{L^2[a,b]}$, where

$$p(-D) = \prod_{j=1}^{m} (-D - \sigma_j I) \tag{3.6}$$

(by Corollary 4 with $c = a$). However, the range of $p(-D)\,y$, $y \in H_0^m[a,b]$, is not all $L^2[a,b]$. If we let

$$p(-D)\,y = w, \qquad w \in L^2[a,b], \tag{3.7}$$

and use the boundary conditions

$$y^{(k)}(a) = 0, \qquad k = 0, 1, \dots, m-1$$

together with the variation of parameters formula, we find that for coefficients $c_j \neq 0$ (not obvious, but readily shown)

$$y(t) = \int_a^t \left( \sum_{j=1}^{m} c_j e^{-\sigma_j(t-s)} \right) w(s)\, ds, \qquad t \in [a,b]. \tag{3.8}$$

Differentiating $m-1$ times, evaluating the results, with (3.8) at $t = b$, where $y^{(k)}(b) = 0$, $k = 0, 1, \dots, m-1$, we see that

$$\sum_{j=1}^{m} c_j(-\sigma_j)^k\, e^{-\sigma_j b} \int_a^b e^{\sigma_j s} w(s)\, ds = 0, \qquad k = 1, 2, \dots, m.$$

Using the fact that the $c_j$ are not zero together with the nonsingularity of the Vandermonde matrix, we conclude that

$$\int_a^b e^{\sigma_j s} w(s)\, ds = 0, \qquad j = 1, 2,..., m, \tag{3.9}$$

whenever $w$ is given by (3.7) for $y \in H_0^m[a, b]$. Let us define

$$H_\Sigma'[a, b] = \left\{ w \in L^2[a, b] \;\middle|\; \int_a^b e^{\sigma_j s} w(s)\, ds = 0, j = 1, 2,..., m \right\}. \tag{3.10}$$

Then, relative to the norm induced from $L^2[a, b]$, $H_\Sigma'[a, b]$ is the dual space of $H_\Sigma[a, b]$, the closed span of the functions $e^{\lambda_k t}$, $\lambda_k \in \Lambda - \Sigma$, i.e., $e^{\lambda_k t}$, $\lambda_k \in \Lambda$, $\lambda_k \neq \sigma_j$, $j = 1, 2,..., m$, which then form a Riesz or strong basis for $H_\Sigma[a, b]$ according as $E(\Lambda)$ is a Riesz or strong basis for $L^2[a, b]$.

Since $p(-D)$ maps $H_0^m[a, b]$ in a one-to-one fashion onto $H_\Sigma'[a, b]$ with $p(-D) y = w$ having norm in $H_\Sigma'[a, b]$ equivalent to $\| y \|_{H^m[a,b]}$, we conclude that, given $z \in H^{-m}[a, b]$, there exists a unique $\zeta \in H_\Sigma[a, b]$ such that

$$l_z(y) = \langle y, z \rangle = \langle p(-D) y, \zeta \rangle_\Sigma, \tag{3.11}$$

where the form $\langle \ , \ \rangle_\Sigma$ is simply the restriction of (3.2) to $y \in H_\Sigma'[a, b]$, $z \in H_\Sigma[a, b]$.

Now the map $\mathscr{P}$: $H_\Sigma[a, b] \to H^{-m}[a, b]$ defined by $\mathscr{P}(\zeta) = z$ is the (Banach space) conjugate or dual to $p(-D)$: $H_0^m[a, b] \to H_\Sigma'[a, b]$. Since the latter is bounded and boundedly invertible, so is $\mathscr{P}$, using a familiar theorem from functional analysis. Now suppose, defining $p(\lambda)$ as in (1.15), we let

$$z = p(\lambda_k)\, e^{\lambda_k t} = p(D)\, e^{\lambda_k t}, \tag{3.12}$$

for $\lambda_k \in \Lambda - \Sigma$. Then $z \in L^2[a, b] \subseteq H^{-m}[a, b]$ and, using the boundary conditions satisfied by $y \in H_0^m[a, b]$, integration by parts yields

$$\begin{aligned} \langle y, p(\lambda_k)\, e^{\lambda_k t} \rangle_{L^2[a,b]} &= \langle y, p(D)\, e^{\lambda_k t} \rangle_{L^2[a,b]} \\ &= \langle p(-D)\, y, e^{\lambda_k t} \rangle_{L^2[a,b]} = \langle p(-D)\, y, e^{\lambda_k t} \rangle_\Sigma . \end{aligned}$$

Comparison with (3.11) shows that when (3.12) is true, we must have $\zeta(t) = e^{\lambda_k t}$. Thus

$$\mathscr{P}: e^{\lambda_k t} \to p(\lambda_k)\, e^{\lambda_k t}, \qquad \lambda_k \in \Lambda - \Sigma.$$

Since $\mathscr{P}$ is bounded and boundedly invertible, the functions $p(\lambda_k)\, e^{\lambda_k t}$, $\lambda_k \in \Lambda - \Sigma$, form the same sort of basis for $H^{-m}[a, b]$ as the $e^{\lambda_k t}$ do for $H_\Sigma[a, b]$. It follows that the $p(\lambda_k)\, e^{\lambda_k t}$, $\lambda_k \in \Lambda - \Sigma$ form a Riesz (strong) basis for $H^{-m}[a,b]$ if the $e^{\lambda_k t}$, $\lambda_k \in \Lambda$ form a Riesz (strong) basis for $L^2[a,b]$. This completes the proof of Theorem 2.

Using differentiation in the sense permitted by the theory of distributions, the map $\mathscr{P}$: $H_\Sigma[a,b] \to H^{-m}[a,b]$ may be identified with $p(D)$. If $\mathscr{P}$ is extended from $H_\Sigma[a,b]$ to $L^2[a,b]$ by defining $\mathscr{P}(e^{\sigma_j t}) = 0$, $j = 1, 2, \dots, m$, this again agrees with $p(D)$. This observation permits one to compute expansion coefficients, relative to $H^{-m}[a,b]$, for an element $z \in H^{-m}[a,b]$, by representing $z$ as

$$z = p(D)\,\zeta, \qquad \zeta \in L^2[a,b].$$

For if

$$\zeta = \sum_{\lambda_k \in \Lambda} \zeta_k e^{\lambda_k t},$$

then

$$z = p(D)\,\zeta = \sum_{\lambda_k \in \Lambda - \Sigma} \zeta_k\, p(\lambda_k)\, e^{\lambda_k t}$$

in $H^{-m}[a,b]$. Consider, in particular, one of the exponentials $e^{\sigma_l t}$, $l = 1, 2, \dots, m$, missing from $E(\Lambda - \Sigma)$. Suppose in $L^2[a,b]$ we have

$$\begin{aligned} te^{\sigma_l t} &= \sum_{\lambda_k \in \Lambda} \zeta_k e^{\lambda_k t} \\ &= \sum_{j=1}^{m} \zeta_j e^{\sigma_j t} + \sum_{\lambda_k \in \Lambda - \Sigma} \zeta_k e^{\lambda_k t}. \end{aligned}$$

Then

$$e^{\sigma_l t} = p(D)(te^{\sigma_l t}) = \sum_{\lambda_k \in \Lambda - \Sigma} \zeta_k\, p(\lambda_k)\, e^{\lambda_k t}.$$

For example, the functions $e^{2\pi i k t}$, $-\infty < k < \infty$, form an orthonormal basis for $L^2[0,1]$. Deleting $k = 0$, and taking $p(\lambda) = \lambda - 0 = \lambda$, we see that the functions $2\pi i k e^{2\pi i k t}$, $k \neq 0$, form a uniform basis for $H^{-1}[0,1]$. Since in $L^2[0,1]$

$$t = \sum_{k=-\infty}^{\infty} \zeta_k e^{2\pi i k t},$$

$$\begin{aligned} \zeta_k = \int_0^1 te^{-2\pi i k y}\, dt &= \tfrac{1}{2}, \qquad && k = 0, \\ &= \frac{-1}{2\pi i k}, \qquad && k = \pm 1, \pm 2, \dots, \end{aligned}$$

we conclude that

$$1 = Dt(=p(D)(t)) = \sum_{k=1}^{m} D\left(\frac{-1}{2\pi ik} e^{2\pi ikt} + \frac{1}{2\pi ik} e^{-2\pi ikt}\right)$$

$$= -\sum_{k=1}^{m} (e^{2\pi ikt} + e^{-2\pi ikt})$$

$$= -2\sum_{k=1}^{m} \cos 2k\pi t,$$

convergent in the topology of $H^{-1}[0, 1]$. This agrees with the result which one obtains with a variety of summation procedures.

## 4. Modifications for Generalized Exponentials

We have restricted our arguments in Sections 2 and 3 to the case where the complex numbers $\sigma_1, \sigma_2,..., \sigma_m$, $\lambda_k$, $k \in K$ (Theorem 1), or $\lambda_k$, $k \in K$ (Theorem 2), are distinct. To have done otherwise would have substantially lengthened our presentation and obscured the simplicity of the proofs. Nevertheless, important instances do arise wherein one must consider generalized exponential bases. We begin this section with a brief discussion of the modifications necessary to treat this case.

Again we let $K$ be a countable index set. We suppose that $\Lambda = \{\lambda_k \mid k \in K\}$ is a set of distinct complex numbers and we let $N = \{\nu_k \mid k \in K\}$ be a set of positive integers. We will assume that there is a positive integer $\nu$ such that $\nu_k \leqslant \nu$, $k \in K$. This is satisfied in practically every case of interest. We suppose then that

$$E(\Lambda, N) = \{(t^j/j!)\, e^{\lambda_k t} \mid k \in K, j = 0, 1,..., \nu_k - 1\} \tag{4.1}$$

forms a Riesz (or strong) basis for $L^2[a, b]$. In modifying Theorem 1 we select $r$ distinct complex numbers $\sigma_1, \sigma_2,..., \sigma_r$, wich may, or may not, already be members of $\Lambda$, and $r$ positive integers $\mu_1, \mu_2,..., \mu_r$ such that $\sum_{j=1}^{r} \mu_j = m$. If $\sigma_j \notin \Lambda$, we propose to adjoin the functions

$$(t^l/l!)\, e^{\sigma_j t}, \qquad l = 0, 1,..., \mu_j - 1, \tag{4.2}$$

to $E(\Lambda, N)$. If $\sigma_j = \lambda_k \in \Lambda$ and the multiplicity associated with $\lambda_k$ is $\nu_k$, then we propose to adjoin

$$(t^{\nu_k + l}/(\nu_k + l)!)\, e^{\sigma_j t}, \qquad l = 0, 1,..., \mu_j - 1. \tag{4.3}$$

The main modification involves replacement of the formula (2.6). All we need do, clearly, is to obtain appropriate particular solutions, for an arbitrary positive integer $q$, of

$$p(D)\,\hat{y}_{k,q}(t) = (t^{q-1}/(q-1)!)\, e^{\lambda_k t}. \tag{4.4}$$

Forming the Laplace transform we have

$$p(\lambda)\,\mathscr{L}(\hat{y}_{k,q}, \lambda) + r_k(\lambda) = \frac{1}{(\lambda-\lambda_k)^q}, \tag{4.5}$$

where $r_k(\lambda)$ is a polynomial in $\lambda$ of degree $\leqslant m-1$ whose coefficients can be selected at will be appropriate choice of the initial conditions on $\hat{y}_{k,q}$. When $\lambda_k \notin \Sigma$ we try a solution of (4.5) of the form

$$\mathscr{L}(\hat{y}_{k,q}, \lambda) = \sum_{j=1}^{q} \frac{\alpha_j}{(\lambda-\lambda_k)^j}. \tag{4.6}$$

Expanding $p(\lambda)$ about $\lambda = \lambda_k$, (4.5) becomes

$$\left(\sum_{l=0}^{m} \frac{p^{(l)}(\lambda_k)}{l!}(\lambda-\lambda_k)^l\right)\left(\sum_{j=1}^{q} \frac{\alpha_j}{(\lambda-\lambda_k)^j}\right) + r_k(\lambda) = \frac{1}{(\lambda-\lambda_k)^q}.$$

Multiplying both sides by $(\lambda-\lambda_k)^q$ we have

$$\left(\sum_{l=0}^{m} \frac{p^{(l)}(\lambda_k)}{l!}(\lambda-\lambda_k)^l\right) \times \left(\sum_{j=1}^{q} \alpha_j(\lambda-\lambda_k)^{q-j}\right) + (\lambda-\lambda_k)^q\, r_k(\lambda) = 1. \tag{4.7}$$

Taking the $\alpha_j$ to be the appropriate coefficients in the Taylor expansion of $1/p(\lambda)$ about $\lambda = \lambda_k$, i.e.,

$$\alpha_j = \frac{1}{(q-j)!}\frac{d^{q-j}}{(d\lambda)^{q-j}}\left(\frac{1}{p(\lambda)}\right)\Bigg|_{\lambda=\lambda_k}, \qquad j = 1, 2, \ldots, q, \tag{4.8}$$

(4.7) assumes the form

$$(\lambda-\lambda_k)^q\,(\rho_k(\lambda) - r_k(\lambda)) = 0$$

where $\rho_k(\lambda)$ is a polynomial in $\lambda$ of degree $\leqslant m-1$. The appropriate choice of $r_k(\lambda)$ reduces this to an identity. Taking the inverse transform of (4.6), we have

$$\hat{y}_{k,q}(t) = \sum_{j=1}^{q} \left(\frac{1}{(q-j)!}\frac{d^{q-j}}{(d\lambda)^{q-j}}\left(\frac{1}{p(\lambda)}\right)\Bigg|_{\lambda=\lambda_k}\right) \frac{t^{j-1}}{(j-1)!}\, e^{\lambda_k t}. \tag{4.9}$$

When we carry this out for $q = 1, 2, \ldots, \nu_k$ we obtain, for

$$p(D)\,\hat{y}_k(t) = \sum_{q=1}^{\nu_k} \beta_q \frac{t^{q-1}}{(q-1)} e^{\lambda_k t}, \tag{4.10}$$

the particular solution

$$\hat{y}_k(t) = \sum_{p=1}^{\nu_k} \alpha_p \frac{t^{p-1}}{(p-1)!} e^{\lambda_k t},$$

where the $\alpha_p$, $\beta_q$ are related by

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\nu_{k-1}} \\ \alpha_{\nu_k} \end{bmatrix} \begin{bmatrix} \frac{1}{p(\lambda_k)} & & & \bigcirc \\ & \frac{1}{p(\lambda_k)} & & \\ & & \ddots & \\ * & & \frac{1}{p(\lambda_k)} & \\ & & & \frac{1}{p(\lambda_k)} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{\nu_{k-1}} \\ \beta_{\nu_k} \end{bmatrix}. \tag{4.11}$$

In (4.11) the element in the $p$th row and $q$th column of the matrix region designated "*" is

$$\frac{1}{p-q} \frac{d^{p-q}}{(d\lambda)^{p-q}} \left( \frac{1}{p(\lambda)} \right) \Bigg|_{\lambda = \lambda_k}. \tag{4.12}$$

Since $1/p(\lambda_k) = \mathcal{O}(1/|\lambda_k|^m)$ as $|\lambda_k| \to \infty$ while all the expressions (4.12) are uniformly $\mathcal{O}(1/|\lambda_k|^{m+1})$ as $|\lambda_k| \to \infty$ we have

$$\alpha_p = \frac{1}{p(\lambda_k)} \beta_p + \mathcal{O}\left( \frac{1}{|\lambda_k|^{m+1}} \right) \left( \sum_{q=1}^{\nu_k} |\beta_q| \right), \qquad |\lambda_k| \to \infty. \tag{4.13}$$

When $\lambda_k \in \Sigma$, i.e., is one of the $\sigma_j$, which can occur for only finitely many $\lambda_k$, a similar computation shows that the solution of (4.4) takes the form

$$\hat{y}_k(t) = \sum_{p=1}^{\nu_k} \alpha_p \frac{t^{\mu_j + p - 1}}{(\mu_j + p - 1)!} e^{\sigma_j t},$$

an expression which incorporates the ast $\nu_k$ terms in (4.3). The first $\mu_j$ terms then come from the solutions

$$\frac{t^{l-1}}{(l-1)!} e^{\sigma_j t}, \qquad l = 1, 2, \ldots, \mu_j,$$

of the homogeneous equation, the coefficients being determined in much the same way as the coefficients of $e^{\sigma_j t}$ were obtained in the proof of Theorem 1.

The rest of the proof of Theorem 1 is largely unchanged. The estimate (4.13) shows that in passing from $L^2[a, b]$ to $H^m[a, b]$ the basis element $(t^l/l!)\, e^{\lambda_k t}$ may be replaced by $1/p(\lambda_k)$ times that same element (for all but finitely many $\lambda_k$) to obtain, along with the adjoined elements indicated in (4.2) and (4.3), a basis for $H^m[a, b]$ of the same character (Riesz or strong). Thus the result for generalized exponentials is essentially the same as that for the case already discussed.

The modifications in Theorem 2 are of much the same variety. We must, of course, assume, since $\sigma_j = \text{some } \lambda_k$, that $\mu_j \leqslant \nu_k$. From $(t^l/l!)\, e^{\lambda_k t}$, $l = 0, \ldots, \nu_k - 1$, we then delete $(t^l/l!)\, e^{\lambda_k t}$, $l = \nu_k - \mu_j, \ldots, \nu_k - 1$. The factor $p(\lambda_k)$ again comes into play in much the same way as in Section 3.

*Additional notes and references.* The author is indebted to the reviewer of this paper for some additional references [23–25] which, while not used directly in our development, are listed here because of their pertinence to the subject material. Pavlov gives a condition for $e^{i\lambda_n t}$ to be a Riesz basis for $L^2[0, a]$ which is both necessary and sufficient, while Hruščev uses this condition to obtain Kadec's result to the effect that $\frac{1}{4}$ is best possible in (1.10). Young's book is a comprehensive introduction to the theory of Riesz bases of complex exponentials.

## References

1. R. A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
2. R. P. Boas, Jr., A trigonometric moment problem, *J. London Math. Soc.* **14** (1939), 242–244.
3. R. . Duffin and J. J. Eachus, Some notes on an expansion theorem of Paley and Wiener, *Bull. Amer. Math. Soc.* **48** (1942), 850–855.
4. R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72** (1952), 341–366.
5. A. E. Ingham, Some trigonometrical inequalities with applications to the theory of series, *Math. Z.* **41** (1936), 367–379.
6. M. I. Kadec, The exact value of the Paley–Wiener constant, *Soviet Math.* **5** (1964), 559–561.
7. P. D. Lax, On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations, *Comm. Pure Appl. Math.* **8** (1955), 615–633.

8. B. JA. LEVIN, "Distribution of Zeros of Entire Functions," Vol. 5, Translations of Mathematical Monographs, Amer. Math. Soc., Providence, R. I., 1964.
9. N. LEVINSON, "Gap and Density Theorems," Colloquium Publications, Vol. 26, Amer. Math. Soc., Providence, R. I., 1940.
10. J. L. LIONS AND E. MAGÈNES, "Problèmes aux limites nonhomogenes et applications," Vol. 1, Dunod, Paris, 1968.
11. R. E. A. C. PALEY AND N. WIENER, "Fourier Transforms in the Complex Domain," Colloquium Publications, Vol. 19, Amer. Math. Soc., Providence, R. I., 1934.
12. R. REDHEFFER, Remarks on the incompleteness of $\{e^{i\lambda_n x}\}$, non-averaging sets, and entire functions, *Proc. Amer. Math. Soc.* **2** (1951), 365–369.
13. R. REDHEFFER, Completeness of sets of complex exponentials, *Adv. in Math.* **24** (1977), 1–62.
14. F. RIESZ AND B. SZ.-NAGY, "Functional Analysis," Ungar, New York, 1955.
15. D. L. RUSSELL, "Closed-loop Eigenvalue Specification for Infinite Dimensional Systems: Augmented and Deficient Hyperbolic Cases," Tech. Summ. Rep. 2021, Mathematics Research Center, University of Wisconsin, Madison, 1979.
16. D. L. RUSSELL, Nonharmonic Fourier series on the control theory of distributed parameter systems, *J. Math. Anal. Appl.* **18** (1967), 542–559.
17. D. L. RUSSELL, Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems, *J. Math. Anal. Appl.* **62** (19788), 186–225.
18. D. L. RUSSELL, Uniform bases of exponentials, neutral groups, and a transform theory for the spaces $H^m[a, b]$, to appear.
19. L. SCHWARTZ, Approximation d'une fonction quelconque, *Ann. Fac. Sci. Univ. Toulouse*, 4$^e$ *Ser.* **6** (1942), 111–174.
20. L. SCHWARTZ, "Etude des sommes d'exponentielles," 2nd ed., Hermann, Paris, 1950.
21. D. ULLRICH, Difference quotients and systems of nonharmonic Fourier series, to appear.
22. N. WIENER, On the closure of certain assemblages of trigonometrical functions, *Proc. Nat. Acad. Sci. USA* **13** (1927), 27–29.
23. S. V. HRUŠČEV, Perturbations theorems for bases of exponentials and Muckenhoupt's condition, *Sov. Math. Dokl.* **20** (1979), 665–669.
24. B. S. PAVLOV, Basicity of an exponential system and Muckenhoupt's condition, *Sov. Math. Dokl.* **20** (1979), 655–659.
25. R. M. YOUNG, "An Introduction to Nonharmonic Fourier Series," Academic Press, New York, 1980.