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Research Article

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Abstract : This paper studies a linear hyperbolic system with boundary conditions that was first studied for someweaker conditions in [8, 11]. Problems on the expansion of a semigroup and a criterion for being a Riesz basis are discussed in the present paper. It is shown that the associated linear system is the infinitesimal generator of a C_0-semigroup ; its spectrum consists of zeros of a sine-type function, and its exponential system $\{e^{\lambda_n t}\}_{n\geq 1}$ constitutes a Riesz basis in $L^2[0,T]$. Furthermore, by the spectral analysis method, it is also shown that the linear system has a sequence of eigenvectors, which form a Riesz basis in Hilbert space, and hence the spectrum-determined growth condition is deduced.

Keywords : Hyperbolic system, eigenvalues, eigenvectors, sine-type function, Riesz basis
MSC 2010 : 15A42, 65F15, 65H17, 46A35, 46B15, 47A55

1 Introduction

In a separable Hilbert space, the most important bases are orthonormal bases. Second in importance are those bases that are bases equivalent to some orthonormal basis. They will be called Riesz bases, and they constitute the largest and most tractable class of bases known. For a linear system, the property that the eigenvectors of the linear system forms a Riesz basis for the state Hilbert space is one of the most important features from both theoretical and practical points of view.
Mathematically, the general Riesz basis property is developed in the context of nonharmonic Fourier series, which originated from the works of Paley and Winer and was developed later by many former Soviet mathematicians ; we can quote most of them [8, 9, 15, 16].

In this paper, we consider on a Hilbert space $\mathcal{H}$ the linear system

$$\begin{cases} \dfrac{d}{dt}\phi(t) = \mathcal{A}\phi(t), t > 0, \\ \\ \phi(0) = \phi_0. \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is an unbounded linear operator on $\mathcal{H}$ and $\phi_0 \in \mathcal{H}$.

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If the operator $\mathcal{A}$ generatesa C_0-semigroup $T(t)$ on $\mathcal{H}$, then the solution of (1.1) can be written in $L^2([0,T],\mathcal{H})$ as $\phi(t) := T(t)\phi_0$.
In this case, we will also claim that this property be essential in our paper. In many applications, the differential operator $\mathcal{A}$ arises from a partial differential equation, for which it is know that (1.1) has a solution. Hence the assumption that $\mathcal{A}$ generates a C_0-semigroup is not strong. However, for our result, the semigroup property does not suffice ; we need that $\mathcal{A}$ generates a group, i.e., (1.1) possesses a unique solution forward and backward in time. Since we also assume that the eigenvalues lie in a strip parallel to the imaginary axis, the group condition is not very restrictive. In order to apply the Pavlov theorem [13] to get a Riesz basis of expansion of eigenvalues of the general linear system in Hilbert space, we need to relate eigenvectors of the system operator with the exponential system through its spectrum.
In spectral theory, to show that system (1.1) satisfies the spectrum-determined growth condition is an interesting but difficult problem. The spectrum-determined growth holds for the C_0-semigroup generated by $\mathcal{A}$. However, verifying this condition is difficult because the semigroup usually may not have a workable expansion. So using the Riesz basis of the eigenvectors of the studied system becomes an attractive alternative.
In this model that we study in this paper, we need to discuss properties of the C_0-semigroup. Since, very frequently, we merely have information about the infinitesimal generator $\mathcal{A}$, it is important to know how to deduce these properties from those of $\mathcal{A}$. It is long this line of thoughts that we try to use spectral information of $\mathcal{A}$ to investigate the spectrum-determined growth. The approach that we take to prove our result is different to the one taken in [4, 6, 8, 10, 11]. Since this Riesz basis property is so important, there is extensive literature on this problem ; we refer to the papers [4, 6, 9, 10] where the same problem is treated for a specific perturbation of a closed linear operator. The basic approach is to estimate eigenvalues, the exponential family of eigenvalues, and then find some transformation to transform the set of the family of a perturbed operator to an orthonormal basis of eigenvectors.
We concentrate our attention in the second part of this paper to giving a unified treatment for the following general class of linear hyperbolic systems :

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u(x,t) \\ w(x,t) \end{pmatrix} + C\begin{pmatrix} u(x,t) \\ w(x,t) \end{pmatrix} = \begin{pmatrix} h(x,t) \\ k(x,t) \end{pmatrix} \\ u(0,t) = \alpha_1 w(0,t), \\ u(L,t) = \beta_1 w(L,t) \\ \alpha_1, \beta_1 \neq \{-1,0,1\} \end{cases} \tag{1.2}$$

where

$$C = \begin{pmatrix} 0 & -c(x)\dfrac{d}{dx} \\ -c(x)\dfrac{d}{dx} & 0 \end{pmatrix} \tag{1.3}$$

with a strictly positive function $c(x)$, which was first studied by Intissar [8] in the case where the function $c(x)$ is considered constant, equal to one, with a unique restrictive termination. A nice summary of the later development can be found in [11] under some transformation on the operator C. Among them, a powerful concept of a Riesz basis was introduced to prove that the eigenvectors of C form a Riesz basis in Hilbert space. However, it appears that a mathematical treatment of this problem has not yet been undertaken in the case where the operator C is given in the above form (1.3). The exact question is to know whether the eigenvectors of this

operator (1.3) form a Riesz basis in Hilbert space. Answering this equation is important in this model. It is the topic that we shall study in this paper.
The main concerns of this paper are
(a) Riesz basis property of exponential family,
(b) expansion of the solution in terms of the eigenvectors under nonharmonic Fourier series,
(c) spectrum-determined growth condition.
We show that the spectrum of system (1.1) consists of zeros of a sine-type function where the expansion family $\{e^{\lambda_n t}\}_{n\geq 1}$ and its eigenvectors form a Riesz basis.
Now let us outline the content of this paper. In Section 2, we recall some preliminary results and definitions that we will be using. In Section 3, we introduce our basic result. Section 4 is devoted to applying the main result for the linear hyperbolic system. In other words, we show that the Riesz basis property as well as the spectrum-determined growth condition hold for the studied system.

2 Preliminary results

For the sake of completeness, we first recall the following notions and preliminary results which will be used in the sequel.
Let $\mathcal{H}$ be a Hilbert space, and let T be a closed operator with dense domain $D(\mathcal{A})$.
Let $\sigma(T)$ (resp. $\rho(T)$) denote the spectrum (resp. the resolvent set) of T. For any $\lambda \in \rho(T)$, $R(\lambda, T) := (\lambda - T)^{-1}$ is called the resolvent of T.
In the remaining part of this section, we will give the characterization of the Riesz basis of an exponential family developed by Pavlov [13]. Before stating this result, we shall first recall some basic facts about functions of sine type due to Avdonin and Ivanov [3] formulated as follows :

Definition 2.1. An entire function $f(z)$ is said to be of exponential type if the inequality $\mid f(z) \mid \leq Ae^{B|z|}$ holds for some positive constants A and B and all complex values of z. The smallest of constants B is said to be the exponential type of $f(z)$.

Definition 2.2. An entire function f of exponential type is said to be of sine type if
(i) the zeros of f lie in a strip $\{z \in \mathbb{C}; \mid \Im m z \mid \leq H\}$ for some $H > 0$,
(ii) there exist $h \in \mathbb{R}$, and positive constants c_1, c_2 such that $c_1 \leq \mid f(x+ih) \mid \leq c_2$ for all $x \in \mathbb{R}$.

We want to mention Pavlov's characterization of exponential Riesz bases [13].

Theorem 2.1 ([13]). Let $\Lambda := \lambda_n$ be a countable set of complex numbers. The family $\{e^{i\lambda_n t}\}_{n=1}^{\infty}$ forms a Riesz basis in $L^2[0, T]$ if and only if the following conditions are satisfied :
(i) the sequence $\{\lambda_n\}_{n=1}^{\infty}$ lies in a strip parallel to the real axis, $sup_{n\geq 1} \mid \Im m\lambda \mid < \infty$
(ii)$\{\lambda_n\}_{n=1}^{\infty}$ is separated in the sense that $\delta := inf_{m,n} \mid \lambda_n - \lambda_m \mid > 0$;
(iii) the generating function of the family $\{e^{i\lambda_n t}\}_{n=1}^{\infty}$ on the interval $[0, T]$ satisfies the Muckenhoupt condition

$$sup_{I\in\mathcal{J}}\{\frac{1}{\mid I \mid^2}\int_I \mid f(x+ih) \mid^2 dx \int_I \mid f(x+ih) \mid^{-2} dx\} < \infty \text{ for some } h \in \mathbb{R}\,;$$

where $\mathcal{J}$ is the set of all intervals of the real axis.

Remark 2.1. It is seen that if the generation function f of $\{e^{i\lambda_n t}\}_{n=1}^{\infty}$ is of sine type, then conditions (i) and (iii) of Theorem 2.1 are always satisfied.

Before recalling the results of the next theorem that are used subsequently, we begin by recalling the useful definition due to M. V. Keldys.

Definition 2.3. Let K be a compact operator in a Hilbert space $\mathcal{H}$. Then K is said to belong to the Carleman class $\mathcal{C}_p$, $p > 0$, with order p if the series $\sum_{n=1}^{+\infty} s_n^p(K)$ converges, where the $s_n(K)$ are the eigenvalues of the operator $\sqrt{K^*K}$

In the particular case $p = 2$, $\mathcal{C}_2$ is exactly the space of Hilbert-Schmidt operators, and for $p = 1, \mathcal{C}_1$ is called the space of nuclear operators or the space of trace class operators on $\mathcal{H}$.

Now let T be a closed operator with dense domain $D(T)$ in a Hilbert space $\mathcal{H}$. We suppose that $0 \in \rho(T)$, and we put $K = T^{-1}$. If K belongs to the Carleman class $\mathcal{C}_p$ for some $p > 0$, then Lidskii [12] obtained estimates for the growth of the entire functions D_λ and $d(\lambda)$ as $\lambda \longrightarrow \infty$. For the operators of this class, Lidskii [12] proved the following result.

Theorem 2.2. If K belongs to the Carleman class $\mathcal{C}_p$ for some $p > 0$, then $(I - \lambda K)^{-1}$ can be represented as the ratio of two entire functions $(I - \lambda K)^{-1} = \dfrac{D_\lambda}{d(\lambda)}$ for which the inequalities $\mid d(\lambda) \mid \leq e^{c_1|\lambda|^p}$ and $\mid\mid D(\lambda) \mid\mid \leq e^{c_2|\lambda|^p}$ are valid for all $\lambda \in \mathbb{C}$ (c_1 and c_2 being positive constants).

In this work, we are interested to proof the Riesz basis property of the eigenvectors associated to unbounded linear operator involving the concept of the exponential Riesz basis family. For this purpose,we need to review the following theorem due to Aimar, Intissar and Paoli [1, 2].

Theorem 2.3. Let T be a closed linear operator with dense domain $D(T)$ on a Hilbert space $\mathcal{H}$. Suppose that (H1) there exists $\lambda \in \rho(T)$such that $K = (T - \lambda I)^{-1}$ is a compact operator,
(H2) there exists $p > 0$ such that $s_n(K) = O(n^{-\frac{1}{p}})$ as $\longrightarrow \infty$;
(H3) for $\mid \lambda \mid$ sufficiently large, the resolvent $R(\lambda, T) := (T - \lambda I)^{-1}$ is bounded on each m rays $\gamma_1, \gamma_2, ..., \gamma_m$ centered at the origin, divided on the all complex plane such that $Arg\gamma_1 = \dfrac{\pi}{2} < Arg\gamma_2 < .... < Arg\gamma_m = \dfrac{3\pi}{2}$ with $Arg\gamma_{j+1} - Arg_j < \dfrac{p}{2}, 0 \leq j \leq m-1$,($m \geq [2p]+$ 1, where $[2p]$ designates the entire part of $2p$),
(H4) $\mid\mid R(\lambda, T) \mid\mid = O(e^{|\lambda|^p})$ as $\mid \lambda \mid \longrightarrow \infty$.
Then the system of generalized eigenvectors of T is complete in $\mathcal{H}$.

3 Main results

In this section, we show that the eigenvectors of system (1.1) form a Riesz basis in Hilbert space $\mathcal{H}$, which makes us conclude its spectrum-determined growth.

Theorem 3.1. Let A be a linear unbounded densely defined operator on a separable Hilbert space $\mathcal{H}$ satisfying that
(i) A generates a C_0-semigroup,
(ii) the resolvent of A is compact,
(iii) the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are simple,
(iv) the family $\{e^{\lambda_n t}\}_{n=1}^{\infty}$ is a Riesz basis for $L^2(0, T), T > 0$,
(v) the system of eigenvectors of A is complete in $\mathcal{H}$.
Then the eigenvectors associated to the operator A form a Riesz basis in $\mathcal{H}$.

Proof. For the convenience of the reader, we give here a short proof of this theorem which

follows as a corollary of the results given by Xu and Yung [16].
We define the $\{T(t)\}$-invariant spectral subspace of $\mathcal{H}$ by $sp(A)$, where we define the set $sp(A)$ as $sp(A) := \{\sum_{k=1}^{m} E(\lambda_k, \mathcal{A})\phi,\ \forall\, \phi \in \mathcal{H}\ \forall\, m \in \mathbb{N}^*\}$, where $E(\lambda_k, \mathcal{A})$ denotes the Riesz projection on $\mathcal{H}$ associated to $\lambda_k \in \sigma(\mathcal{A})$.

Since the system of the generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$, then, for any $f \in \mathcal{H}$ and $\phi \in sp(A)$, the function $< T(t)\phi, f >$ is in the subspace $\bar{E}_1 := L^2[0,T]$, where we define E_1 by the span generated by the family $\{e^{\lambda_n t}\}_{n=1}^{\infty}\}$ as

$E_1 := span\{\sum_{k=1}^{n} a_k e^{\lambda_k t},\ \forall\, n \geq 1\}.$

Hence, for any $\phi \in \mathcal{H}$, $< T(t)\phi, f > \in L^2[0,T]$, and we can write them as

$< T(t)\phi, f > := \{\sum_{n=1}^{+\infty} E(\lambda_n, \mathcal{A})\phi e^{\lambda_n t}.$

Moreover, we infer from the Riesz property of the exponential family $\{e^{\lambda_n t}\}_{n=1}^{\infty}\}$ in $L^2[0,T]$ that there exist two positive constants K_1 and K_2 such that

$$K_1 \sum_{n=1}^{+\infty} |< E(\lambda_n, \mathcal{A})\phi, f >|^2 \leq \int_0^T |< T(t)\phi, f >|^2\, dt \leq K_2 \sum_{n=1}^{+\infty} |< E(\lambda_n, \mathcal{A})\phi, f >|^2.$$

According to assumption (i) with the property of the semigroup that $\| T(t) \| \leq Me^{wt}$ for $M \geq 1$ and $w \geq 0$ (see [14, Theorem 2.2, p. 4]), we get

$K_1 \sum_{n=1}^{+\infty} |< E(\lambda_n, \mathcal{A})\phi, f >|^2 \leq M^2[\frac{e^{2wT} - 1}{2w}] \| \phi \|^2 \| f \|^2.$

Consequently,

$K_1 \sum_{n=1}^{+\infty} \| E(\lambda_n, \mathcal{A})\phi \|^2 \leq M^2[\frac{e^{2wT} - 1}{2w}] \| \phi \|^2.$

Since $T(t)$ is the generator of a C_0-semigroup $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, then the adjoint $\mathcal{A}^*$ of $\mathcal{A}$ has the same property [7, p. 2354]; in other terms, $T^*(t)$ is the generator of a C_0-semigroup $\mathcal{A}^*$ with the same conjugate eigenvalues $\bar{\lambda}_n$ as $\mathcal{A}$, and each $\bar{\lambda}_n$is an isolated eigenvalue of $\mathcal{A}^*$ with $E^*(\lambda_n, \mathcal{A}) = E(\bar{\lambda}_n, \mathcal{A}^*)$.

For any $f, \phi \in \mathcal{H}$, $< T^*(t)\phi, f > = < \phi, T(t)f > \in L^2[0,T]$ and $< T^*(t)\phi, f > = \sum_{n=1}^{+\infty} < E(\bar{\lambda}_n, \mathcal{A}^*)\phi, f > e^{\bar{\lambda}_n t}$.

Reasoning in a similar way as above, we can easily check that

$K_1 \sum_{n=1}^{+\infty} \| E^*(\lambda_n, \mathcal{A})\phi \|^2 \leq M^2[\frac{e^{2wT} - 1}{2w}] \| \phi \|^2.$

For any $f \in sp(\mathcal{A})$, we can write them as

$f = \sum_{n=1}^{m} E(\lambda_k, \mathcal{A})f.$

So a short computation reveals that

$\| f \|^2 \leq \sum_{n=1}^{m} \| E(\lambda_k, \mathcal{A})f \|^2\ M^2[\frac{e^{2wT} - 1}{2wK_1}].$

. Taking the limit, we get

$\| f \|^2 \leq M^2[\frac{e^{2wT} - 1}{2wK_1}] \sum_{n=1}^{+\infty} \| E(\lambda_k, \mathcal{A})f \|^2$ for all $f \in \mathcal{H}$.

. Hence we deduce that

$$\frac{2wK_1}{M^2[e^{2wT}-1]}\sum_{n=1}^{+\infty} \| E(\lambda_k,\mathcal{A})f \|^2 \leq \| f \|^2 \leq M^2[\frac{e^{2wT}-1}{2wK_1}]\sum_{n=1}^{+\infty} \| E(\lambda_k,\mathcal{A})f \|^2 \text{ for all } f\in\mathcal{H}.$$
.

This makes us conclude that the series $\sum_{n=1}^{+\infty} E(\lambda_k,\mathcal{A})f$ converges in $\mathcal{H}$ with the sum equal to f and yields that $\{E(\lambda_n,\mathcal{A})f\}_{n=1}^{+\infty}$ forms a Riesz basis in $\mathcal{H}$.

As a consequence of the foregoing result from the Riesz basis property of eigenvectors, we have the following result.

Corollary 3.1. System (1.1) satisfies the spectrum-determined growth assumption, i.e., $S(A)=w(A)$, where $S(\mathcal{A}):=sup_{\lambda\in\sigma(\mathcal{A}}\Re e\lambda$ and $w(\mathcal{A}):=inf\{w;\exists c\geq 1, \| e^{t\mathcal{A}} \|\leq Ce^{wt},\forall t\geq 0\}$.

To illustrate the applicability of our main result, we concerns our attention in the next section to present an application of Theorem 3.1 to prove the Riesz basis property of the linear hyperbolic system.

4 Application to a linear hyperbolic system

In this section, we shall show that system (1.2) has a set of eigenvectors which form a Riesz basis in Hilbert space $\mathcal{H}$. The key idea is to apply some basic result of Pavlov [13], which provides a useful way to verify the Riesz basis property for the eigenvectors of a linear operator in Hilbert space.

Since the matrix $\begin{pmatrix} 0 & -c(x) \\ -c(x) & 0 \end{pmatrix}$ is symmetric with distinct eigenvalues $\lambda_i=(-1)^{i+1}c(x), i=1,2,$ introducing the change of variables :

$\begin{pmatrix} p \\ v \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, $\begin{pmatrix} f \\ g \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$, $\alpha=\frac{1-\alpha_1}{1+\alpha_1}$ and $\beta=\frac{1-\beta_1}{1+\beta_1}$.

Our system becomes

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} p(x,t) \\ v(x,t) \end{pmatrix} + \begin{pmatrix} c(x) & 0 \\ 0 & -c(x) \end{pmatrix}\frac{d}{dx}\begin{pmatrix} p(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} f(x,t) \\ g(x,t) \end{pmatrix} \\ \qquad p(0,t)=\alpha v(0,t), \\ \qquad p(L,t)=\beta w(L,t) \end{cases} \tag{4.1}$$

To turn system (1.2) into a framework of semigroups, we introduce the underlying state Hilbert space $\mathcal{H}:=L^2[0,L]\times L^2[0,L]$.
System (4.1) is then written as an evolutionary equation in $\mathcal{H}$,

$$\frac{d}{dt}W(t)=\mathcal{A}W(t), t>0, \quad \text{with } W(t)=\begin{pmatrix} (p(.,t) \\ v(.,t)) \end{pmatrix}$$

and $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}$ defined by

$$\mathcal{A} := C(x)\frac{d}{dx} := \begin{pmatrix} c(x) & 0 \\ 0 & -c(x) \end{pmatrix} \frac{d}{dx} \tag{4.2}$$

with domain

$$D(\mathcal{A}) = \{ \begin{pmatrix} p(x) \\ v(x) \end{pmatrix} \in H^1[0,L] \times H^1[0,L] : p(0) = \alpha v(0), p(L) = \beta v(L)\} \tag{4.3}$$

In this part, our goal is to give some important properties about a linear hyperbolic system to verify the assumptions of Theorem 3.1 in several steps.We shall assume without loss of generality that $p(x) := p(x,t)$, $v(x) := v(x,t)$, $f(x) := f(x,t)$ and $g(x) := g(x,t)$. In the following, we establish some preliminary results about the operator $\mathcal{A}$.

Proposition 4.1. Let $\mathcal{A}$ be the operator given in (4.2) and (4.3). Then the following assertions are true :
(i) $D(\mathcal{A})$ is dense on $\mathcal{H}$.
(ii) The canonical injection i from $D(\mathcal{A})$ into $\mathcal{H}$ is compact.

Proof. (i) It is seen that $D(\mathcal{A})$ is a closed subset of the Sobolev space $H^1[0,L] \times H^1[0,L]$ which is dense in $\mathcal{H}$.
Then assertion (i) follows immediately.
(ii) On the other hand, we infer from the Sobolev theorem [5, Theorem VIII.7, p. 129] that the injection from $H^1[0,L]$ into $L^2[0,L]$ is compact. So the injection from $D(\mathcal{A})$ into $\mathcal{A}$ is too.

Theorem 4.1. The following statements hold :
(i) The operator $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$.
(ii) The resolvent expression $R(\lambda, \mathcal{A})$ of $\mathcal{A}$ can be represented as

$$R(\lambda,\mathcal{A}) \begin{pmatrix} f \\ g \end{pmatrix} (x) = M(x,0,\lambda) \begin{pmatrix} \alpha v(0) \\ v(0) \end{pmatrix} - \int_0^x M(x,s,\lambda) C^{-1}(s) \begin{pmatrix} f(s) \\ g(s) \end{pmatrix} ds$$

for all $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}$.

where

$$M(x,s,\lambda) = \begin{pmatrix} e_1(x,s,\lambda) & 0 \\ 0 & e_2(x,s,\lambda) \end{pmatrix}, \; e_i(x,s,\lambda) = e^{(-1)^{i+1}\lambda \int_s x C^{-1}(y)dy}, \; i = 1,2,$$

and

$$\begin{cases} v(0) = -\frac{1}{\Delta(\lambda)}[\int_0^L < M(L,s,\lambda) \begin{pmatrix} -c^{-1}(s) & 0 \\ 0 & c^{-1}(s) \end{pmatrix} \begin{pmatrix} f(s) \\ g(s) \end{pmatrix}, \begin{pmatrix} 1 \\ -\beta \end{pmatrix} > ds], \\ \Delta(\lambda) = \alpha e_1(L,0,\lambda) - \beta e_2(L,0,\lambda) \end{cases}$$

Proof. (i) Since the entries of the matrix $C(x)$ are real-valued C^1-functions in x, $\mathcal{A}$ generates an evolution operator.
(ii) First, to get the resolvent expression of the operator $\mathcal{A}$, we consider the inhomogeneous equation expressed as

$$(\lambda I - \mathcal{A})U = F, \tag{4.4}$$

where $\lambda \in \mathbb{C}$, $U := \begin{pmatrix} p(x) \\ v(x) \end{pmatrix} \in D(\mathcal{A})$ and $F := \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$ is arbitrary in $\mathcal{H}$.

An elementary calculation by the variation of constants reveals that the general solution of equation (4.4) is given by

$$R(\lambda, \mathcal{A}) \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = M(x,0,\lambda) \begin{pmatrix} \alpha v(0) \\ v(0) \end{pmatrix} - \int_0^x M(x,s,\lambda) C^{-1}(s) \begin{pmatrix} f(s) \\ g(s) \end{pmatrix} ds,$$

where we designate by $M(x,s,\lambda)$ the matrix

$$M(x,s,\lambda) = \begin{pmatrix} e_1(x,s,\lambda) & 0 \\ 0 & e_2(x,s,\lambda) \end{pmatrix}, \; e_i(x,s,\lambda) = e^{(-1)^{i+1}\lambda \int_s xC^{-1}(r)dr}, \; i = 1,2,$$

As a preparation for attacking the expression of $v(0)$, it is sufficient to solve $p(L) = \beta v(L)$ by recalling the resolvent of $\mathcal{A}$. So we get

$$[\alpha e_1(L,0,\lambda) - \beta e_1(L,0,\lambda)]v(0) = \int_0^L [e_1(L,s,\lambda)c^{-1}(s)f(s) + e_1(L,s,\lambda)c^{-1}(s)g(s)]ds$$

Denote by $\Delta(\lambda)$ the expression

$$\Delta(\lambda) = \alpha e_1(L,0,\lambda) - \beta e_1(L,0,\lambda)$$

which is an entire function on λ. Using the above equality with this notation, we get

$$\Delta(\lambda)v(0) = -\int_0^L < M(L,s,\lambda) \begin{pmatrix} -c^{-1}(s) & 0 \\ 0 & c^{-1}(s) \end{pmatrix} \begin{pmatrix} f(s) \\ g(s) \end{pmatrix}, \begin{pmatrix} 1 \\ -\beta \end{pmatrix} > ds$$

A short computation shows that a complex number λ belongs to the resolvent set of $\mathcal{A}$ if and only if $\Delta(\lambda) \neq 0$; in this case,

$$v(0) = -\frac{1}{\Delta(\lambda)} \int_0^L < M(L,s,\lambda) \begin{pmatrix} -c^{-1}(s) & 0 \\ 0 & c^{-1}(s) \end{pmatrix} \begin{pmatrix} f(s) \\ g(s) \end{pmatrix}, \begin{pmatrix} 1 \\ -\beta \end{pmatrix} > ds$$

which completes the proof.

Theorem 4.2. $\mathcal{A}$ is a discrete operator in $\mathcal{H}$; in other words, for any $\lambda \in \rho(\mathcal{A})$, $R(\lambda, \mathcal{A})$ is compact on $\mathcal{H}$. Therefore, the spectrum $\sigma(\mathcal{A})$ consists only of isolated eigenvalues.

Proof. It follows from Proposition 4.1 together with the fact $\rho(\mathcal{A}) \neq 0$, that $\mathcal{A}$ has a compact resolvent on $\mathcal{H}$. Thus its spectrum consists only of isolated eigenvalues, which are exactly the zeros set of $\Delta(\lambda)$.

For each $\lambda \in \sigma(\mathcal{A})$, all eigenvectors associated with λ can be represented as

$$\begin{pmatrix} p(x,\lambda) \\ v(x,\lambda \end{pmatrix} = \begin{pmatrix} \alpha e_1(x,0,\lambda)v(0) \\ e_2(x,0,\lambda)v(0) \end{pmatrix} \text{ for all non-zero } v(0) \text{ satisfying } \Delta(\lambda)v(0) = 0.$$

The next theorem gives information about the distribution of the eigenvalues of the operator $\mathcal{A}$.

Theorem 4.3. Each eigenvalue of $\mathcal{A}$ is simple and given by

$$\lambda_n = \frac{1}{2c_L} log \mid \frac{(1-\beta_1)(1+\alpha_1)}{(1-\alpha_1)(1+\beta_1)} \mid + i\frac{n\pi}{c_L},\ n \in \mathbb{Z}$$

Proof. An elementary calculation reveals that the zeros set of ?(?) consists of an infinite number of complex eigenvalues of the operator $\mathcal{A}$, which are formulated by the form

$$\lambda_n = \frac{1}{2c_L} log \mid \frac{\beta}{\alpha} \mid + i\frac{n\pi}{c_L},\ n \in \mathbb{Z}$$

where α, β are defined as above and c_L is defined by $c_L = \int_0^L c^{-1}(r)dr$

Obviously, we observe from the above formula that each eigenvalue λ_n is simple.
The corresponding eigenvectors associated to λ_n are

$$\begin{pmatrix} p(x,\lambda_n) \\ v(x,\lambda_n \end{pmatrix} = \begin{pmatrix} \alpha e_1(x,0,\lambda_n)v(0) \\ e_2(x,0,\lambda_n)v(0) \end{pmatrix} = \begin{pmatrix} \alpha e^{c_x[\frac{1}{2c_L}log|\frac{\beta}{\alpha}|+i\frac{n\pi}{c_L}]}v(0) \\ e^{-c_x[\frac{1}{2c_L}log|\frac{\beta}{\alpha}|+i\frac{n\pi}{c_L}]}v(0) \end{pmatrix}$$

where $c_x = \int_0^x c^{-1}(r)dr$

Therefore, the eigenvectors associated to λ_n have the asymptotic form

$$\begin{pmatrix} p(x,\lambda_n) \\ v(x,\lambda_n \end{pmatrix} = \begin{pmatrix} \alpha e^{\frac{c_x}{2c_L}log|\frac{\beta}{\alpha}|}v(0) & 0 \\ 0 & e^{-\frac{c_x}{2c_L}log|\frac{\beta}{\alpha}|}v(0) \end{pmatrix} \begin{pmatrix} e^{i\frac{c_x}{c_L}n\pi} \\ e^{-i\frac{c_x}{c_L}n\pi} \end{pmatrix}$$

It is easily seen that

$$H(x) := \begin{pmatrix} \alpha e^{\frac{c_x}{2c_L}log|\frac{\beta}{\alpha}|}v(0) & 0 \\ 0 & e^{-\frac{c_x}{2c_L}log|\frac{\beta}{\alpha}|}v(0) \end{pmatrix} \text{ is invertible in } x \text{ and } \mid H(x) \mid \neq 0.$$

The motivation for our studies is based on proving the Riesz basis property of $\{e^{\lambda_n t}\}$ in $L^2[0, 2cL]$. To obtain such property, explicit estimates for sine-type functions turn out to be good.

Lemma 4.1. $\{e^{\lambda_n t}\}$ forms a Riesz basis for $L^2[0, 2c_L]$.

Proof. It should be noted that our results are aimed to provide a description for the Riesz basis property of $\{e^{\lambda_n t}\}$ since the **real part** of the eigenvalues $\{\lambda_n\}$ is independent of n.

To claim this, let $g(\mu)$ be defined by $g(\mu) := sin(c_L\mu)$. The zeros set $\{\mu_n\}$ of $g(\mu)$ is related by $\mu_n = \frac{n\pi}{c_L}$.
. Obviously, $g(\mu)$ is uniformly bounded on the real axis, and hence g belongs to the Cartwright class. Thus its indicator diagram is an interval. Furthermore, it is easy to show that $g(\mu)$ is of exponential type with type $c_L = \frac{T}{2}$.

In fact, let $z := x + iy \in \mathbb{C}$. Then

$$| g(z) |= \sqrt{sin^2(c_L x)cosh^2(c_L y) + cos^2(c_L x)sinh^2(c_L y)} \leq| cosh(c_L y) |\leq e^{c_L|y|} \leq e^{c_L|z|} \ .$$

Moreover, from the last expression, we have

$$C_1 e^{c_L|y|} \leq| g(x + iy) |\leq C_1 e^{c_L|y|},$$

where $0 < C_1 < \frac{1}{2}, , C_2 \geq 1$ and all $y \in \mathbb{R}$ whenever y is sufficiently large. In what follows, item (ii) of Definition 2.2 is satisfied.
Moreover, it is seen that the zeros $\{\mu_n\}$ are separated. Indeed, let $n \in \mathbb{Z}$. Then we have

$$| \mu_{n+1} - \mu_n |=| \frac{(n+1)\pi}{c_L} - \frac{(n\pi}{c_L} = \frac{\pi}{c_L}$$

. As a consequence of the above sentences, we get the Riesz basis property by Theorem 2.1 for $\{e^{i\frac{n\pi}{c_L}t}\}$ in $L^2[0, 2c_L]$ and hence for $\{e^{\lambda_n t}\}$ in $L^2[0, 2c_L]$.

Proposition 4.2. Let $\mathcal{A}$ be the generator of a C_0-semigroup in a Hilbert space $\mathcal{H}$. Then the spectrum of its adjoint is given by $\sigma(\mathcal{A}^*) = \{\bar{\lambda} : \lambda \in \sigma(\mathcal{A})\}$.

Proof. Since $\mathcal{H}$ is a Hilbert space and $\mathcal{A}$ is the generator of a C_0-semigroup $T(t)$, then $T^*(t)$ is also a C_0-semigroup, and its generator is $\mathcal{A}^*$. Moreover, its spectrum consists of an isolated eigenvalue $\bar{\lambda}$, where $\lambda \in \sigma(\mathcal{A})$.

Theorem 4.4. The resolvent of the operator $\mathcal{A}$ belongs to the Carleman class $\mathcal{C}_{1+\epsilon}$ for every $\epsilon > 0$.

Proof. A short computation from the expression of the eigenvalues λ_n of the operator $\mathcal{A}$, which is given in Theorem 4.3, shows that

$$| \lambda_n |= \frac{n\pi}{| c_L |}\sqrt{1 + \frac{1}{4n^2\pi^2} log | \frac{\beta}{\alpha} |}$$

Therefore $| \lambda_n |\sim \frac{n\pi}{| c_L}$ as $n \longrightarrow +\infty$

Obviously, $0 \in \rho(\mathcal{A})$, so let $K := \mathcal{A}^{-1}$, and let $s_n(\mathcal{A}^{-1})$ be the eigenvalue of the operator $\sqrt{K^*K}$. It is clear that

$$\sum_{n=1}^{+\infty} \frac{1}{| \lambda_n |^{1+\epsilon}} < +\infty \text{ for every } \epsilon > 0.$$

Then $\mathcal{A}^{-1}$ belongs to the Carleman class $\mathcal{C}_{1+\epsilon}$ for every $\epsilon > 0$.

As a consequence of Theorem 2.2, we have the following corollary.

Corollary 4.1. The resolvent of $\mathcal{A}$ can be represented as two entire functions $R(\lambda, A) = \frac{F(\lambda, A)}{\Delta(\lambda)}$.

Proof. Let $\lambda \in \rho(\mathcal{A})$. Then we have

$R(\lambda, A) = (\lambda I - \mathcal{A})^{-1} = ([\lambda \mathcal{A}^{-1} - I]^{-1}\mathcal{A})^{-1} = \mathcal{A}^{-1}(\lambda \mathcal{A}^{-1} - I)^{-1}.$

Denoting $K := \mathcal{A}^{-1}$, then

$$R(\lambda, \mathcal{A}) = K(\lambda K - I)^{-1} = KR_\lambda(K). \tag{4.5}$$

Since the resolvent of $\mathcal{A}$ belongs to the class$\mathcal{C}_{1+\epsilon}$ for $\epsilon > 0$, then we infer from Theorem2.2 with equation (4.5) that the resolvent of $\mathcal{A}$ can be represented under the form $\dfrac{F(\lambda, \mathcal{A})}{\Delta(\lambda)}$, where $F(\lambda, \mathcal{A})$ and $\Delta(\lambda)$ are two entire functions.

Now we are in the position to provide the denseness of the eigenvectors of the operator $\mathcal{A}$.

Theorem 4.5. The system of the eigenvectors of the operator $\mathcal{A}$ is dense in $\mathcal{H}$.

Proof. From Theorem 4.4, the resolvent of $\mathcal{A}$ belongs to the Carleman class $\mathcal{C}_{1+\epsilon}$ for $\epsilon > 0$ since $R(\lambda, \mathcal{A})$ is an entire function of λ and the orders of both entire functions of $F(\lambda, \mathcal{A})$ and $\Delta(\lambda)$ are less than or equal to 1. That is, there is $\epsilon > 0$ such that

$|| R(\lambda, \mathcal{A}) ||= O(e^{|\lambda|^{1+\epsilon}})$ as $| \lambda |\longrightarrow +\infty$.

As $\mathcal{A}$ generates an analytic semigroup, it follows that $|| R(\lambda, \mathcal{A}) ||$ is uniformly bounded in both real and imaginary axis.We may assume without loss of generality that $|| R(\lambda, \mathcal{A}) ||$ is uniformly bounded in the right complex plane, particularly on the imaginary axis.

Set $\mathcal{S}_j = \{\lambda \in \mathbb{C} : Arg\gamma_j \leq Arg\lambda Arg\gamma_{j+1}\}, j = 1, 2.$

By assumption, $R(\lambda, \mathcal{A})$ is bounded on the boundary $\mathcal{S}_j$, and $|| R(\lambda, \mathcal{A}) ||= O(e^{|\lambda|^{|1+\epsilon}})$ for all $\lambda \in \mathcal{S}_j$, where $\epsilon > 0$ is chosen so that $1 + \epsilon < 2$. Applying the Phragmen-Lindelof theorem to $R(\lambda, \mathcal{A})$ in each $\mathcal{S}_j$ (see [17, Theorem 10, p. 80]), we known that $R(\lambda, \mathcal{A}$ is uniformly bounded in $\mathcal{S}_j$ and so in the whole complex plane.

Therefore, all conditions of Theorem 2.3 are satisfied with $p = 1 + \epsilon$ for $\epsilon > 0$, $m = 3$ and $\gamma_2 = \{\lambda : Arg\lambda = \pi\}$.

This result in the completeness of the generalized eigenvectors of $\mathcal{A}$ on $\mathcal{H}$.

We summarize all the results proved in this section by mentioning the following result.

Theorem 4.6. The eigenvectors associated to the operator $\mathcal{A}$ form a Riesz basis in $\mathcal{H}$.

Proof. From the previous results presented in this part, we infer that all the hypotheses of Theorem 3.1 are fulfilled, which makes us conclude that the system of eigenvectors associated to $\mathcal{A}$ forms a Riesz basis in the state Hilbert space $\mathcal{H}$.

As a consequence of the spectral mapping theorem for the C_0-semigroup generated by $\mathcal{A}$, we have the following result.

Corollary 4.2. Our system satisfies the spectrum-determined growth assumption, i.e.,
$S(\mathcal{A}) = w(\mathcal{A})$.

Remark. All the results in this paper cover the main results of [8, 11] as a special case and remove some additional conditions imposed there. The exponential family of the eigenvalues is then readily established from an expression of the eigenvalues, which is also obtained in the process of verification of the Riesz basis property.

Conclusion

Our work represents a specific case of the results given in [8, 11]. Our new contribution in this paper is considering a perturbed operator matrix A in terms of the variable x. New results and techniques are given to reach the Riesz basis property of the eigenvectors associated to an unbounded operator matrix A. However, this kind of study makes us conclude its spectrum-determined growth. More precisely, the study involves an elegant use of the notion of Pavlov's characterization of the exponential Riesz basis properties in order to deduce the Riesz basis property for the eigenvectors associated to the operator A. The abstract result is illustrated by an example of a linear hyperbolic system.

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References

[1] M.-T. Aimar, A. Intissar and J.-M. Paoli, Densité des vecteurs propres généralisés d'une classe d'opérateurs non auto-adjoints à résolvante compacte, C. R. Acad. Sci. Paris Sér. I Math. 315 (1992), no. 4, 393-396.
[2] M.-T. Aimar, A. Intissar and J.-M. Paoli, Densité des vecteurs propres généralisés d'une classe d'opérateurs compacts non auto-adjoints et applications, Comm. Math. Phys. 156 (1993), no. 1, 169-177.
[3] S. A. Avdonin and S. A. Ivanov, Families of Exponentials : The Method of Moments in Controllability Problems for Distributed Parameter Systems, Cambridge University, Cambridge, 1995.
[4] N. Ben Ali and A. Jeribi, On the Riesz basis of a family of analytic operators in the sense of Kato and application to the problem of radiation of a vibrating structure in a light fluid, J. Math. Anal. Appl. 320 (2006), no. 1, 78-94.
[5] H. Brézis, Analysis Functional Theory and Applications, Masson, Paris, 1999.
[6] S. Charfi, A. Jeribi and I. Walha, Riesz basis property of families of nonharmonic exponentials and application to a problem of a radiation of a vibrating structure in a light fluid, Numer. Funct. Anal. Optim. 32 (2011), no. 4, 370-382.
[7] N. Dunford and J. T. Schwartz, Linear Operators. Part III : Spectral Operators, John Wiley & Sons, New York, 1971.
[8] A. Intissar, Analyse functionnelle et théorie spectrale pour les opérateurs compacts non auto-adjoints et exercices avec solutions, Cépaduâs, Toulouse, 1997.
[9] A. Jeribi, Denseness, Bases and Frames in Banach Spaces and Applications, De Gruyter, Berlin, 2018.
[10] A. Jeribi and A. Intissar, On an Riesz basis of generalized eigenvectors of the nonselfadjoint problem deduced from a perturbation method for sound radiation by a vibrating plate in a light fluid, J. Math. Anal. Appl. 292 (2004), no. 1, 1-16.
[11] J. Kergomard and V. Debut, Resonance modes in a one-dimensional medium with two purely resistive boundaries : Calulation methods, orthogonality, and completeness, Acoust. Soc. Amer. 119 (2006), no. 3, 1356-1367.
[12] V. B. Lidskii, Summability of series in the principal vectors of non-selfadjoint operators, Amer. Math. Soc. Transl. Ser. 2 40 (1964), 193-228.

[13] B. S. Pavlov, Basicity of an exponential systems and Muckenhoupt ?s condition, Soviet Math. Dokl. 20 (1979), no. 4, 655-659.
[14] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci. 44, Springer, New York, 1983.
[15] J.-M. Wang, G.-Q. Xu and S.-P. Yung, Exponential stability of variable coefficients Rayleigh beams under boundary feedback controls : A Riesz basis approach, Systems Control Lett. 51 (2004), no. 1, 33-50.
[16] G. Q. Xu and S. P. Yung, The expansion of a semigroup and a Riesz basis criterion, J. Differential Equations 210 (2005), no. 1, 1-24.
[17] R. M. Young, An Introduction to Nonharmonic Fourier Series, Pure Appl. Math. 93, Academic Press, New York, 1980.